

Unit

15

PROJECTION OF A SIDE OF A TRIANGLE

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- ◆ In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- ◆ In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- ◆ In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

15.1 Projection of a side of a Triangle

Understand the following theorems along with their corollaries and their applications to solve the allied problems.

Theorem 15.1.1

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given:

$\triangle ABC$ with an obtuse angle at vertex A.

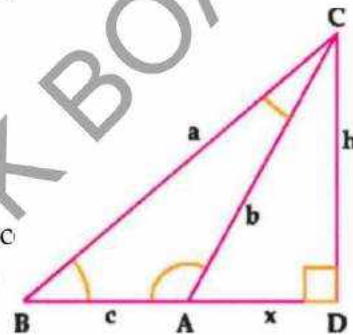
To prove:

$$\text{i.e. } a^2 = b^2 + c^2 + 2cx$$

Construction:

Draw perpendicular $\overline{CD}$ on $\overline{BA}$ produced meeting at point D, so that AD is the projection of $\overline{AC}$ on $\overline{BA}$ produced.

Taking, $m\overline{BC} = a$, $m\overline{CA} = b$, and also $m\overline{AD} = x$ and $m\overline{CD} = h$.



Proof:

Statements	Reasons
In right angled $\triangle CDA$ $m\angle CDA = 90^\circ$	Construction
so, $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{DC})^2$	By Pythagoras theorem.
i.e. $b^2 = x^2 + h^2 \dots (i)$	By supposition
In right angled $\triangle CDB$ $m\angle CDB = 90^\circ$	Construction
so, $(m\overline{BC})^2 = (m\overline{BD})^2 + (m\overline{CD})^2$	By Pythagoras theorem.
i.e. $a^2 = (c+x)^2 + h^2 \dots (ii)$	$m\overline{BD} = m\overline{BA} + m\overline{AD}$
or $a^2 = c^2 + 2cx + x^2 + h^2 \dots (ii)$	$b^2 = x^2 + h^2$
$a^2 = c^2 + 2cx + b^2$	
Thus $(BC)^2 = (AB)^2 + 2(AB)(AD) + (AC)^2$	

Q.E.D

Example: In a $\triangle ABC$ with obtuse angle is at vertex A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced, meeting at point D, and $m\overline{AC} = m\overline{AB}$. Then, $(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD})$.

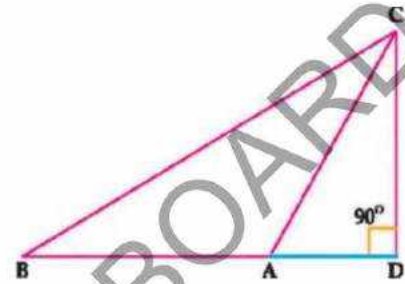
Given:

In a $\triangle ABC$, $m\angle A$ is an obtuse,
 $m\overline{AC} = m\overline{AB}$ and $\overline{CD}$ being altitude on
 $\overline{BA}$ produced, meeting at point D.

To prove:

$$(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD}).$$

Proof:



Statements	Reasons
In a $\triangle ABC$	
$(m\overline{BC})^2 = (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD}) + (m\overline{AC})^2$	Theorem 15.1.1
$(m\overline{BC})^2 = (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD}) + (m\overline{AB})^2$	Given that $m\overline{AC} = m\overline{AB}$
$(m\overline{BC})^2 = 2(m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$	Taking $2m\overline{AB}$ as common
$(m\overline{BC})^2 = 2m\overline{AB}(m\overline{AB} + m\overline{AD})$	$m\overline{BD} = m\overline{AD} + m\overline{AB}$
$(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD})$	

Q.E.D

Exercise 15.1

- Find the length of $\overline{AB}$ and area of the triangle ABC, when
 - $m\overline{AC} = 3\text{ cm}$, $m\overline{BC} = 6\text{ cm}$ and $m\angle C = 120^\circ$, where
 $m\overline{CD} = m\overline{BC} \cos(180^\circ - m\angle C)$
 Hint: $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$
 - $m\overline{AC} = 40\text{ mm}$, $m\overline{BC} = 80\text{ mm}$ and $m\angle C = 120^\circ$, where
 $m\overline{CD} = m\overline{BC} \cos(180^\circ - m\angle C)$
 Hint: $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$
- Find the length of $m\overline{AC}$ in the $\triangle ABC$ when $m\overline{BC} = 6\text{ cm}$, $m\overline{AB} = 4\sqrt{2}\text{ cm}$ and $m\angle ABC = 135^\circ$. If possible, find the area of the $\triangle ABC$.
- Find the length of $m\overline{AC}$ in the $\triangle ABC$ when $m\overline{BC} = 6\sqrt{2}\text{ cm}$, $m\overline{AB} = 8\text{ cm}$ and $m\angle ABC = 135^\circ$. If possible, find the area of the $\triangle ABC$.

Theorem 15.1.2

In any triangle, the square on the side opposite to an acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given:

$\triangle ABC$ with an acute $\angle CAB$ at vertex A.

