

ANGLE IN A SEGMENT OF A CIRCLE

In this unit the students will be able to

To prove the following theorems along with corollaries and apply them to solve appropriate problems.

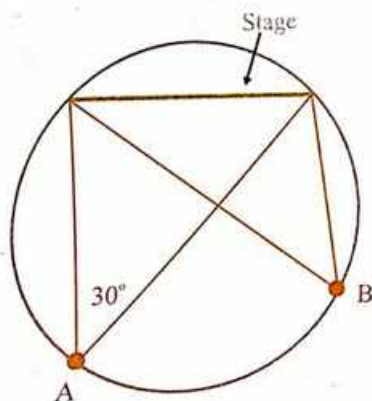
- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle in a semi-circle is a right angle,
- The angle in a segment greater than a semi-circle is less than a right angle,
- The angle in a segment less than a semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Why it's important

A person's effective field of vision is about 30° . In the diagram of the amphitheater, a person sitting at point A can see the entire stage. What is the measure of $\angle B$? Can the person sitting at point B view the entire stage?

Angles A and B intercept the same arc. By Theorem 2 of this unit, the angles must have the same measure, so $m\angle A = m\angle B = 30^\circ$.

The person sitting at point B can view the entire stage.



APPLICATION OPTICS



Theorem 12.1

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.

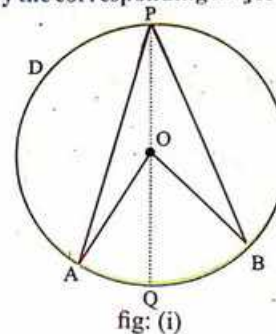


fig: (i)

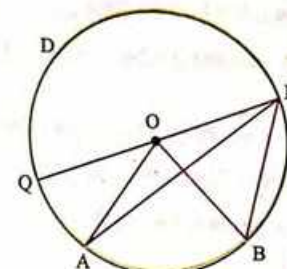


fig: (ii)

Given

A circle with centre O, $\widehat{AB}$ is a minor arc whose central angle is $\angle AOB$.

P is any point on the major arc $\widehat{ADB}$. $\angle APB$ is the angle subtended by the arc at P.

To prove

$$m\angle AOB = 2m\angle APB$$

Case 1: When P is an arbitrary point of the major arc (General Case) figure (i).

Case 2: When PQ is an extremity of the semi-circle containing the minor arc $\widehat{AB}$ figure (ii).

Construction

Draw $\overrightarrow{PO}$ to meet the circumference at Q (for both cases).

Proof

Statements	
Case (i)	
In the $\triangle OAP$	
$m\overline{OA} = m\overline{OP}$	Radii of the same circle.
$\therefore \triangle OAP$ is an isosceles triangle	Definition of an isosceles triangle.
and $m\angle OAP = m\angle OPA$	$\therefore$ If two sides of a triangle are equal, the angles which are opposite to them are also equal.

Also $m\angle AOQ = m\angle OAP + m\angle OPA$

$\therefore m\angle AOQ = 2m\angle OPA$ (I)

Similarly $m\angle BOQ = 2m\angle OPB$ (II)

$\therefore m\angle AOQ + m\angle BOQ = 2m\angle OPA + 2m\angle OPB$

or $m\angle AOB = 2[m\angle OPA + m\angle OPB]$

or $m\angle AOB = 2m\angle APB$

Case (ii)

All the statements and reasons are same up to obtaining the equations I and II.

Subtracting I from II we get

$m\angle BOQ - m\angle AOQ = 2m\angle OPB - 2m\angle OPA$

$m\angle BOQ - m\angle AOQ = 2[m\angle OPB - m\angle OPA]$

$\therefore m\angle AOB = 2m\angle APB$

$\therefore$ The measure of exterior angle of a triangle is equal to the sum of the measures of opposite interior angles.

$\therefore m\angle OAP = m\angle OPA$ (Proved above)

Adding I and II.

Postulate of addition of angles.

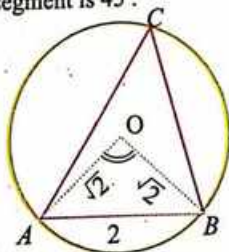
Postulate of subtraction of angles.

Corollary

The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.

Example 1

The radius of a circle is $\sqrt{2}$ cm. A chord 2 cm in length divides the circle into two segments. Prove that the angle of larger segment is 45° .

