

A TEXTBOOK OF

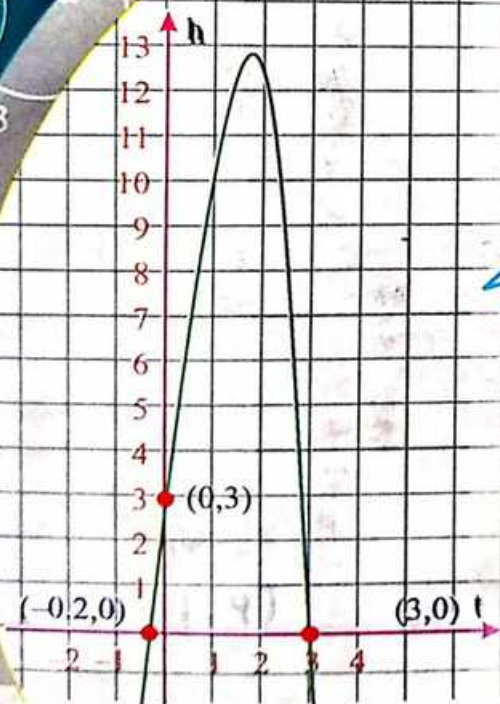
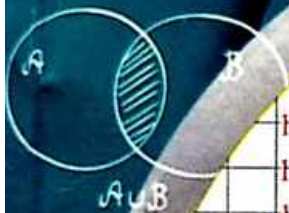
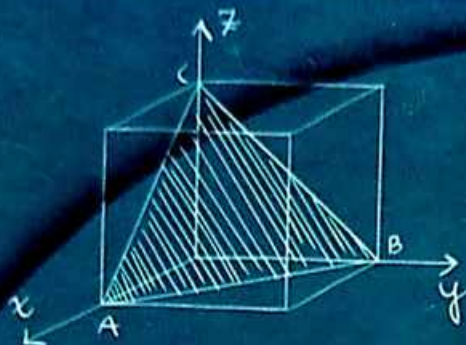
Mathematics

Grade

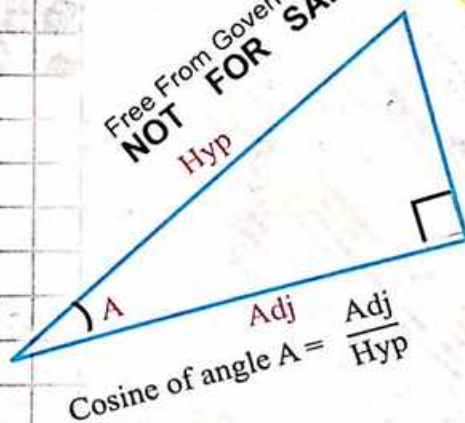
X



$$V = \pi r^2 h$$



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Unit

1

QUADRATIC EQUATIONS

In this unit the students will be able to

- Define a quadratic equation.
- Solve a quadratic equation in one variable by
 - Factorization
 - Completing square method
- Use method of completing square to derive the quadratic formula.
- Use the quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type: $ax^4 + bx^2 + c = 0$.
- Solve the equations of the type: $ap(x) + \frac{b}{p(x)} = c$.
- Solve reciprocal equations of the type $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$.
- Solve exponential equations in which the variables occur in exponents.
- Solve equations of the type $(x+a)(x+b)(x+c)(x+d) = k$, where $a+b=c+d$.
- Solve equations of the type:
 - $\sqrt{ax+b} = cx+d$.
 - $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$.
 - $\sqrt{x^2 + px + m} + \sqrt{x^2 + px + n} = q$.

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$0 = ax^2 + bx + c$$

Expression set equal to 0

Quadratic term
 $a \neq 0$

Linear term

Constant term

Why it's important

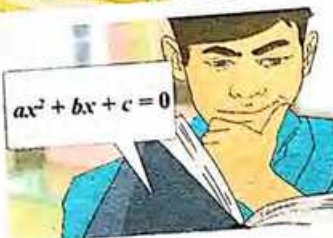
Balls, Arrows, Missiles and Stones

When you throw a ball (or shoot an arrow, fire a missile or throw a stone) it goes up into the air, slowing as it travels, then comes down again faster and faster ...

... and a Quadratic Equation tells you its position at all times!

Quadratic Equations are useful in many other areas: For a parabolic mirror, a reflecting telescope or a satellite dish, the shape is defined by a quadratic equation.

Quadratic equations are also needed when studying lenses and curved mirrors. Many questions involving time, distance and speed need quadratic equations.



1.1 Quadratic equation

A quadratic equation in one variable is an equation that can be written in the form $ax^2 + bx + c = 0$, where a , b , and c , are real numbers and $a \neq 0$.

— this makes it Quadratic

$$5x^2 - 3x + 3 = 0$$

Examples of quadratic equations include $x^2 + x + 1 = 0$, $2x^2 - 3 = 0$, $x^2 + 2x = 0$, and $x^2 = 3x + 1$.

1.2 Solution of quadratic equations

All those values of the variable for which the given equation is true are called solution or roots of the equation, and the set of all solutions is called solution set.

For example, the quadratic equation $x^2 - 9 = 0$ is true only for $x = 3$ and $x = -3$, hence $x = 3$ and $x = -3$ are the solutions or roots of the quadratic equation $x^2 - 9 = 0$ and $\{3, -3\}$ is the solution set.

