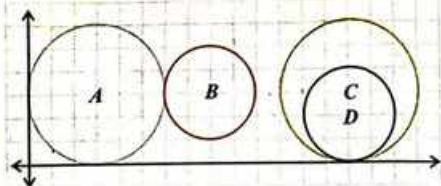


Activity

Lengths in circles in a coordinate plane

Use the diagram for lengths.

- Radius of $\odot A$
- Diameter of $\odot A$
- Radius of $\odot B$
- Diameter of $\odot B$
- Radius of $\odot C$
- Diameter of $\odot C$
- Radius of $\odot D$
- Diameter of $\odot D$



Summary

- A secant to a circle is a line that intersects the circle at two distinct points. If a straight line and a circle have only one point of contact then that line is called a tangent and the point of intersection is known as point of tangency/contact.
- A line is tangent to a circle if and only if the line is perpendicular to a radius of the circle at its end point on the circle.
- If a line is perpendicular to a radius at its end points the line is a tangent
- Tangent segments from a common external point are congruent.
- The tangents subtend equal angles at the centre.
- The line joining the external point to the centre of the circle bisects the angle between the tangents
- If two circles touch externally, the distance between their centres is equal to the sum of their radii.
- If two circles touch internally, the distance between their centres is the difference of their radii.

Did You Know?

$$3025 = (30 + 25)^2$$

and

$$2025 = (20 + 25)^2$$



Unit

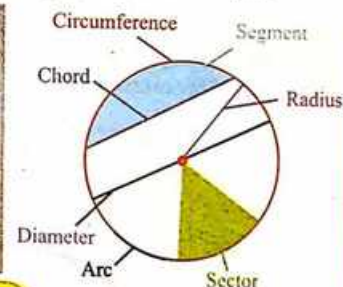
11

CHORDS AND ARCS

In this unit the students will be able to

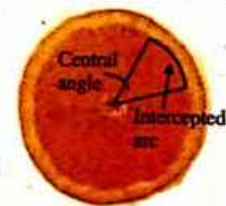
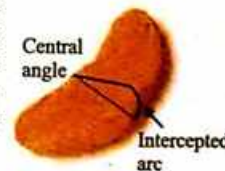
To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Why it's important

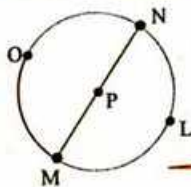
An orange may consist of nine wedges, seen in cross section here. Thus, an average wedge would form a central angle of about one-ninth of the full circle, or 40° . (When you look at a typical orange wedge, does it seem to be about 40° ?)



Major and Minor Arcs

An arc is an unbroken part of a circle. Any two distinct points on a circle divide the circle into two arcs. The points are called the endpoints of the arcs.

$\widehat{MO}$ (red) is a minor arc of $\odot P$.



$\widehat{MON}$ and $\widehat{MLN}$ are semicircles of $\odot P$.

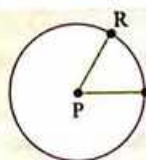
$\widehat{MLO}$ (blue) is a major arc of $\odot P$.

A **semicircle** is an arc whose endpoints are endpoints of the diameter of the circle. A semicircle is informally called a half-circle. A semicircle is named by its endpoints and another point that lies on the arc.

A **minor arc** of a circle is an arc that is shorter than a semicircle of that circle. A minor arc is named by its endpoints.

A **major arc** of a circle is an arc that is longer than a semicircle of that circle. A major arc is named by its endpoints and another point that lies on the arc.

Central angles of circles are used to find the measures of arcs.



$\widehat{RS}$ is the intercepted arc of central angle $\angle RPS$.

A **central angle** of a circle is an angle in the plane of a circle whose vertex is the centre of the circle. An arc whose endpoints lie on the sides of the angle and whose other points lie in the interior of the angle is the **intercepted arc** of the central angle.

Math Fun

$$1 \times 9 + 2 = 11$$

$$12 \times 9 + 3 = 111$$

$$123 \times 9 + 4 = 1111$$

$$1234 \times 9 + 5 = 11111$$

$$12345 \times 9 + 6 = 111111$$

$$123456 \times 9 + 7 = 1111111$$

$$1234567 \times 9 + 8 = 11111111$$

$$12345678 \times 9 + 9 = 111111111$$

$$123456789 \times 9 + 10 = 1111111111$$

Amazing



Theorem 11.1

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.

