

# CHORDS AND ARCS

Unit

27

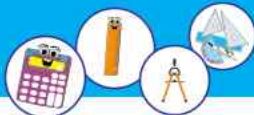
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## Students Learning Outcomes (SLOs)

**After completing this unit, students will be able to:**

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
  - ❖ If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
  - ❖ If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi-circular) are congruent.
  - ❖ Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
  - ❖ If the angles subtended by the two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

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### Introduction:

In this unit, we discuss the theorems related to chords and arcs of a circle.

### Theorem 27.1

If two arcs of a circle (or congruent circles) are congruent then the corresponding chords are equal.

We shall prove the theorem:

- For one circle
- For two congruent circles

#### i. For one circle

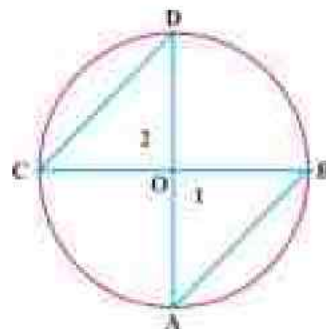
**Given:** A circle with center O whose  $\widehat{AB}$  and  $\widehat{CD}$  are congruent arcs i.e.  $\widehat{AB} \cong \widehat{CD}$ .

$\overline{AB}$  and  $\overline{CD}$  are the corresponding chords of the given congruent arcs.

**To prove:**  $\overline{AB} \cong \overline{CD}$

**Construction:** Join the points O with A, B, C and D.

**Proof:**



| Statement                                        | Reason                                     |
|--------------------------------------------------|--------------------------------------------|
| In $\triangle OAB \leftrightarrow \triangle OCD$ |                                            |
| $\overline{OA} \cong \overline{OC}$              | Radii of same circle                       |
| $\overline{OB} \cong \overline{OD}$              | Radii of same circle                       |
| $m\angle 1 = m\angle 2$                          | Central angles of two congruent arcs       |
| $\triangle OAB \cong \triangle OCD$              | S.A.S postulate                            |
| $\overline{AB} \cong \overline{CD}$              | Corresponding sides of congruent triangles |

Q.E.D

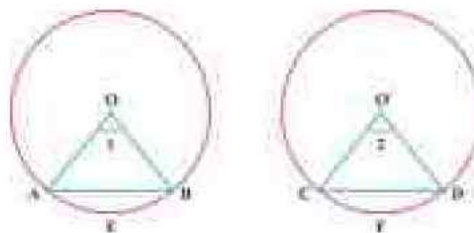
#### ii. For two congruent circles

**Given:** Two congruent circles with centers O and O' respectively.  $\widehat{AB}$  and  $\widehat{CD}$  are congruent arcs of these circles where  $\overline{AB}$  and  $\overline{CD}$  are the corresponding chords.

**To prove:**  $\overline{AB} \cong \overline{CD}$

**Construction:** Join O to A and B. Join O' to C and D.

**Proof:**



| Statement                                         | Reason                                        |
|---------------------------------------------------|-----------------------------------------------|
| In $\triangle OAB \leftrightarrow \triangle O'CD$ |                                               |
| $\overline{OA} \cong \overline{O'C}$              | Radii of same circle                          |
| $\overline{OB} \cong \overline{O'D}$              | Radii of same circle                          |
| $m\angle 1 = m\angle 2$                           | Central angles of two congruent arcs          |
| $\therefore \triangle OAB \cong \triangle O'CD$   | S.A.S postulate                               |
| $\therefore \overline{AB} \cong \overline{CD}$    | Corresponding sides of two congruent triangle |



### Theorem 27.2 (Converse of theorem 1)

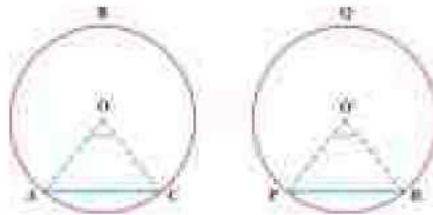
If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi-circle) are congruent.

**Given:** Two congruent circles with center  $O$  and  $O'$  respectively having two congruent arcs i.e.,  $\widehat{AC} \cong \widehat{PR}$ .