We may solve quadratic equations by the following methods.

- by factorization
- by completing the square
- by quadratic formula

(a) Solution of a quadratic equation by factorization to its

A quadratic equation can easily be solved by splitting it in factors. The factorization method is illustrated in the following examples:

Example 1 Solving a quadratic equation with factorization.

Solve each quadratic equation by factorization. Check your results

(i) $2x^2 + 2x - 11 = 1$

(ii) $12t^2 = t + 1$

Solution

(i) Start by writing the equation in the form $ax^2 + bx + c = 0$.

$$2x^2 + 2x - 11 = 1$$

given equation

$$2x^2 + 2x - 12 = 0$$

subtract 1 from each side

$$x^2 + x - 6 = 0$$

divide each side by 2 (optional step)

$$(x + 3)(x - 2) = 0$$

factor

$$x + 3 = 0$$

$$\text{or } x - 2 = 0$$

zero-product property

$$x = -3$$

$$\text{or } x = 2$$

solve

These solution can be checked by substituting them in the given equation.

$$2(-3)^2 + 2(-3) - 11 = 1$$

$$1 = 1 \text{ (true)}$$

$$2(2)^2 + 2(2) - 11 = 1$$

$$1 = 1 \text{ (true)}$$

(ii) Write the equation in the form $at^2 + bt + c = 0$.

$$12t^2 = t + 1$$

given equation

$$12t^2 - t - 1 = 0$$

subtract t and 1

$$(3t - 1)(4t + 1) = 0$$

factorize

$$3t - 1 = 0$$

$$\text{or } 4t + 1 = 0$$

zero-product property

$$t = \frac{1}{3}$$

$$\text{or } t = -\frac{1}{4}$$

solve

To check these solutions, substitute them into the given equation.

$$12\left(\frac{1}{3}\right)^2 = \frac{1}{3} + 1$$

$$\frac{4}{3} = \frac{4}{3} \text{ (true)}$$

$$12\left(-\frac{1}{4}\right)^2 = -\frac{1}{4} + 1$$

$$\frac{3}{4} = \frac{3}{4} \text{ (true)}$$

C Try This

Solve the following equations

(i). $a^2 + 7a - 8 = 0$

(ii). $6p^2 + 7p - 20 = 0$

A ball is thrown straight up, from 3 m above the ground with a velocity of 14 m/s. When does it hit the ground?

Solution

We can work out its height by adding up these three things:

The height starts at 3 m: 3

It travels upwards at 14 meters per second (14 m/s): $14t$

Gravity pulls it down, changing its position by about 5 m per second squared: $-5t^2$

Add them up and the height h at any time t is:

$$h = 3 + 14t - 5t^2$$

And the ball will hit the ground when the height is zero:

$$3 + 14t - 5t^2 = 0$$

$$\text{Or } 5t^2 - 14t - 3 = 0.$$

Which is a quadratic equation.

Let us solve it ...

Replace the middle term with -15 and 1

$$\text{i.e. } 5t^2 - 15t + t - 3 = 0$$

$$5t(t-3) + 1(t-3) = 0$$

$$(5t+1)(t-3) = 0$$

And the two solutions are:

$$5t+1=0 \text{ or } t-3=0$$

$$t=-0.2 \text{ or } t=3$$

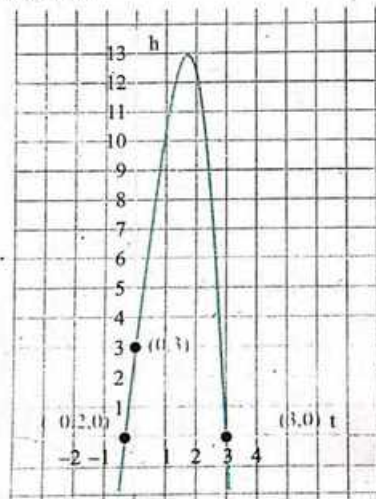
The " $t=-0.2$ " is a negative time which is impossible in our case.

The " $t=3$ " is the answer we want:

The ball hits the ground after 3 seconds!

Note

t is time in seconds

**Note**

(0,3) says when $t=0$ (at the start) the ball is at 3 m.

(3,0) says that at 3 seconds the ball is at ground level.

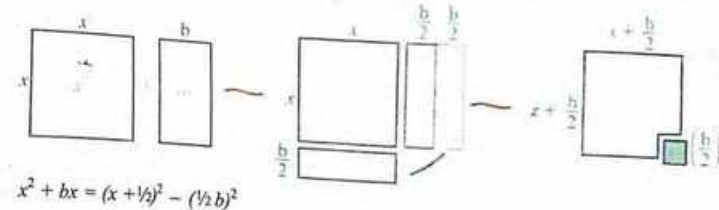
Study Tip

The examples in the text are carefully chosen to prepare you for success with the exercise sets. Study the step-by-step solutions of the examples, noting the substitutions and explanations. The time you spend studying the examples will save your valuable time when you do your homework.

(b) Solution of a quadratic equation by completing square

Some quadratic equations are not easily factorized and hence it is not easy to find their solution by factorization.

For example, the quadratic equation $x^2 + bx = -c$ cannot be easily factorized. Such quadratic equation can be solved by completing the square method



$$x^2 + bx = (x + \frac{1}{2}b)^2 - (\frac{1}{2}b)^2$$

$$\left(x + \frac{1}{2}b\right)^2 = \left(\frac{1}{2}b\right)^2 - c$$

$$x + \frac{1}{2}b = \pm \sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

**Try This**

What must be added to obtain a perfect square?