**Solution****Given**

A circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm,
length of chord $AB = 2$ cm divides the circle into two segments with ACB as
one.

To prove

$m\angle ACB = 45^\circ$

Construction

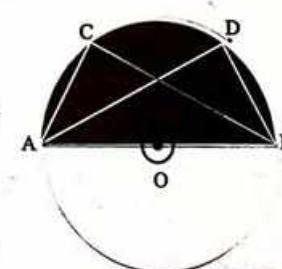
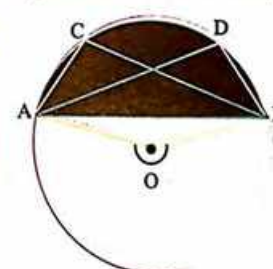
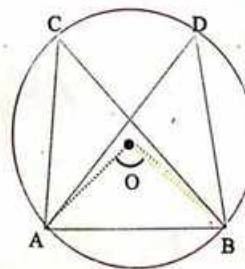
Join O with A and B .

Proof

Statements	
In $\triangle OAB$	
$(OA)^2 + (OB)^2 = (\sqrt{2})^2 + (\sqrt{2})^2$	$m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm
$= 2 + 2$	
$= 4 = (2)^2 = (AB)^2$	$\therefore m\overline{AB} = 2$ cm
$\therefore \triangle OAB$ is right angled triangle.	
With $m\angle AOB = 90^\circ$	Which being a central angle standing on an arc AB .
$m\angle ACB = \frac{1}{2} m\angle AOB$	By Theorem 12.1
$= \frac{1}{2} (90^\circ) = 45^\circ$	Circum-angle is half of the central angle.

**Theorem 12.2**

Any two angles in the same segment of a circle are equal.



Given

A circle with centre O. $\angle ACB$ and $\angle ADB$ are any two angles in the same segment ACDB (Shaded) of the circle.

To prove

$$m\angle ACB = m\angle ADB$$

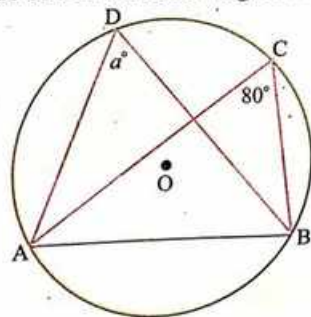
Construction

Join O to A and B respectively.

Proof

Statements	Reasons
In all the three figures, we have $m\angle AOB = 2m\angle ACB$ (I)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
Similarly $m\angle AOB = 2m\angle ADB$ (II)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$\therefore 2m\angle ACB = 2m\angle ADB$	The RHS of (I) and (II) are equal to the same quantity.
$\therefore m\angle ACB = m\angle ADB$	Halves of equal quantities are again equal.

Example 2 Find the value of a in the following circle centred at O.



Solution

$$a = 80$$

(Angles in the same segment).

Example 3 Find the value of each of the variables in the following circle centred at O.

Solution

$$2a = 104 \quad (\text{Angle at centre theorem})$$

$$\frac{2a}{2} = \frac{104}{2}$$

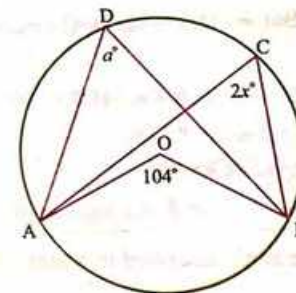
$$a = 52$$

$$\text{Also, } 2x = a \quad (\text{Angles in the same segment})$$

$$\therefore 2x = 52$$

$$\frac{2x}{2} = \frac{52}{2}$$

$$x = 26$$



Theorem 12.3(a)

The angle in a semi-circle is a right angle.

Given

A circle with centre O, $\overline{AB}$ is a diameter of the circle and $\angle ACB$ is any angle in the semi-circle.

To prove

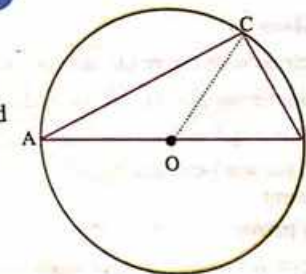
$$\angle ACB \text{ is a right angle i.e. } m\angle ACB = 90^\circ.$$

Construction

Join O to C.