**To prove:**  $\widehat{AC} \cong \widehat{PR}$

**Construction:** Join  $O$  with  $A$  and  $C$ ,  $O'$  with  $P$  and  $R$ .

**Proof:**



| Statement                                         | Reason                                        |
|---------------------------------------------------|-----------------------------------------------|
| In $\triangle AOC \leftrightarrow \triangle PO'R$ |                                               |
| $\overline{OA} \cong \overline{O'P}$              | Radii of congruent circles                    |
| $\overline{OC} \cong \overline{O'R}$              | Radii of congruent circles                    |
| $\widehat{AC} \cong \widehat{PR}$                 | Given                                         |
| $\triangle OAC \cong \triangle PO'R$              | S.S.S $\cong$ S.S.S                           |
| $m\angle AOC = m\angle PO'R$ (i)                  | Corresponding angles of congruent $\Delta$ s. |
| Hence $\widehat{AC} \cong \widehat{PR}$           | From eq: (i)                                  |

**Q.E.D**

### Example 1:

A point  $P$  on the circumference of a circle is equidistant from the radial segments  $\overline{OA}$  and  $\overline{OB}$ . Prove that  $m\widehat{AP} = m\widehat{BP}$

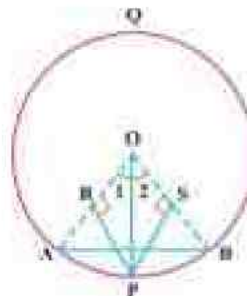
**Solution:**

**Given:**  $\overline{AB}$  is the chord of a circle with center  $O$ . Point  $P$  on the circumference of the circle is equidistant from the radial segment  $\overline{OA}$  and  $\overline{OB}$ . i.e.  $m\widehat{PR} = m\widehat{PS}$

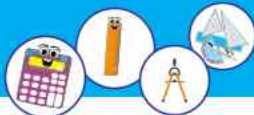
**To prove:**  $m\widehat{AP} = m\widehat{BP}$

**Construction:** Join  $O$  with  $P$ . Write  $m\angle 1$  and  $m\angle 2$  as shown in the figure.

**Proof:**



| Statement                                                    | Reason                                       |
|--------------------------------------------------------------|----------------------------------------------|
| In right angle $\triangle OPR \leftrightarrow \triangle OPS$ |                                              |
| $m\widehat{OP} = m\widehat{OP}$                              | Common                                       |
| $m\widehat{PR} = m\widehat{PS}$                              |                                              |
| $\therefore \triangle OPR \cong \triangle OPS$               | H.S $\cong$ H.S                              |
| So $m\angle 1 = m\angle 2$ (i)                               | Corresponding angles of congruent $\Delta$ s |
| $\Rightarrow \widehat{AP} \cong \widehat{BP}$                | From (i)                                     |
| Hence $m\widehat{AP} = m\widehat{BP}$                        | By definition of congruent arcs              |



### Theorem 27.3

Equal chords of a circle (or of congruent circles) subtend equal angles at the center (at the corresponding centers).

We shall prove theorem:

i) For one circle

ii) For two circles

#### i) For one circle

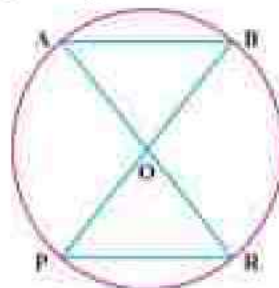
**Given:** Circle with centre O, having two congruent chords i.e.,

$\overline{AB} \cong \overline{PR}$ .

**To prove:**  $m\angle AOB = m\angle POR$

**Construction:** Join O with A and B and also O with P and R.

**Proof:**



| Statement                                     | Reason                                                                                     |
|-----------------------------------------------|--------------------------------------------------------------------------------------------|
| $\overline{AB} \cong \overline{PR}$ (i)       | Given                                                                                      |
| $\overline{OB} \cong \overline{OR}$ (ii)      | $\overline{OB}, \overline{OR}, \overline{OA}$ and $\overline{OP}$ are radii of same circle |
| and $\overline{OA} \cong \overline{OP}$ (iii) |                                                                                            |
| $\triangle AOB \cong \triangle POR$           | S.S.S $\cong$ S.S.S                                                                        |
| $m\angle AOB = m\angle POR$                   | Corresponding angles of congruent triangles                                                |

#### Q.E.D

#### ii) For two congruent circles

**Given:**

Two congruent circles with center O and O' respectively so that  $m\widehat{AC} = m\widehat{PR}$ . Where  $\widehat{AC}$  and  $\widehat{PR}$  are the chords of circles.