(i). $x^2 + 5x$ (ii). $q^2 - 4q$

Steps involved in completing the square

- Write the quadratic equation in its standard form.
- Divide both sides of the equation by the co-efficient of x^2 if it is other than 1.
- Shift the constant term to the right-hand side of the equation.
- Square half the co-efficient of x and add the square to both sides.
- Write the left-hand side of the equation as a perfect square and simplify the right-hand side.
- Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

Example 3 Completing the square.

Solve each equation

$$x^2 - 8x + 9 = 0$$

SolutionStart by writing the equation in the form $x^2 + kx = d$.

$$x^2 - 8x + 9 = 0$$

$$x^2 - 8x = -9$$

$$x^2 - 8x + 16 = -9 + 16$$

$$(x - 4)^2 = 7$$

$$x - 4 = \pm \sqrt{7}$$

$$x = 4 \pm \sqrt{7}$$

given equation

subtract 9 from each side

$$\text{add } \left(\frac{x}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$$

the perfect square

square root property

add 4 to each side

! Avoid Errors

When completing the square to solve an equation make sure that

- the coefficient of x^2 is 1
- you add the term $(b/2)^2$ to both sides of the equation.

1.3 Quadratic formula

The general form of quadratic equation is $ax^2 + bx + c = 0$, where a , b and c are real numbers and " a " is not equal to zero. By using completing the square method we can derive the quadratic formula for the solution of all quadratic equations.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c \quad \text{shift the constant term to the right of the equation}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{divide all terms of both sides by } a.$$

$$\text{Add } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \text{ to both sides of the equation.}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Which is the required quadratic formula.}$$

**Example 4** Using the quadratic formula.Solve the equation $3x^2 - 6x + 2 = 0$ **Solution**Let $a = 3$, $b = -6$ and $c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

quadratic formula

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)}$$

substitute for a , b and c

$$x = \frac{6 \pm \sqrt{12}}{6}$$

simplify

$$x = 1 \pm \frac{1}{\sqrt{3}}$$



Use an online calculator
To solve the following
quadratic equations step
by step

$$2x^2 - 9x + 4 = 0$$

Try This

Solve

(i). $4x^2 + 3x - 2 = 0$

(ii). $9x^2 - 42x + 49 = 0$

(iii). $5x^2 - 10x + 13 = 0$

Example 5

A company is making frames as part of a new product they are launching. The frame will be cut out of a piece of steel. To keep the weight down, the final area should be 28 cm^2 . The inside of the frame has to be 11 cm by 6 cm . What should the width x of the metal be?

Solution Area of steel before cutting:

$$\text{Area} = (11 + 2x) \times (6 + 2x) \text{ cm}^2 = 4x^2 + 34x + 66 \text{ cm}^2$$

Area of steel after cutting out the 11×6 middle:

$$\text{Area} = 4x^2 + 34x + 66 - 66 = 4x^2 + 34x$$

Since the area equals 28 cm^2 ,

$$4x^2 + 34x = 28.$$

Or $2x^2 + 17x - 14 = 0$. Here $a = 2$, $b = 17$ and $c = -14$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

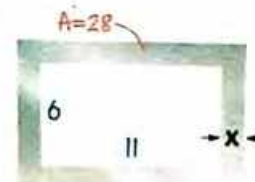
quadratic formula

$$x = \frac{-(17) \pm \sqrt{(17)^2 - 4(2)(-14)}}{2(2)}$$

substitute the values

$$x = \frac{-17 \pm \sqrt{401}}{4}$$

simplify

 x is about -9.3 or 0.8 The negative value of x make no sense, so the answer is: $x = 0.8 \text{ cm}$ (approx.)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1.1

1. Solve each of the following equations by factorization.

(i). $x^2 + 5x + 4 = 0$

(ii). $(x-3)^2 = 4$

(iii). $x^2 + 3x - 10 = 0$

(iv). $6x^2 - 13x + 5 = 0$

(v). $3(x^2 - 1) = 4(x + 1)$

(vi). $x(3x - 5) = (x - 6)(x - 7)$

2. Solve each of the following equations by completing the square.

(i). $x^2 + 6x - 40 = 0$

(ii). $x^2 - 10x + 11 = 0$

(iii). $4x^2 + 12x = 0$

(iv). $5x^2 - 10x - 840 = 0$

(v). $9x^2 - 6x + \frac{5}{9} = 0$

(vi). $(x-1)(x+3) = 5(x+2) - 3$

3. Solve each of the following equations by quadratic formula.

(i). $x^2 - 8x + 15 = 0$

(ii). $x^2 - 2x - 4 = 0$

(iii). $4x^2 + 3x = 0$

(iv). $3x(x-2) + 1 = 0$

(v). $6x^2 - 17x + 12 = 0$

(vi). $\frac{x^2}{3} - \frac{x}{12} = \frac{1}{24}$

4. Find all solutions to the following equations

(i). $t^2 - 8t + 7 = 0$

(ii). $72 + 6x = x^2$

(iii). $t^2 + 4t + 1 = 0$

(iv). $x(x+10) = 10(-10-x)$

5. The equation $(y+13)(y+a)$ has no linear term. Find value of a .6. The equation $ax^2 + 5x = 3$ has $x = 1$ as a solution. What is the other solution?7. What is the positive difference of the roots of $x^2 - 7x - 9 = 0$?

