

Unit

9

CONGRUENT TRIANGLES

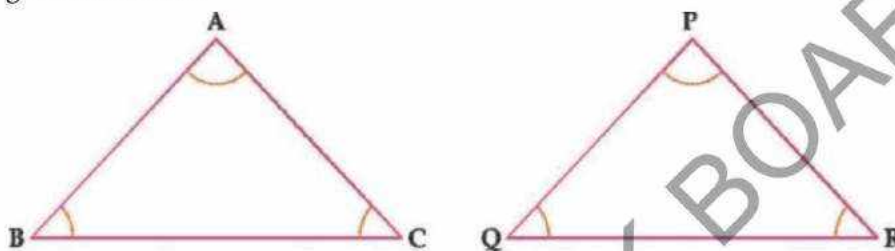
Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- ◆ Understand the following theorems along with their corollaries and apply them to solve allied problems.
- ◆ In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.
- ◆ If two angles of a triangle are congruent then the sides opposite to them are also congruent.
- ◆ In the correspondence of the two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent or similar triangles.
- ◆ If in the correspondence of two right angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

Introduction

A triangle has six elements, three sides and three angles. If we are given two triangles ABC and PQR, we can associate their vertices to establish a (1-1) correspondence between the sides and angles of these triangles in six different way given as under:



In the correspondence $\triangle ABC \leftrightarrow \triangle PQR$ it means

- (i) $\angle A \leftrightarrow \angle P$ ($\angle A$ corresponds to $\angle P$).
- (ii) $\angle B \leftrightarrow \angle Q$ ($\angle B$ corresponds to $\angle Q$).
- (iii) $\angle C \leftrightarrow \angle R$ ($\angle C$ corresponds to $\angle R$).
- (iv) $\overline{AB} \leftrightarrow \overline{PQ}$ ($\overline{AB}$ corresponds to $\overline{PQ}$).
- (v) $\overline{BC} \leftrightarrow \overline{QR}$ ($\overline{BC}$ corresponds to $\overline{QR}$).
- (vi) $\overline{CA} \leftrightarrow \overline{RP}$ ($\overline{CA}$ corresponds to $\overline{RP}$).

9.1 Congruent triangles

“Sameness of size and shape” in the mathematics called congruence.

Two cars have different colours and their position is different in the given figures. But they have same size and shape. These two cars are said to be congruent. If we keep the picture of one car on the other car then they two will overlap each other.

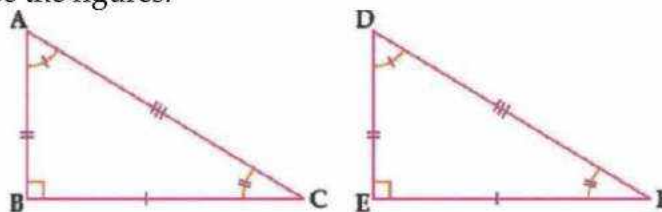


ACTIVITY

Exploration

Can you identify any congruent figures or objects in your classroom or school?
 Make a list of these congruent figures by drawing or taking photos.

Two triangles are said to be congruent if their corresponding angles and sides are congruent.
Let's see the figures.



These two triangles ABC and DEF are congruent and written as:

$$\triangle ABC \cong \triangle DEF$$

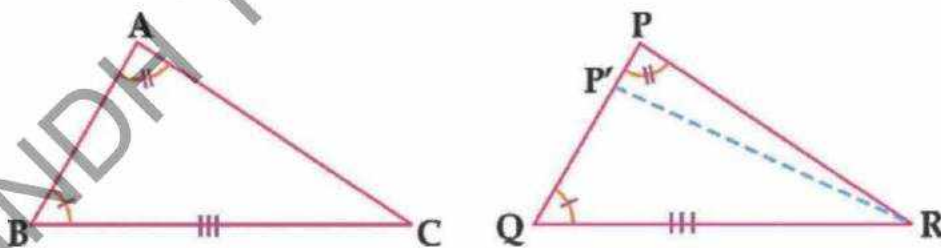
The $\triangle ABC$ and $\triangle DEF$ having their corresponding sides and angles are equal, in measure.