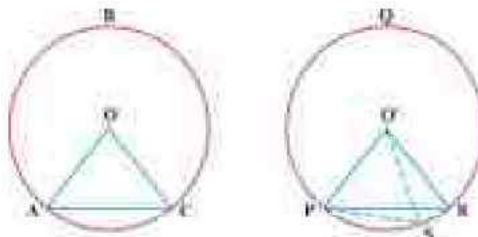
**To prove:**

$m\angle AOC = m\angle PO'S$

**Construction:**

Take a point S on arc PR and join S to P and O' such that  $\angle AOC \cong \angle PO'S$ .

**Proof:**



| Statement                                        | Reason                                                      |
|--------------------------------------------------|-------------------------------------------------------------|
| Let $m\angle AOC = m\angle PO'R$                 |                                                             |
| But $m\angle AOC \cong m\angle PO'S$             | Construction                                                |
| $\therefore \widehat{AC} \cong \widehat{PS}$ (i) | Arcs subtended by equal central angles of congruent circles |
| $\therefore m\widehat{AC} = m\widehat{PS}$ (ii)  | From theorem (1)                                            |
| But $m\widehat{AC} = m\widehat{PR}$ (iii)        | Given                                                       |
| $\therefore m\widehat{PR} = m\widehat{PS}$       | Using (ii) and (iii)                                        |
| Which is only possible, if R coincides with S.   | Because P is common point                                   |
| $\Rightarrow m\angle AOC = m\angle POR$          | Our supposition is working                                  |



**Corollary 1:**

In congruent circles or in same circle, if central angles are equal then corresponding sectors are equal.

**Corollary 2:**

In congruent circles or in same circle, arcs will subtend unequal central angles for unequal chords.

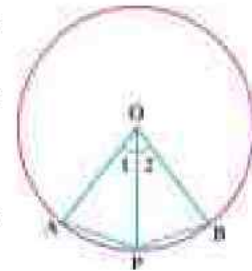
**Example 1:**

Prove that the internal bisector of a central angle in a circle bisects the corresponding arc.

**Given:** A circle with center O.  $\overline{OP}$  is an internal bisector of central angle AOB. i.e.  $\angle 1 \cong \angle 2$

**To prove:**  $\widehat{AP} \cong \widehat{PB}$

**Construction:** Draw  $\overline{AP}$  and  $\overline{BP}$ , then write  $\angle 1$  and  $\angle 2$  are corresponding angles of  $\widehat{AP}$  and  $\widehat{BP}$  respectively.

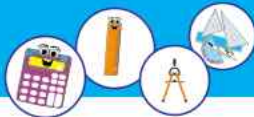


**Proof:**

| Statement                                        | Reason                                          |
|--------------------------------------------------|-------------------------------------------------|
| In $\triangle OAP \leftrightarrow \triangle OBP$ |                                                 |
| $\overline{OA} = \overline{OB}$                  | Radii of same circle                            |
| $m\angle 1 = m\angle 2$                          | Given                                           |
| and $\overline{OP} = \overline{OP}$              | Common                                          |
| $\therefore \triangle OAP \cong \triangle OBP$   | S.A.S postulate                                 |
| Hence $\overline{AP} \cong \overline{BP}$        | Corresponding sides of congruent $\triangle$ s  |
| $\Rightarrow \widehat{AP} \cong \widehat{BP}$    | Arcs corresponding to equal chords in a circle. |

Q.E.D

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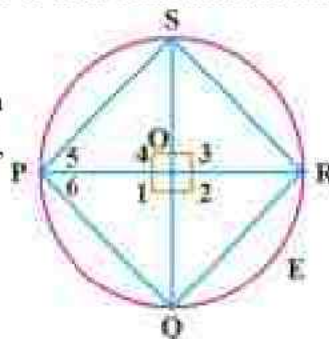


### Example 2:

Prove that in a circle, if two diameters are perpendicular to each other then the lines joining their ends form a square.

#### Given:

Given  $\overline{PR}$  and  $\overline{QS}$  are two perpendicular diameters of a circle with center O. So PQRS is a quadrilateral. Naming  $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$  and  $\angle 6$  as shown in figure.