Taking, $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$

Construction:

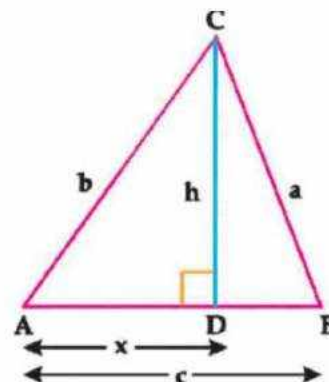
Draw $\overline{CD} \perp \overline{AC}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also $m\overline{AD} = x$ and $m\overline{CD} = h$

To prove:

$$(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$$

$$\text{i.e. } a^2 = b^2 + c^2 - 2cx$$



Proof:

Statements	Reasons
In right angled $\triangle CDA$, $m\angle CDA = 90^\circ$, so, $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{CD})^2$ i.e. $b^2 = x^2 + h^2$... (i)	Construction Using Pythagoras Theorem. By supposition
In right angled $\triangle CDB$, $m\angle CDB = 90^\circ$ $(m\overline{BC})^2 = (m\overline{BD})^2 + (m\overline{CD})^2$ so, i.e. $a^2 = (c-x)^2 + h^2$ $a^2 = c^2 - 2cx + x^2 + h^2$ or $a^2 = c^2 - 2cx + b^2$... (ii) $a^2 = c^2 + b^2 - 2cx$	Construction By Pythagoras Theorem. From the figure.
Hence $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$	$\therefore m\overline{BD} = m\overline{AB} - m\overline{AD}$ Using equation... (i)

Q.E.D

Apollonius and the theorem of Apollonius:

Apollonius was a great geometer and astronomer.

Now we state and prove one of his well-known theorem "the Apollonius theorem".

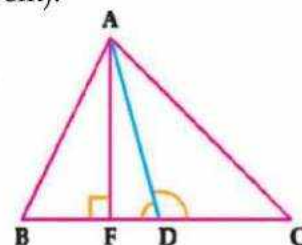
Theorem 15.1.3 (Apollonius theorem)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

Given:

In $\triangle ABC$, the median $\overline{AD}$ bisects $\overline{BC}$ at point D.

such that $m\overline{BD} = m\overline{CD}$.



To prove:

$$(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$$

Construction:Draw $\overline{AF} \perp \overline{BC}$.**Proof:**

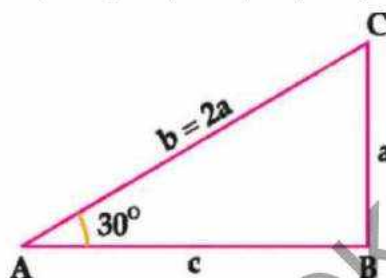
Statements	Reasons
In $\triangle ADB$ Since, $\angle ADB$ is acute	$\triangle ADF$ is right angled triangle with right angled at F (construction)
So, $(m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2 - 2(m\overline{BD})(m\overline{FD}) \dots (i)$	Using theorem 15.1.2
Now, In $\triangle ADC$ We $\angle ADC$ is an obtuse angle at point D.	Supplement of in $\angle ADB$
So, $(m\overline{AC})^2 = (m\overline{CD})^2 + (m\overline{AD})^2 + 2(m\overline{CD})(m\overline{FD})$	Theorem 15.1.1
$(m\overline{AC})^2 = (m\overline{BD})^2 + (m\overline{AD})^2 + 2(m\overline{BD})(m\overline{FD}) \dots (ii)$	$m\overline{CD} = m\overline{BD}$
Thus, $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$	Adding eqs. (i) and (ii)

Q.E.D

Exercise 15.2

1. In $\triangle ABC$, $m\angle A = 30^\circ$, prove that

$$(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - \sqrt{3}(m\overline{AB})(m\overline{AC}).$$



2. In a $\triangle ABC$, calculate when $m\overline{BC} = 6\text{cm}$, $m\overline{AC} = 5\text{cm}$, and $m\angle A = 30^\circ$
 3. In a $\triangle ACB$, $a = 6\text{cm}$, $b = 8\text{cm}$ and $c = 10\text{cm}$. Find $m\angle A$
 4. Whether the triangle with sides 3cm, 4cm and 5cm is acute, obtuse or right angled.
 5. Whether the triangle with sides 4cm, 7cm and 8cm is acute, obtuse or right angled.

Review Exercise 15

1. Fill in the blanks.

(i) In rt. $\triangle ABC$, $(m\overline{AB})^2 + \underline{\hspace{2cm}} = (m\overline{AC})^2$,

(ii) In $\triangle ABC$, two sides are equal to 4cm it is called _____ triangle.

(iii) In $\triangle ABC$, with $m\angle B = 90^\circ$ then $m\angle B = 90^\circ$ then $(m\overline{AB})^2 + (m\overline{BC})^2 = \underline{\hspace{2cm}}$

(iv) 8cm, 15cm and 17 cm are the sides of _____

2. In $\triangle ABC$, $m\overline{AC} = 3\text{cm}$, $m\overline{BC} = 6\text{cm}$ and $m\angle C = 120^\circ$. Compute $m\overline{AB}$

 Summary

- ◆ In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- ◆ In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- ◆ In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side, (Apollonius theorem).