Note: Following results are useful.

- (i) Identity congruence i.e $\triangle ABC \cong \triangle ABC$.
- (ii) Symmetric property i.e $\triangle ABC \cong \triangle PQR$ then $\triangle PQR \cong \triangle ABC$.
- (iii) Transitive property of congruence, if $\triangle ABC \cong \triangle PQR$ and $\triangle PQR \cong \triangle DEF$, then $\triangle ABC \cong \triangle DEF$.

Theorem 9.1.1 (A.S.A. $\cong$ A.S.A.)

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.



Given:

In $\triangle ABC \leftrightarrow \triangle PQR$, then
 $\angle B \cong \angle Q$, $m\overline{BC} \cong m\overline{QR}$,
 and $\angle A \cong \angle P$.

To prove:

$$\triangle ABC \cong \triangle PQR$$

Construction:

Suppose, $\overline{AB} \not\cong \overline{PQ}$ take a point P' on PQ such that $AB \cong P'Q$.

Join P' to R .

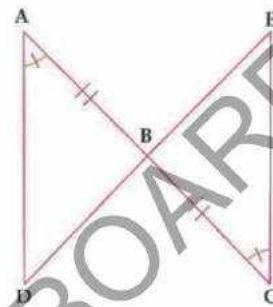
Proof:

Statements	Reasons
In $\triangle ABC \leftrightarrow \triangle PQR$	
i. $\angle A \cong \angle P$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
$\therefore \angle C \cong \angle R$	Two angles of both triangles are congruents.
If $\overline{BA} \not\cong \overline{QP}$, take a point P' on $\overline{QP}$ (or $\overline{QP}$ produced) such that:	Assumption
$\overline{QP'} \cong \overline{BA}$	
In $\triangle ABC \leftrightarrow \triangle P'QR$	
$\overline{BC} \cong \overline{QR}$	i. Given
$\angle B \cong \angle Q$	ii. Given
$\overline{BA} \cong \overline{QP'}$	iii. By supposition
$\therefore \triangle ABC \cong \triangle P'QR$	S.A.S postulate
$\therefore \angle C \cong \angle QRP'$	By the congruence of $\triangle$ s.
But $\angle C \cong \angle QRP$	Proved in 2 (above).
$\therefore \angle QRP' \cong \angle QRP$	Transitive property of congruence
This is possible only when points P' & P coincide and	By angle construction postulate
$\overline{RP'} \cong \overline{RP}$	
Hence $\overline{BA} \cong \overline{QP}$	As P and P' coincide.
In $\triangle ABC \leftrightarrow \triangle PQR$	
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP}$	iii. Proved above
$\therefore \triangle ABC \cong \triangle PQR$	S.A.S Postulate.

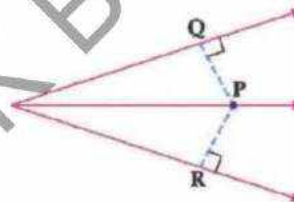
Q.E.D.

Exercise 9.1

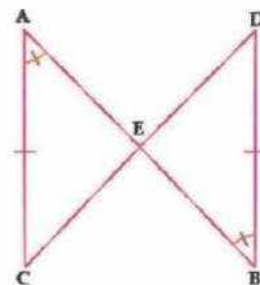
1. In the given figure, $m\overline{AB} = m\overline{CB}$ and $\angle A \cong \angle C$ prove that $\triangle ABD \cong \triangle CBE$



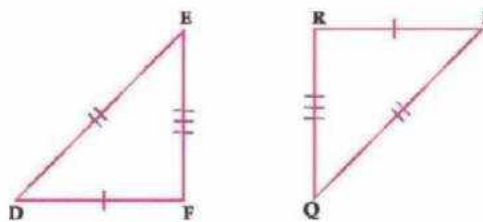
2. From a point on the line bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.