Case (a) For one circle

Given

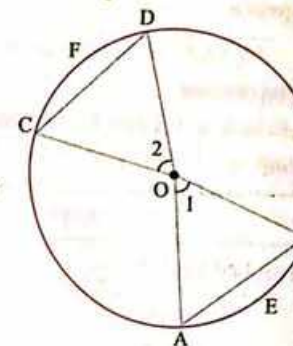
A circle with centre O. $\widehat{AEB}$ and $\widehat{CFD}$ are congruent arcs i.e. $\widehat{AEB} \cong \widehat{CFD}$. $\overline{AB}$ and $\overline{CD}$ are the corresponding chords of the given congruent arcs.

To prove

$$\overline{AB} \cong \overline{CD}.$$

Construction

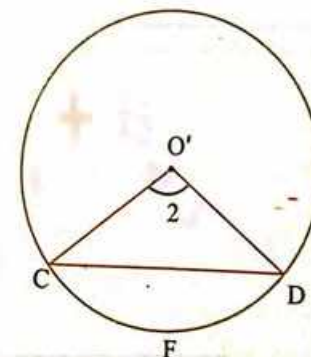
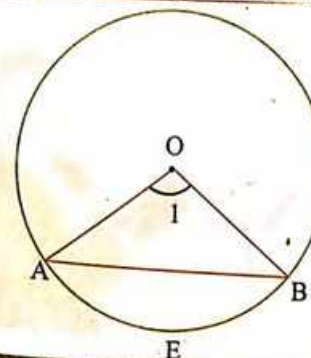
Join O to A, B, C and D respectively and name the central $\angle 1$ and $\angle 2$.



Proof

Statements	Reasons
In $\triangle OAB \leftrightarrow \triangle OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\angle 1 \cong \angle 2$	Central angles of two congruent arcs.
$\therefore \triangle OAB \cong \triangle OCD$	S.A.S Postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.

Case (b) For two congruent circles



Given

Two congruent circles with centres O and O' respectively. AEB and CFD are two congruent arcs of these circles where $\overline{AB}$ and $\overline{CD}$ are the corresponding chords.

To prove

$$\overline{AB} \cong \overline{CD} \text{ or } m\overline{AB} = m\overline{CD}$$

Construction

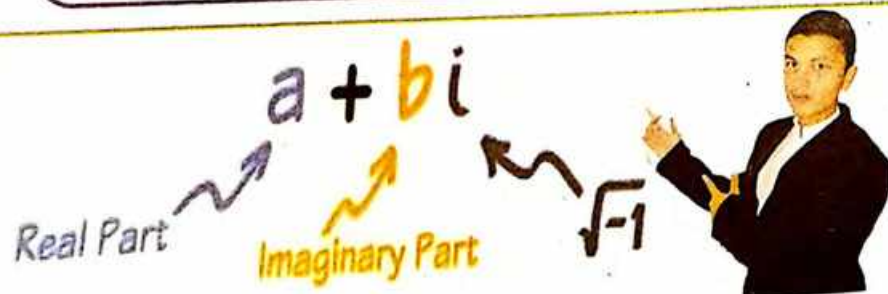
Join O to A and B and O' to C and D respectively.

Proof

Statements	Reasons
In $\triangle OAB \leftrightarrow \triangle O'CD$	
$\overline{OA} \cong \overline{OC}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{OD}$	Radii of two congruent circles.
$\angle 1 \cong \angle 2$	Central angles of two congruent arcs
$\therefore \triangle OAB \cong \triangle O'CD$	S.A.S Postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles
Or $m\overline{AB} = m\overline{CD}$	

Tidbit

The sum of any two odd numbers is even;
the product of any two odd numbers is odd.

**Theorem 11.2**

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.

Case (a) For one circle**Given**

A circle with centre O having two chords $\overline{AB}$ and $\overline{CD}$ such that $\overline{AB} \cong \overline{CD}$.