#### To prove:

PQRS is a square.

#### Proof:

| Statement                                                                                      | Reason                                                  |
|------------------------------------------------------------------------------------------------|---------------------------------------------------------|
| $\overline{PR}$ and $\overline{QS}$ are two perpendicular diameters of a circle with center O. | Given                                                   |
| $m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^\circ$                                     | Diameters are perpendicular to each other               |
| $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{SP}$ (i)                        | Arcs opposite to the equal central angles in a circle   |
| $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{SP}$ (ii)                       | Chords corresponding to equal arcs.                     |
| In right $\triangle POS$ ,                                                                     |                                                         |
| $m\overline{PO} = m\overline{OS}$ (iii)                                                        | $\overline{PO}$ and $\overline{OS}$ are radii of circle |
| $\triangle POS$ is right isosceles, triangle                                                   | From (iii)                                              |
| $m\angle 5 = 45^\circ$ (iv)                                                                    | $\triangle POS$ is right isosceles, triangle            |
| Similarly, $\triangle POQ$ is right isosceles triangle                                         | By the above process                                    |
| $m\angle 6 = 45^\circ$ (v)                                                                     |                                                         |
| Moreover,                                                                                      |                                                         |
| $m\angle P = m\angle 5 + m\angle 6$                                                            | Angle addition postulate                                |
| $m\angle P = 45^\circ + 45^\circ$                                                              | By using (iv) and (v)                                   |
| $m\angle P = 90^\circ$ (vi)                                                                    |                                                         |
| Similarly, $m\angle Q = m\angle R = m\angle S = 90^\circ$ (vii)                                | By the above process                                    |
| PQRS is a square                                                                               | Using eq (ii), (vi) and (vii)                           |

Q.E.D



#### Theorem 27.4

If the angles subtended by two chords of a circle (or congruent circles) at the centres (corresponding centres) are equal, the chords are equal.

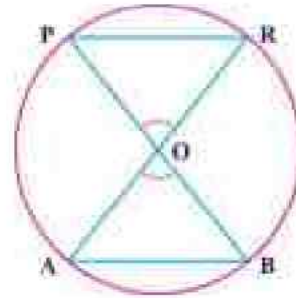
**(a) For one circle**

**Given:**

Circle with centre O, having chords  $\overline{PR}$  and  $\overline{AB}$ . The angles subtended by both arcs are congruent i.e.  $\angle POR \cong \angle AOB$ .

**To prove:**  $m\overline{AB} \cong m\overline{PR}$

**Construction:** Join O with P and R, and also with A and B.



**Proof:**

| Statement                                    | Reason                                    |
|----------------------------------------------|-------------------------------------------|
| $\overline{OR} \cong \overline{OB}$ (i)      | Radius of same circle                     |
| and $\overline{OP} \cong \overline{OA}$ (ii) | Radii of same circle                      |
| $\angle POR \cong \angle AOB$ (iii)          | Given                                     |
| $\triangle POR \cong \triangle AOB$          | S.A.S $\cong$ S.A.S (Postulate)           |
| $\overline{AB} \cong \overline{PR}$          | Corresponding side of congruent triangles |

**Q.E.D**

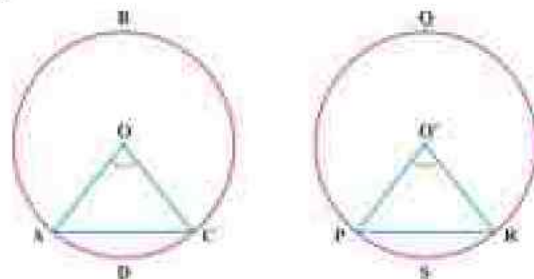
**(b) For two congruent circles**

**Given:**

Two congruent circles with center O and O' with  $\overline{AC}$  and  $\overline{PR}$  are chords. Subtending equal angles at centre i.e.  $\angle AOC \cong \angle PO'R$ .