3. In the given figure, we have, $\triangle ACE \cong \triangle BDE$, such that:
 $m\overline{AC} = m\overline{BD} = 3 \text{ cm}$, $\angle A = (3x + 1)^\circ$, $m\angle E = (3y - 2)^\circ$ and
 $m\angle B = (x + 35)^\circ$. Find the values of x and y .



4. In the given figure, $\triangle DEF \cong \triangle PQR$, such that:
 $m\overline{DE} = (6x + 1)\text{cm}$, $m\overline{EF} = 8\text{cm}$,
 and $m\overline{RQ} = (5y - 7)\text{cm}$
 and $m\overline{PQ} = (10x - 19)\text{cm}$.
 Find the values of x and y .



Theorem 9.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.

Given:

In $\triangle ABC$,

We have, $\angle B \cong \angle C$

To prove:

$$\overline{AB} \cong \overline{AC}$$

Construction: Draw $\overline{AD}$ the bisector of $\angle A$, meeting $\overline{BC}$ at point D.

Proof:

Statement	Reason
In $\triangle ADE \leftrightarrow \triangle ADC$	
i. $\overline{AB} \cong \overline{AC}$	i. Given
ii. $\angle 1 \cong \angle 2$	ii. Construction
i. $\overline{AD} \cong \overline{AD}$	iii. Common (Identity congruence)
$\therefore \triangle ABD \cong \triangle ADC$	S.A.S Postulate.
$\therefore m\angle B = m\angle C$	By the congruence of $\triangle$ s.

Q.E.D

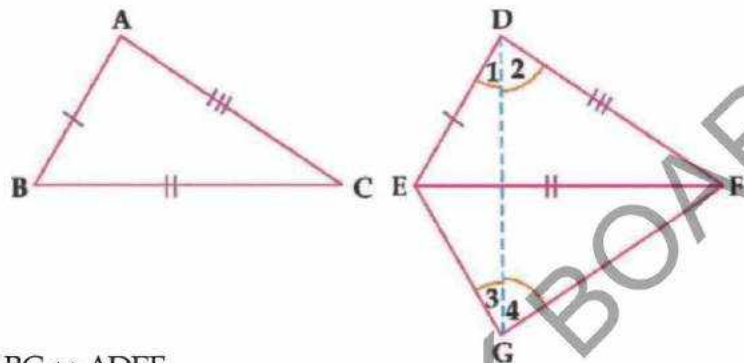
Exercise 9.2

1. ABC is a triangle in which $m\angle A = 35^\circ$ and $m\angle B = 100^\circ$, $\overline{BD} \perp \overline{AC}$. Prove that $\triangle BDC$ is an isosceles triangle.
2. If the bisector of an angle of a triangle is perpendicular to its opposite side, then the triangle is an isosceles triangle.
3. ABC is a triangle in which $m\angle A = 25^\circ$, $m\angle B = 45^\circ$ and $\overline{CD} \perp \overline{AB}$. Prove that $\triangle DBC$ is an isosceles $\triangle$.

Theorem 9.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.

Proof:



Given:

In $\triangle ABC \leftrightarrow \triangle DEF$

$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To prove that: $\triangle ABC \leftrightarrow \triangle DEF$

Construction: Suppose $\overline{BC}$ greatest of all the three sides of $\triangle ABC$. Construct $\triangle GEF$ such that:

- Point G
 - $\angle FEG \cong \angle B$
 - $\overline{EG} \cong \overline{BA}$
- Join D and G.

Proof:

Statements	Reasons
In $\triangle ABC \leftrightarrow \triangle GEF$	
i. $\overline{BC} \cong \overline{EF}$	i. Given
ii. $\angle B \cong \angle GEF$	ii. Construction
iii. $\overline{BA} \cong \overline{GE}$	iii. Construction
$\therefore \triangle ABC \cong \triangle GEF$	S.A.S. postulate.
$\therefore \overline{AC} \cong \overline{GF}$ and $\angle A \cong \angle G$	By the congruence of triangles.
But $\overline{DF} \cong \overline{AC}$	Given
$\therefore \overline{GF} \cong \overline{DF}$	Transitive property.
$\therefore$ In $\triangle DEG$, $m\angle 1 = m\angle 3$	Opposite sides congruent
	$\overline{EG} \cong \overline{BA} \cong \overline{ED}$