Activity

Verify the answers of Exercise 1.1 by using an online calculator.

Activity

Solve the following quadratic equation $x^2 - 8x + 15 = 0$, by

(i) Factorization

(ii) Completing square

Verify the answer by using an online calculator.

1.4 Solution of equations reducible to quadratic form

The following types of equations can be reduced to the quadratic form and can be solved by any of the method learnt above.

Type

Equation of the form $ax^4 + bx^2 + c = 0$.Example 6 Solve $12x^4 - 11x^2 + 2 = 0$ Solution By making substitution $y = x^2$ the equation becomes $12y^2 - 11y + 2 = 0$, which is a quadratic equation in terms of y and can be solved by factorization,

$$\begin{aligned} (3y-2)(4y-1) &= 0 \\ 3y-2 &= 0 \text{ or } 4y-1 = 0 \\ \Rightarrow y &= \frac{2}{3} \text{ or } y = \frac{1}{4} \end{aligned}$$

To find x , use the fact that $y = x^2$, therefore,

$$\begin{aligned} x^2 &= \frac{2}{3} \text{ or } x^2 = \frac{1}{4} \\ \Rightarrow x &= \pm\sqrt{\frac{2}{3}} \text{ or } x = \pm\sqrt{\frac{1}{4}} \\ \Rightarrow x &= \pm\sqrt{\frac{2}{3}} \text{ or } x = \pm\frac{1}{2} \end{aligned}$$

The given equation $12x^4 - 11x^2 + 2 = 0$ has four solutions.Thus the solution set is $\left\{\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{2}\right\}$.

Type

Equation of the form $a p(x) + \frac{b}{p(x)} = c$,

Example 7

Solve $2x + \frac{4}{x} = 9$

Solution

$$2x + \frac{4}{x} = 9$$

Multiplying both sides by x , we get

$$\begin{aligned} x\left(2x + \frac{4}{x}\right) &= 9x \\ \Rightarrow 2x^2 + 4 &= 9x \end{aligned}$$

Polynomial of degree four is called biquadratic.



$\Rightarrow 2x^2 - 9x + 4 = 0$
 which is a quadratic equation in x and can be solved by factorization

$$(2x-1)(x-4) = 0$$

$$\Rightarrow x = \frac{1}{2} \text{ or } x = 4$$

Hence the solutions are $\frac{1}{2}, 4$ and the solution set is $\left\{\frac{1}{2}, 4\right\}$.

Example 8 Solve $\frac{x-1}{x+3} + \frac{x+3}{x-1} = \frac{13}{6}$

Solution

$$\frac{x-1}{x+3} + \frac{x+3}{x-1} = \frac{13}{6}$$

Let $y = \frac{x-1}{x+3}$. Then $\frac{1}{y} = \frac{x+3}{x-1}$

Therefore, the given equation reduces to $y + \frac{1}{y} = \frac{13}{6}$.

Multiplying both sides by $6y$, we get

$6y^2 - 13y + 6 = 0$, which is a quadratic equation in terms of y and can be solved by factoring as follows:

$$(2y-3)(3y-2) = 0$$

$$\Rightarrow 2y-3=0 \text{ or } 3y-2=0$$

$$\Rightarrow y = \frac{3}{2} \text{ or } y = \frac{2}{3}$$

If $y = \frac{3}{2}$, then $\frac{x-1}{x+3} = \frac{3}{2}$

$$\Rightarrow 2x-2 = 3x+9$$

$$\Rightarrow x = -11$$

If $y = \frac{2}{3}$, then $\frac{x-1}{x+3} = \frac{2}{3}$

$$\Rightarrow 3x-3 = 2x+6$$

$$\Rightarrow x = 9$$

Thus the solution set is $\{-11, 9\}$.

WARNING

When you perform a substitution of variable, you must remember to go back and to express the answers in terms of the original variable



Type III

Reciprocal equation of the form: $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$

Example 9 Solve the following equations.

(i) $2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$ (ii) $8\left(x^2 + \frac{1}{x^2}\right) - 42\left(x - \frac{1}{x}\right) + 29 = 0$

Solution

(i) The given equation is $2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$

Let $x + \frac{1}{x} = y$, square both sides $x^2 + \frac{1}{x^2} + 2 = y^2$ or $x^2 + \frac{1}{x^2} = y^2 - 2$

Therefore, the given equation reduces as follows:

$$2(y^2 - 2) - 9y + 14 = 0$$

$$\Rightarrow 2y^2 - 4 - 9y + 14 = 0$$

$$\text{or } 2y^2 - 9y + 10 = 0$$

The equation can be factorized

$$\therefore (2y-5)(y-2) = 0$$

$$\therefore y = \frac{5}{2} \text{ or } y = 2$$

Now $y = \frac{5}{2}$

$$\Rightarrow x + \frac{1}{x} = \frac{5}{2}$$

$$\Rightarrow 2x^2 + 2 = 5x$$

$$\text{or } 2x^2 - 4x - x + 2 = 0$$

$$\Rightarrow 2x^2 - 4x - x + 2 = 0$$

$$\Rightarrow 2x(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x-2)(2x-1) = 0$$