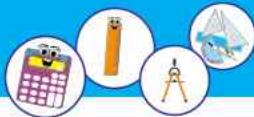
**To prove:**  $m\overline{AC} = m\overline{PR}$

**Proof:**



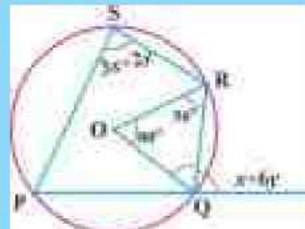
| Statement                                       | Reason                                         |
|-------------------------------------------------|------------------------------------------------|
| In $\triangle AOC \cong \triangle PO'R$         |                                                |
| $\overline{OA} \cong \overline{O'A}$            | Radii of congruent circles                     |
| $\angle AOC \cong \angle PO'R$                  | Given                                          |
| $\overline{OC} \cong \overline{O'R}$            | Radii of congruent circle                      |
| $\therefore \triangle AOC \cong \triangle PO'R$ | S.A.S postulate                                |
| Hence $m\overline{AC} = m\overline{PR}$         | Corresponding sides of congruent $\triangle$ s |

**Q.E.D**

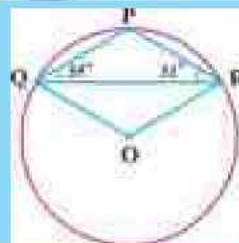


### EXERCISE 27.1

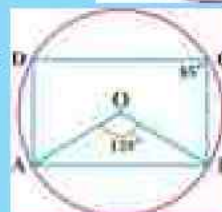
1. In a circle prove that the arcs between two parallel and equal chords or equal.
2. Prove that in equal circles, equal central angles have equal arcs.
3. In the following figure, O is the centre of the circle. Find the value of  $x$  and  $y$ .



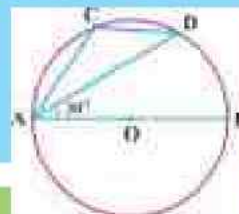
4. In the following figure  $m\angle PQR = 35^\circ$  and  $m\angle PRQ = 31^\circ$ . Find  $m\angle QPR$  and  $m\angle OQR$ .



5. In the figure O is the center of circle, given that  $m\angle AOB = 120^\circ$  and  $m\angle BCD = 85^\circ$ . Find  $m\angle OAD$



6. In the following figure,  $\overline{AB}$  is diameter of the circle given that  $m\angle DAB = 30^\circ$  Find  $m\angle ACD$



### REVIEW EXERCISE 27

#### 1. Multiple Choice Question: Tick the correct option.

- i. If a chord of a circle subtends a central angle of  $60^\circ$ . Then the chord and the radial segment are \_\_\_\_\_.  
(a) Parallel (b) Perpendicular (c) Congruent (d) Incongruent
- ii. An arc subtends a central angle of  $45^\circ$  then the corresponding chords will subtend a central angle of ?  
(a)  $15^\circ$  (b)  $30^\circ$  (c)  $45^\circ$  (d)  $60^\circ$
- iii. A pair of chords of a circle subtending two congruent central angle are.  
(a) Perpendicular (b) Non congruent (c) Congruent (d) None of these



- iv. The arcs opposite to congruent central angles of a circle are always.  
(a) Parallel (b) Congruent (c) Perpendicular (d) None of these
- v. A 6cm long chord subtends a central angle of  $60^\circ$ . The radial segment of this circle is  
(a) 4cm (b) 6cm (c) 5cm (d) 8cm
- vi. The chord length of a circle subtending a central angle of  $180^\circ$  is always  
(a) equal to the radial segment  
(b) less than radial segment  
(c) double of radial segment  
(d) half of the radial segment.
- viii. The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:  
(a)  $60^\circ$  (b)  $75^\circ$  (c)  $90^\circ$  (d)  $45^\circ$
- ix. Out of two congruent arcs of a circle, if one arc makes a central angle of  $30^\circ$  then the other arc will subtend the central angle:  
(a)  $60^\circ$  (b)  $90^\circ$  (c)  $75^\circ$  (d)  $30^\circ$
- x. Diameter divides the circle into \_\_\_\_\_ parts.  
(a) Two (b) Three (c) Four (d) All of these

### SUMMARY

- The circles are congruent if their radii are equal.
- Equal chords of a circle subtend equal angles at the centre.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The boundary traced by a moving point in a circle is called its circumference.
- An arc of the circumference of a circle is called an arc of the circle.
- A straight line drawn from the centre of a circle bisecting a chord is perpendicular to the chord.
- A perpendicular drawn from the centre of a circle on a chord, it bisects the chord.
- If two circles (or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angle subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.