Similarly, in $\triangle GFE$, $m\angle 2 = m\angle 4$

$$\therefore m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$$

$$\text{or } m\angle D = m\angle G$$

$$\text{But } m\angle G = m\angle A$$

$$\therefore m\angle A = m\angle D$$

In $\triangle ABC \leftrightarrow \triangle DEF$

$$\text{i. } \overline{AB} \cong \overline{DE}$$

$$\text{ii. } \angle A \cong \angle D$$

$$\text{iii. } \overline{AC} \cong \overline{DF}$$

$$\therefore \triangle ABC \cong \triangle DEF$$

$$\overline{DF} \cong \overline{GF}$$

Addition property of equation

$$m\angle 1 + m\angle 2 = m\angle D$$

$$m\angle 3 + m\angle 4 = m\angle G$$

Proved above

Transitive property

i. Given

ii. Proved above

iii. Given

S.A.S Postulate

Q.E.D

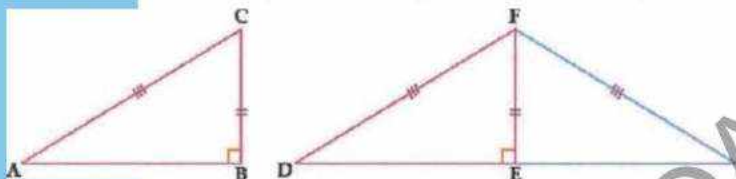
Corollary: The angles of an equilateral triangle are also equal in measurement.

Exercise 9.3

1. ABC is an isosceles triangle. D is the mid-point of base $\overline{BC}$. Prove that $\overline{AD}$ bisects $\angle A$ and $\overline{AD} \perp \overline{BC}$.
2. ABC and DBC are two isosceles triangles on the same side of a common base $\overline{BC}$. Prove that $\overline{AD}$ is the right bisector of $\overline{BC}$.
3. PQRS is a square. X, Y and Z are the mid-points of $\overline{PQ}$, $\overline{QR}$ and $\overline{RS}$ respectively. Prove that $\triangle PXY \cong \triangle SZY$.

Theorem 9.1.4

If in the correspondence of two right-angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent (H.S $\cong$ H.S).



Given: In correspondence

$$\triangle ABC \leftrightarrow \triangle DEF$$

$$\angle B \cong \angle E \text{ (rt } \angle\text{s)} \quad \overline{AC} \cong \overline{DF} \text{ (Hyp)} \text{ and } \overline{BC} \cong \overline{EF}$$

To Prove: $\triangle ABC \cong \triangle DEF$

Construction: Produce $\overline{DE}$ to point G such that $\overline{EG} \cong \overline{AB}$. Then join F and G.

Proof:

Statements	Reasons
$m\angle DEF + m\angle GEF = 180^\circ$	Supplement postulate
But $m\angle DEF = 90^\circ$	Given
$\therefore m\angle GEF = 90^\circ$	$\therefore 180^\circ - 90^\circ = 90^\circ$
In $\triangle GEF \leftrightarrow \triangle ABC$	Construction
i. $\overline{GE} \cong \overline{AB}$	Each is rt angle
ii. $\angle GEF \cong \angle ABC$	Given
iii. $\overline{EF} \cong \overline{BC}$	S.A.S. $\cong$ S.A.S.
$\therefore \triangle GEF \cong \triangle ABC$	By the congruence of Δ s.
$\therefore \overline{FG} \cong \overline{AC}$ and $\angle G \cong \angle A$	$\therefore \overline{AC} \cong \overline{DF}$ (Given)
$\therefore \overline{FG} \cong \overline{DF}$	Opposite sides congruents
In $\triangle DFG$, $\angle D \cong \angle G$	Each is congruent to $\angle G$
$\therefore \angle D \cong \angle A$	
In $\triangle ABC \leftrightarrow \triangle DEF$	
i. $\angle A \cong \angle D$	i. Proved
ii. $\angle ABC \cong \angle DEF$	ii. rt Δ s
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
$\therefore \triangle ABC \cong \triangle DEF$	A.A.S. $\cong$ A.A.S