Given $x = 2$ or $x = \frac{1}{2}$

Hence the solutions are $2, \frac{1}{2}, 1$ and 1 . The solution set is $\left\{2, \frac{1}{2}, 1\right\}$.



and $y = 2$

$$\Rightarrow x + \frac{1}{x} = 2$$

$$\Rightarrow x^2 + 1 = 2x$$

$$\Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)^2 = 0$$

$$\therefore x = 1, 1$$

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(ii) The given equation is $8\left(x^2 + \frac{1}{x^2}\right) - 42\left(x - \frac{1}{x}\right) + 29 = 0$

Let $x - \frac{1}{x} = y$. Then $\left(x - \frac{1}{x}\right)^2 = y^2 \Rightarrow x^2 + \frac{1}{x^2} - 2 = y^2$
 $\Rightarrow x^2 + \frac{1}{x^2} = y^2 + 2$.

Therefore, the given equation reduces to

$$\begin{aligned} 8(y^2 + 2) - 42y + 29 &= 0 \\ 8y^2 + 16 - 42y + 29 &= 0 \\ \Rightarrow 8y^2 - 42y + 45 &= 0 \\ \text{or } 8y^2 - 42y + 30y + 45 &= 0 \\ \Rightarrow 4y(2y - 3) - 15(2y - 3) &= 0 \\ \Rightarrow (2y - 3)(4y - 15) &= 0 \\ \Rightarrow 2y - 3 = 0 \text{ or } 4y - 15 &= 0 \end{aligned}$$

give $y = \frac{3}{2}$ or $y = \frac{15}{4}$

Now $y = \frac{3}{2}$

$$\Rightarrow x - \frac{1}{x} = \frac{3}{2}$$

$$\therefore \frac{x^2 - 1}{x} = \frac{3}{2}$$

or $2(x^2 - 1) = 3x$

$$\Rightarrow 2x^2 - 3x - 2 = 0$$

$$(x - 2)(2x + 1) = 0$$

$$(x - 2) = 0 \text{ or } (2x + 1) = 0$$

$$x = 2 \text{ or } x = -\frac{1}{2}$$

and $y = \frac{15}{4}$

$$\Rightarrow x - \frac{1}{x} = \frac{15}{4}$$

$$\therefore \frac{x^2 - 1}{x} = \frac{15}{4}$$

or $4(x^2 - 1) = 15x$

$$\Rightarrow 4x^2 - 15x - 4 = 0$$

$$(x - 4)(4x + 1) = 0$$

$$(x - 4) = 0 \text{ or } (4x + 1) = 0$$

$$x = 4 \text{ or } x = -\frac{1}{4}$$

Hence the required solution set is $2, \left\{2, -\frac{1}{2}, 4, -\frac{1}{4}\right\}$.

math
is fun



Type IV

Equations that involve terms of the form a^x where $a > 0$, $a \neq 1$ are called **exponent equations**. These equations can be reduced to quadratic equations by making the substitution $y = a^x$, which changes the equations into quadratic equations in term of y .

Example 10 Solve $4 \cdot 2^{2x} - 10 \cdot 2^x + 4 = 0$

Solution

$$4 \cdot 2^{2x} - 10 \cdot 2^x + 4 = 0$$

We may write the given equation as

$$4 \cdot (2^x)^2 - 10 \cdot 2^x + 4 = 0$$

Let $2^x = y$. The above equation reduces to

$$4y^2 - 10y + 4 = 0$$

$$\Rightarrow 2y^2 - 5y + 2 = 0$$

$$\Rightarrow (2y - 1)(y - 2) = 0$$

$$\Rightarrow 2y - 1 = 0 \text{ or } y - 2 = 0$$

gives $y = \frac{1}{2}$ or $y = 2$

$$\Rightarrow 2^x = \frac{1}{2} \text{ or } 2^x = 2$$

$$\Rightarrow 2^x = 2^{-1} \text{ or } 2^x = 2^1$$

$$\Rightarrow x = -1 \text{ or } x = 1$$

Thus the solution set is $\{-1, 1\}$.

Example 11 Solve the equation $2^{2+x} + 2^{2-x} = 10$

Solution

$$2^{2+x} + 2^{2-x} = 10$$

$$\Rightarrow 2^2 \cdot 2^x + 2^2 \cdot 2^{-x} - 10 = 0$$

Let $2^x = y$. Then $2^{-x} = \frac{1}{y}$

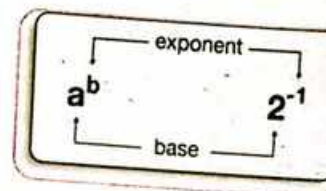
The above equation reduces to

$$4y + \frac{4}{y} - 10 = 0$$

$$\Rightarrow 4y^2 - 10y + 4 = 0$$

$$\Rightarrow 2y^2 - 5y + 2 = 0$$

taking 2 common



One-to-One Property of Exponential Functions

If $b^n = b^m$
then $n = m$

factorizing

$$\begin{aligned} \Rightarrow (2y-1)(y-2) &= 0 \\ \Rightarrow 2y-1 &= 0 \quad \text{or} \quad y-2 = 0 \\ \Rightarrow y &= \frac{1}{2} \quad \text{or} \quad y = 2 \\ \Rightarrow 2^x &= \frac{1}{2} \quad \text{or} \quad 2^x = 2 \\ \Rightarrow 2^x &= 2^{-1} \quad \text{or} \quad 2^x = 2^1 \\ \Rightarrow x &= -1 \quad \text{or} \quad x = 1 \end{aligned}$$

Thus the solution set is $\{-1, 1\}$.