Proof

Statements	Reasons
In $\triangle OAC$, $\overline{OA} = \overline{OC}$	Radii of the same circle Definition of an isosceles triangle.
$\therefore \triangle OAC$ is an isosceles triangle and $m\angle OAC = m\angle OCA$ (I)	If two sides of a triangle are equal, the angles which are opposite to them are also equal.
Similarly in the $\triangle OCB$ $\overline{OB} = \overline{OC}$	Radii of a circle.
$\therefore m\angle OBC = m\angle OCB$ (II)	
$\therefore m\angle OAC + m\angle OBC = m\angle OCA + m\angle OCB$	Adding (I) and (II).



$$m\angle OAC + m\angle OBC = m\angle ACB \quad (\text{III})$$

$$\text{But } m\angle OAC + m\angle OBC + m\angle ACB = 180^\circ$$

$$\text{Or } m\angle ACB + m\angle ACB = 180^\circ$$

$$\Rightarrow 2m\angle ACB = 180^\circ$$

$$\therefore m\angle ACB = 90^\circ$$

or $\angle ACB$ is a right-angle.

The angle inscribed in a semicircle is always a right angle (90°).

Theorem 12.3(b)

The angle in a segment greater than a semi-circle is less than a right angle.

Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies below the diameter $\overline{EF}$. Thus the segment AECDFB is greater than the semi-circular region. $\angle ADB$ is any angle in the segment.

To prove

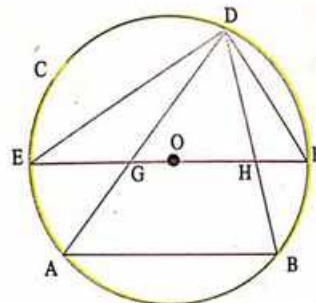
$\angle ADB$ is less than a right angle i.e. $m\angle ADB < 90^\circ$.

Construction

Join D with E and F such that $\angle EDF$ is in the semi-circle.

Proof

	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi-circle is a right angle.
$\therefore m\angle EDG + m\angle GDH + m\angle HDF = 90^\circ$	$\therefore m\angle EDG + m\angle GDH + m\angle HDF = m\angle EDF$
$\therefore m\angle GDH = 90^\circ$	
$\quad - [m\angle EDG + m\angle HDF]$	
$\Rightarrow m\angle GDH < 90^\circ$	
or $m\angle ADB < 90^\circ$	$\therefore \angle GDH$ and $\angle ADB$ are same.



Theorem 12.3(c)

The angle in a segment less than a semi-circle is greater than a right angle.

Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies above the diameter $\overline{EF}$. The segment ACDB is less than the semi-circular region. $\angle ADB$ is any angle in the segment.

To prove

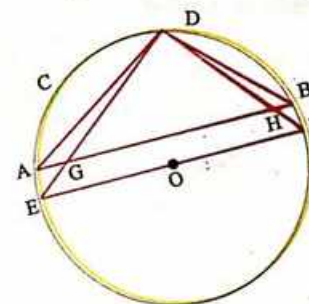
$\angle ADB$ is greater than a right angle i.e. $m\angle ADB > 90^\circ$.

Construction

Join D to E and F. $\angle EDF$ is in the semi-circle.

Proof

Statements	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi-circle is a right angle.
or $m\angle GDH = 90^\circ$	$\therefore \angle EDF$ and $\angle GDH$ are same.
$m\angle ADG + m\angle GDH + m\angle HDB =$	Adding equal quantities on both sides of an equation.
or $m\angle ADG + 90^\circ + m\angle HDB$	Postulate of addition of angles.
$\Rightarrow m\angle ADB = 90^\circ + m\angle ADG + m\angle HDB$	
$\Rightarrow m\angle ADB > 90^\circ$	



Theorem 12.4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given

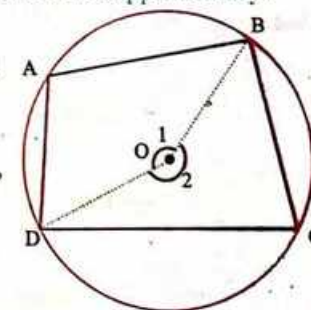
A circle with centre at O and ABCD is a quadrilateral inscribed in the given circle.

To prove

$m\angle BCD + m\angle BAD = 180^\circ$ and $m\angle ABC + m\angle ADC = 180^\circ$

Construction

Join O to B and D.