Q.E.D

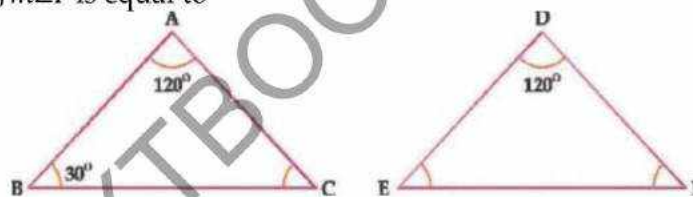
Exercise 9.4

1. Prove that:
The perpendiculars from the vertices of the base to opposite sides of an isosceles triangle are congruent.
2. Prove that, if the bisector of an angle of a triangle bisects its opposite side, then the triangle will be an isosceles triangle.
3. Prove that, in an equilateral triangle any two median are congruent.
4. Prove that the median bisecting the base of an isosceles triangle bisect the vertical angle and is perpendicular to the base.
5. Prove that if three altitudes of a triangle are congruent, then the triangle is equilateral.

Review Exercise 9

1. If $\triangle ABC \cong \triangle DEF$, $m\angle F$ is equal to

- A. 90°
- B. 60°
- C. 30°
- D. 20°



2. Identify true and false statements in the following:

- (i) The sum of the measure of all angles in a quadrilateral is 360° .
- (ii) The sum of the measure of all angles in a triangle is 270° .
- (iii) In an equilateral triangle, angles are of the same measurement.
- (iv) There are two right angles in a triangle.
- (iv) In an isosceles triangle, corresponding angles and corresponding sides are equal in measure.

3. Fill in the blanks to make the sentences true sentences:

- (i) In $\triangle ABC \leftrightarrow \triangle DEF$, then $\overline{AC}$ corresponds to _____.
- (ii) In $\triangle KLM \leftrightarrow \triangle PQR$, then $\angle MKL$ corresponds to _____.
- (iii) In an isosceles triangle, the base angles are _____.
- (iv) If the measure of each of the angles of a triangle is 60° , then the triangle is _____.
- (v) In a right-angled triangle, side opposite to right angle is called _____.
- (vi) The sum of the measures of acute angles of a right triangle is _____.



4. Encircle the corresponding letters a, b, c or d for correct answer:
- Which of the following is not a sufficient condition for congruence of two triangles?

(a) $A.S.A \cong A.S.A$	(b) $H.S.H \cong H.S.H$
(c) $S.A.A \cong S.A.A$	(d) $A.A.A \cong A.A.A$
 - In $\triangle ABC$, if $\angle A \cong \angle B$, then the bisector of ____ angle divides the triangle into congruent triangles:

(a) $\angle A$	(b) $\angle B$
(c) $\angle C$	(d) any one of its angles.
 - The diagonal of ____ does not divide it into two congruent triangles:

(a) Rectangle	(b) Trapezium
(c) Parallelogram	(d) Square
 - How many acute angles are there in an acute angled triangle?

(a) 1	(b) 2
(c) 3	(d) not more than 2.

Summary

In this unit we stated and proved the following theorem:

- ◆ In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent. ($A.S.A \cong A.S.A$)
- ◆ If two angles of a triangle are congruent, then the side opposite to them are also congruent.
- ◆ In the correspondence of two triangles, if three sides of two triangles are congruent to the corresponding three sides of other, then the two triangles are congruent ($S.S.S \cong S.S.S$).
- ◆ If in the correspondence of the two right-angled triangles the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of other, then the triangles are congruent. ($H.S \cong H.S$).
- ◆ Two triangles are said to be congruent, if there exists a correspondence between them such that all the corresponding sides and triangles are congruent. ($S.S.S$).