Type Equations of the form $(x+a)(x+b)(x+c)(x+d)=k$
where $a+b=c+d$

Example 12 Solve $(x+1)(x+3)(x-2)(x-4)=24$

Solution As $1+(-2)=3+(-4)$
So re-arranging the factors on the left side of the given equation, we have

$$\begin{aligned} &[(x+1)(x-2)][(x+3)(x-4)] = 24 \\ \Rightarrow (x^2-x-2)(x^2-x-12) &= 24 = 0 \end{aligned}$$

Let $x^2-x=y$. The above equation becomes

$$\begin{aligned} (y-2)(y-12) &= 24 = 0 \\ \Rightarrow y^2-14y+24 &= 0 \\ \Rightarrow y^2-14y &= 0 \\ \Rightarrow y(y-14) &= 0 \\ \Rightarrow y = 0 \quad \text{or} \quad y = 14 \end{aligned}$$

If $y=0$, then $x^2-x=0$

$$\Rightarrow x(x-1)=0$$

gives $x=0$ or $x=1$. If $y=14$, then $x^2-x=14$

$$\Rightarrow x^2-x-14=0$$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-14)}}{2(1)} = \frac{1 \pm \sqrt{1+56}}{2} = \frac{1 \pm \sqrt{57}}{2} = \frac{1+\sqrt{57}}{2}, \frac{1-\sqrt{57}}{2}$$

Thus the solution set is $\left\{0, 1, \frac{1+\sqrt{57}}{2}, \frac{1-\sqrt{57}}{2}\right\}$.

Exercise 1.2

1. Solve the following equations.

(i). $x^4 - 5x^2 + 4 = 0$

(ii). $x^4 - 7x^2 + 12 = 0$

(iii). $6x^4 - 13x^2 + 5 = 0$

(iv). $x+2 - \frac{1}{x} = \frac{3}{2}$

(v). $\frac{4}{x} = 2$

(vi). $\frac{x-2}{x-2} - \frac{x-2}{x+2} = \frac{5}{6}$

(vii). $3\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 26 = 0$

(viii). $\left(x + \frac{1}{x}\right)^2 - 10\left(x + \frac{1}{x}\right) + 16 = 0$

(ix). $\left(x^2 + \frac{1}{x^2}\right) - \left(x - \frac{1}{x}\right) - 4 = 0$

(x). $3^{2x} - 10 \cdot 3^x + 9 = 0$

(xi). $3 \cdot 3^{2x+1} - 10 \cdot 3^x + 1 = 0$

(xii). $5^{x+1} + 5^{2-x} = 126$

(xiii). $(x-3)(x+9)(x+5)(x-7) = 385$

(xiv). $(x-1)(x+2)(x+3)(x+4) + 1 = 0$

(xv). $(x+1)(x+3)(x+5)(x+7) + 16 = 0$

2. Solve the equation $x^4 - x^3 - 2x^2 + 2x + 1 = 0$

Math Fun

Take any four digit number, follow these steps, and you'll end up with 6174.

1. Choose a four digit number (the only condition is that it has at least two different digits).
2. Arrange the digits of the four digit number in descending then ascending order.
3. Subtract the smaller number from the bigger one.
4. Repeat.

Eventually you'll end up at 6174, which is known as Kaprekar's constant. If you then repeat the process you'll just keep getting 6174 over and over again.

1.5 Radical equations

An equation in which the variable appears in one or more radicands is called a radical equation.

For example,

$$\sqrt{x+2}=3, \sqrt{2x+3}=2x+5, \sqrt{x+5}=\sqrt{2x-1}, 3\sqrt{x^2+x+1}=2 \text{ are radical equations.}$$

To solve radical equations, we transform the given equation into an equation that contains no radicals by squaring it. A solution of the transformed equation that does not satisfy the original radical equation is called an extraneous solution.

Type 1 Equation of the form $\sqrt{ax+b}=cx+d$

Example 14 Solve $\sqrt{27-3x}=x-3$

Solution

$$\sqrt{27-3x}=x-3$$

Squaring both sides, we get $(\sqrt{27-3x})^2=(x-3)^2$

$$\begin{aligned} 27-3x &= x^2-6x+9 \\ \Rightarrow 0 &= x^2-6x+9-27+3x \\ \text{or } x^2-3x-18 &= 0 \\ (x-6)(x+3) &= 0 \end{aligned}$$

$$\Rightarrow x-6=0 \text{ or } x+3=0$$

gives $x=6$ or $x=-3$

Now it is necessary to check the solutions in the original equation,

$$\begin{aligned} \sqrt{27-3x} &= x-3 \\ \text{If } x=6, \text{ then } \sqrt{27-3x} &= x-3 \\ \sqrt{27-3(6)} &= 6-3 \\ \sqrt{27-18} &= 3 \\ \sqrt{9} &= 3 \\ 3 &= 3 \quad (\text{true}) \end{aligned}$$

On checking, we find that $x=-3$ is an extraneous root. Thus the solution set is $\{6\}$.

Note Check all roots (solutions) of the transformed equation in the original equation to exclude extraneous roots.