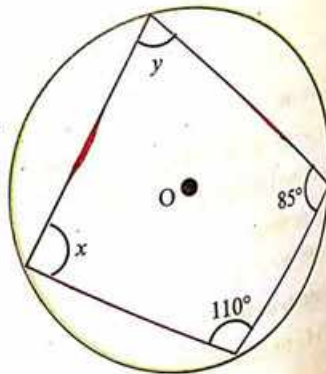
Proof

Statements	Reasons
$m\angle 1 = 2m\angle BCD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 2 = 2m\angle BAD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 1 + m\angle 2 = 2m\angle BCD + 2m\angle BAD$ (I)	Adding the above two results.
But $m\angle 1 + m\angle 2 = 360^\circ$	$\therefore$ circumference of a circle subtends an angle equal to four right angles at the centre.
$\therefore$ (I) becomes	
$2[m\angle BCD + m\angle BAD] = 360^\circ$	Dividing both sides of an equation by 2.
$\therefore m\angle BCD + m\angle BAD = \frac{360^\circ}{2}$	
or $m\angle BCD + m\angle BAD = 180^\circ$	Definition of supplementary angles.
$\therefore \angle BCD$ and $\angle BAD$ are supplementary.	
Similarly we can prove that $m\angle ABC$ and $m\angle ADC$ are supplementary.	

Example 4 O is the centre of the circle. Find the unknowns in the figure.

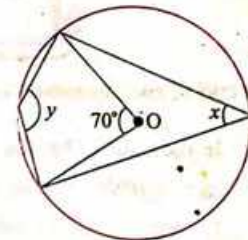
Solution Since

$$\begin{aligned}x + 85^\circ &= 180^\circ \\ \therefore x &= 180^\circ - 85^\circ = 95^\circ \\ y + 110^\circ &= 180^\circ \\ \therefore y &= 180^\circ - 110^\circ = 70^\circ\end{aligned}$$

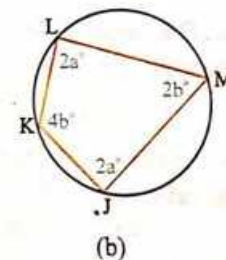
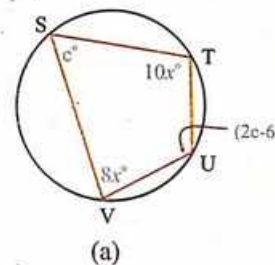


Exercise 12

1. O is the centre of the circle.
Find the unknowns in the figure.

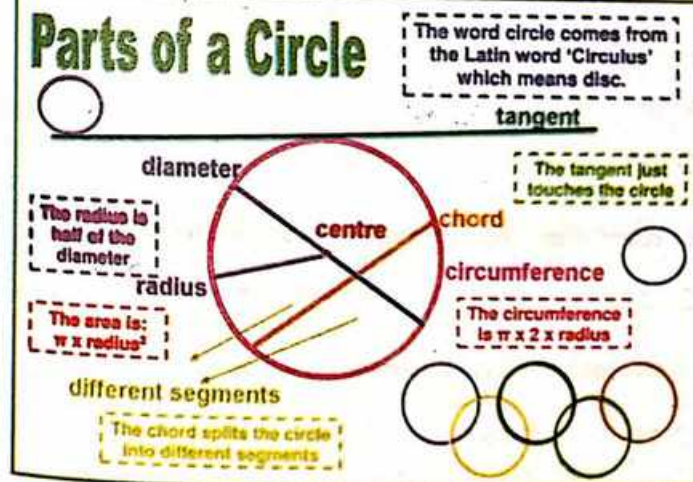


2. A regular hexagon is inscribed in a circle. Each side of the hexagon is $5\sqrt{3}$ units from centre of the circle. Find the radius of the circle.
3. Find the value of each variable.



4. Show that a parallelogram inscribed in a circle will be a rectangle.

Parts of a Circle



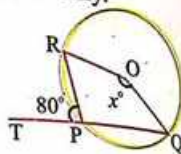
Let us
Revise



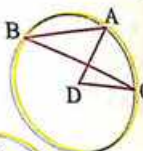
Review Exercise 12

1. At the end of each question, four circles are given. Fill in the correct circle only.

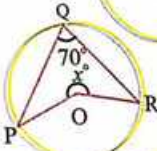
- (i). In the figure, O is the centre of the circle, $\angle TPR = 80^\circ$ and $\angle QPS = x^\circ$. Find x .
☐ 80° ☐ 160° ☐ 100° ☐ 120°