TIP We can get rid of a square root by squaring.

$$\begin{aligned} \text{If } x=-3, \text{ then } \sqrt{27-3x} &= x-3 \\ \sqrt{27-3(-3)} &= -3-3 \\ \sqrt{27+9} &= -6 \\ \sqrt{36} &= -6 \\ 6 &= -6 \quad (\text{false}) \end{aligned}$$

Type 2 Equation of the form $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 14 Solve $\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$

Solution

$$\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$$

Squaring both sides, we get

$$(\sqrt{x+2}+\sqrt{x+7})^2=(\sqrt{x+23})^2$$

$$x+2+x+7+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$$

$$\Rightarrow 2x+9+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$$

$$\Rightarrow 2\sqrt{(x+2)(x+7)}=14-x$$

Squaring both sides again, we get

$$\Rightarrow 4(x+2)(x+7)=(14-x)^2$$

$$\Rightarrow 4(x^2+9x+14)=196-28x+x^2$$

$$\Rightarrow 4x^2+36x+56=196-28x+x^2$$

$$\Rightarrow 4x^2-x^2+36x+28x+56-196=0$$

$$\Rightarrow 3x^2+64x-140=0$$

$$\Rightarrow 3x^2-6x+70x-140=0$$

$$\Rightarrow 3x(x-2)+70(x-2)=0$$

$$\Rightarrow (x-2)(3x+70)=0$$

$$\Rightarrow x-2=0 \text{ or } 3x+70=0$$

$$x=2 \text{ or } x=-\frac{70}{3}$$

On checking, we find that $-\frac{70}{3}$ is an extraneous root.

Thus the solution set is $\{2\}$.

WARNING

You can perform addition only with identical radical forms. Adding unlike radicals is one of the most common mistakes made by students in algebra! You can easily verify that

$$\sqrt{9}+\sqrt{16}=3+4=7$$

$$\sqrt{9+16}=\sqrt{25}=5$$

$$7=\sqrt{9}+\sqrt{16}\neq\sqrt{9+16}=5$$



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Type III Equation of the form $\sqrt{x^2 + px + m} + \sqrt{x^2 + px + n} = q$

Example 13

Solve the equation $\sqrt{x^2 + 3x + 5} + \sqrt{x^2 + 3x + 1} = 2$

Solution

$$\sqrt{x^2 + 3x + 5} + \sqrt{x^2 + 3x + 1} = 2$$

The given equation can be written as

Squaring both sides, we get

$$x^2 + 3x + 5 = 4 + x^2 + 3x + 1 - 2(2)\sqrt{x^2 + 3x + 1}$$

$$\Rightarrow x^2 + 3x + 5 = x^2 + 3x + 5 - 4\sqrt{x^2 + 3x + 1}$$

$$\Rightarrow 0 = -4\sqrt{x^2 + 3x + 1}$$

$$\Rightarrow \sqrt{x^2 + 3x + 1} = 0$$

$$\Rightarrow x^2 + 3x + 1 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{9 - 4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

Thus the solution set is $\left\{ \frac{-3 \pm \sqrt{5}}{2} \right\}$.

Did You Know?

Beautiful Number Relationships.

$$135 = 1^3 + 3^3 + 5^3$$

$$175 = 1^3 + 7^3 + 5^3$$

$$518 = 5^3 + 1^3 + 8^3$$

$$598 = 5^3 + 9^3 + 8^3$$

WOW!



so easy

Exercise 1.3

1. Solve the following equations.

(i). $\sqrt{5x+21} = x+3$

(ii). $\sqrt{2x-1} = x-2$

(iii). $\sqrt{4x+5} = 2x-5$

(iv). $\sqrt{29-4x} = 2x+3$

(v). $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

(vi). $\sqrt{x} + \sqrt{3x+1} = \sqrt{5x+1}$

(vii). $\sqrt{6x+40} - \sqrt{x+21} = \sqrt{x+5}$

(viii). $\sqrt{2x-3} + \sqrt{2x+4} = \sqrt{6x+13}$

(ix). $\sqrt{x^2+2x+4} + \sqrt{x^2+2x+9} = 5$

(x). $\sqrt{2x^2+3x+5} + \sqrt{2x^2+3x+1} = 2$

2. Find $2x+5$ if x satisfies $\sqrt{40-9x} - 2\sqrt{7-x} = \sqrt{-x}$

Review Exercise 1

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If $(x+1)(x-5) = 0$, then the solutions are

- ☐ $x = 1, -5$ ☐ $x = 1, 5$ ☐ $x = -1, -5$ ☐ $x = -1, 5$

(ii). If $x^2 - x - 1 = 0$, then $x =$

- ☐ $\frac{-1 \pm \sqrt{5}}{2}$ ☐ $-1 \pm \frac{\sqrt{5}}{2}$ ☐ $\frac{1 \pm \sqrt{5}}{2}$ ☐ $1 \pm \frac{\sqrt{5}}{2}$

(iii). $\frac{-1 \pm \sqrt{5}}{2}$ in simplified form is

- ☐ $1 \pm \sqrt{24}$ ☐ $1 \pm \sqrt{6}$ ☐ $2 \pm \sqrt{6}$ ☐ cannot be simplified

(iv). To apply the quadratic formula to $2x^2 - x = 3$

- ☐ $a = 2, b = -1, c = 3$ ☐ $a = 2, b = 1, c = 3$
☐ $a = 2, b = -1, c = -3$ ☐ $a = 2, b = -1, c = 0$