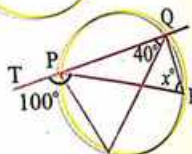
- (ii). In the diagram, $\angle ADC$ is a central angle and $m\angle ADC = 60^\circ$. What is $m\angle ABC$?
☐ 15° ☐ 60° ☐ 120° ☐ 30°



- (iii). In the figure, O is centre of the circle and $\angle PQR = 70^\circ$. Calculate the value of x .
☐ 140° ☐ 220° ☐ 290° ☐ 110°



- (iv). In the figure, $\angle SPT = 100^\circ$, $\angle PQS = 40^\circ$ and $\angle PRQ = x^\circ$. Calculate the value of x .
☐ 30° ☐ 40° ☐ 50° ☐ 60°



- (v). In the adjacent figure if $m\angle 3 = 75^\circ$, then find $m\angle 1$ and $m\angle 2$.
☐ $37\frac{1}{2}^\circ, 37\frac{1}{2}^\circ$ ☐ $37\frac{1}{2}^\circ, 75^\circ$
☐ $75^\circ, 37\frac{1}{2}^\circ$ ☐ $75^\circ, 75^\circ$



- (vi). Given that O is the centre of the circle. The angle marked x will be:

☐ $12\frac{1}{2}^\circ$ ☐ 25° ☐ 50° ☐ 75°



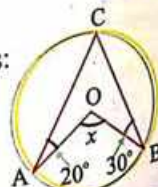
- (vii). Given that O is the centre of the circle, the angle marked y will be:

☐ $12\frac{1}{2}^\circ$ ☐ 25° ☐ 50° ☐ 75°

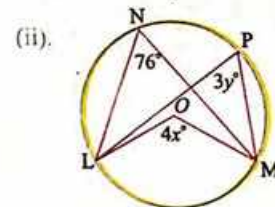
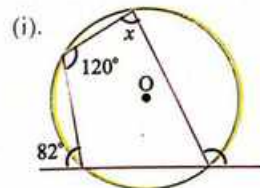


- (viii). In the figure, O is the centre of the circle then the angle x is:

☐ 50° ☐ 75° ☐ 100° ☐ 125°

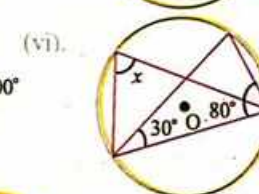
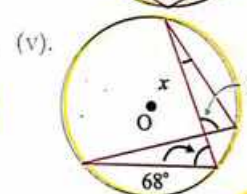
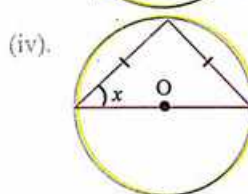
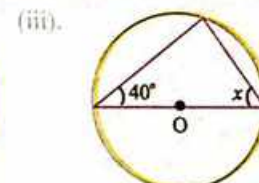
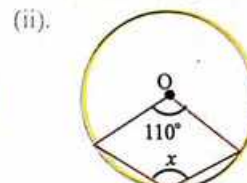
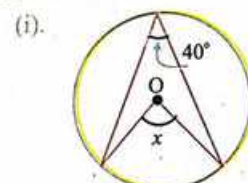


2. O is the centre of the circle. Find the unknowns in the figure.



3. ABCD is a quadrilateral circumscribed about a circle. Show that $m\widehat{AB} + m\widehat{CD} = m\widehat{BC} + m\widehat{DA}$

4. Find the value of x in each of the following figures where O is the centre of the circle.



Summary

- ☑ The inscribed angle is an angle whose vertex is on a circle and whose sides contain chords of the circle.
- ☑ The arc that lies in the interior of an inscribed angle and has end points on the angle is called the intercepted arc of the angle.
- ☑ The measure of an inscribed angle is one half the measure of its intercepted arc.
- ☑ If two inscribed angles of a circle intercept the same arc, then the angles are congruent.
- ☑ A polygon is an inscribed polygon if all its vertices lie on a circle.
- ☑ The circle that contains the vertices is a circumscribed circle.
- ☑ A quadrilateral with its four vertices lying on the circumference of the circle is called a cyclic quadrilateral.
- ☑ A quadrilateral can be inscribed in a circle if and only if the opposite angles are supplementary.
- ☑ In a circle:
 - the angle in a semicircle is right
 - the angle in a segment greater than a semicircle is acute
 - the angle in a segment less than a semicircle is obtuse.