(v). If $x^2 - 3x - 4 = 0$, then the solutions are

- ☐ $x = 4, -1$ ☐ $x = -4, 1$ ☐ $x = 4, 1$ ☐ $x = -4, -1$

(vi). If $2x^2 + 4x - 9 = 0$, the solutions are

- ☐ $x = \frac{2 \pm \sqrt{22}}{2}$ ☐ $x = \frac{-2 \pm \sqrt{22}}{2}$ ☐ $x = 2 \pm \frac{\sqrt{22}}{2}$ ☐ $x = -2 \pm \frac{\sqrt{22}}{2}$

(vii). $x^2 - \frac{1}{4} = 0$, the solution are

- ☐ $x = \pm \frac{1}{2}$ ☐ $x = \pm \frac{1}{4}$ ☐ $x = \pm \frac{1}{8}$ ☐ $x = \pm \frac{1}{16}$

- (viii). What are the solutions of the equation $x^2 + 7x - 18 = 0$?
☐ 2 or -9 ☐ -2 or 9 ☐ -2 or -9 ☐ 2 or 9
- (ix). Which of the following values of x are roots of the equation $x^2 - 8x + 15 = 0$?
☐ $x = 1$ or $x = -7$ ☐ $x = 2$ or $x = 4$
☐ $x = -2$ or $x = 4$ ☐ $x = 3$ or $x = 5$

2. Solve $2w^4 - 5w^2 + 2 = 0$.

3. Find the constants a and b such that $x = -1$ and $x = 1$ are both solutions to the equation $ax^2 + bx + 2 = 0$.

4. Find all values of x such that $x^2 + 5x + 6$ and $x^2 + 19x + 34$ are equal.

Challenge!

5. Find the solutions to the equation: $49x^2 - 316x + 132 = 0$
 If you can factorize this successfully, you have probably mastered the art of factorizing.



Find some internet sites which solve quadratic equations. Which site do you think is better and why?

Activity

Juwaria tried to solve the quadratic equation $x^2 + 5x - 2 = 0$ by completing the square, but she made a mistake. In which line of her working, shown below, did she make the mistake?

$$x^2 + 5x - 2 = 0$$

$$[\text{Line 1}] \Rightarrow x^2 + 5x = 2$$

$$[\text{Line 2}] \Rightarrow x^2 + 5x + (5/2)^2 = 2 + (5/2)^2$$

$$[\text{Line 3}] \Rightarrow (x + 5/2)^2 = 2 + 10/4$$

$$[\text{Line 4}] \Rightarrow (x + 5/2)^2 = 18/4$$

$$[\text{Line 5}] \Rightarrow (x + 5/2) = \pm\sqrt{18/4} = \pm(\sqrt{18})/(\sqrt{4}) = \pm\sqrt{18/2}$$

$$[\text{Line 6}] \Rightarrow x = \pm\sqrt{18/2} - 5/2$$

$$[\text{Line 7}] \Rightarrow x = -4.62 \text{ or } -0.38 \text{ to 2 decimal places.}$$



Summary

- The following table summarizes important topics related to quadratic equations.

Concept	Explanation	Notes
Quadratic equation	$ax^2 + bx + c = 0$, where a , b , and c are constants with $a \neq 0$.	A quadratic equation can have zero, one, or two real solutions. $x^2 = -5$ No real solutions $(x-2)^2 = 0$ One real solution $x^2 - 4 = 0$ Two real solutions
Factorizing	A symbolic technique for solving equations, based on the zero-product property: if $ab = 0$, then either $a = 0$ or $b = 0$.	$x^2 - 3x + 2 = 0$ $x^2 - 3x + 2 = 0$ $(x-1)(x-2) = 0$ $x-1 = 0$ or $x-2 = 0$ $x = 1$ or $x = 2$
Square root property	The solutions to $x^2 = k$ are $x = \pm\sqrt{k}$, where $k \geq 0$.	$x^2 = 9$ is equivalent to $x = \pm 3$. $x^2 = 11$ is equivalent to $x = \pm\sqrt{11}$.
Completing the square	To solve $x^2 + kx = d$ symbolically, add $(\frac{k}{2})^2$ to each side to obtain to perfect square trinomial. Then apply the square root property.	$x^2 - 6x = 1$ $x^2 - 6x + 9 = 1 + 9$ $(\frac{-6}{2})^2 = 9$ $x^2 - 6x + 9 = 1 + 9$ $(x-3)^2 = 10$ $x-3 = \pm\sqrt{10}$ $x = 3 \pm\sqrt{10}$
Quadratic formula	The solutions to $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Always gives the exact solutions.	To solve $2x^2 - x - 4 = 0$, let $a = 2$, $b = -1$, and $c = -4$. $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)}$ $= \frac{1 \pm \sqrt{33}}{4} \approx 1.69, -1.19$

- The following table outlines important concepts in this section.

Solving radical equations	The solutions to $a = b$ are among the solutions to $a^n = b^n$ when n is a positive integer. Check your results.	Solve $\sqrt{2x+3} = x$ $2x+3 = x^2$ Square each side. $x^2 - 2x - 3 = 0$ Rewrite equations $x = -1$ or $x = 3$ Factor and solve. Checking reveals that 3 is the only solution.
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To complete the square of a quadratic equation

Add and Subtract

$$\left(\frac{\text{Coefficient of "x"}}{2} \right)^2$$

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