To prove

$AEB \cong CFD$ (These are minor arcs of the chords $\overline{AB}$ and $\overline{CD}$ respectively).

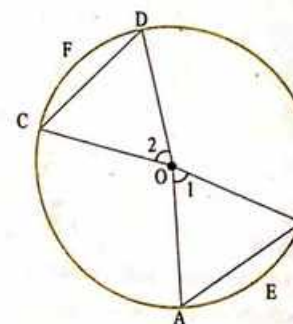
and $AFB \cong CED$ (These are major arcs of the chords $\overline{AB}$ and $\overline{CD}$ respectively).

Construction

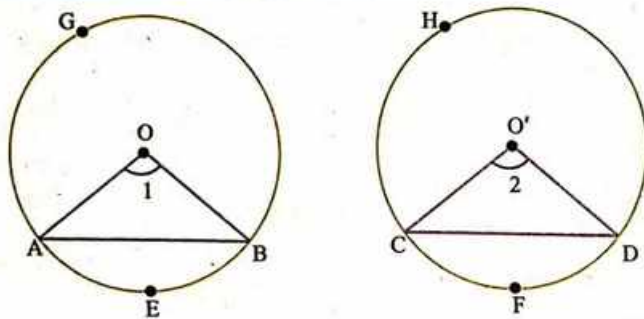
Join O with A , B , C and D respectively.

Proof

Statements	Reasons
In the $\triangle AOB \leftrightarrow \triangle COD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore \triangle AOB \cong \triangle COD$	S.S.S $\cong$ S.S.S
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of two congruent triangles.
but $\angle 1$ and $\angle 2$ are central angles	
$\therefore \widehat{AEB} \cong \widehat{CFD}$	Definition of equal or congruent arcs.
Thus the corresponding minor arcs of two equal chords $\overline{AB}$ and $\overline{CD}$ of a circle are congruent.	



Case (b) Let's prove the same result for two congruent circles

**Given**

Two congruent circles with centres O and O' with two chords $\overline{AB}$ and $\overline{CD}$ respectively which are not the diameters such that $m\overline{AB} = m\overline{CD}$ or $\overline{AB} \cong \overline{CD}$.

To prove

$\overline{AEB} \cong \overline{CFD}$ (these are minor arcs corresponding to the chords $\overline{AB}$ and $\overline{CD}$) and

$\overline{AGB} \cong \overline{CHD}$ (these are the major arcs corresponding to the chords $\overline{AB}$ and $\overline{CD}$)

Construction

Join O with A and B and O' with C and D . This gives central angles labeled 1 and 2.

Proof

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\overline{AB} \cong \overline{CD}$	Given
$\triangle OAB \cong \triangle O'CD$	S.S.S $\cong$ S.S.S
$\angle 1 \cong \angle 2$	Corresponding angles of two congruent triangles.
$m\angle 1 = m\angle 2$	Condition of equality of two arcs.
$m\overline{AEB} = m\overline{CFD}$	
$\overline{AEB} \cong \overline{CFD}$	

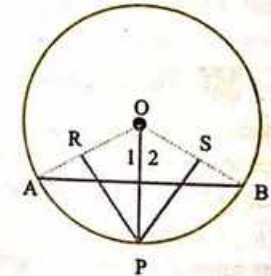
∴ the corresponding minor arcs of two equal chords $\overline{AB}$ and $\overline{CD}$ of two congruent circles are congruent.

Example 1 A point P on the circumference is equidistant from the radii $\overline{OA}$ and $\overline{OB}$.
Prove that $m\overline{AP} = m\overline{BP}$

Given

AB is the chord of a circle with centre O . Point P on the circumference of the circle is equidistant from the

radii $\overline{OA} = \overline{OB}$
so that $m\overline{PR} = m\overline{PS}$

**To prove**

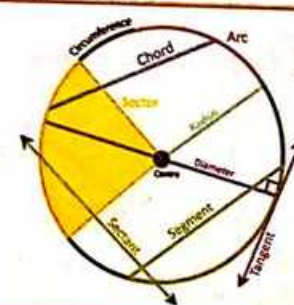
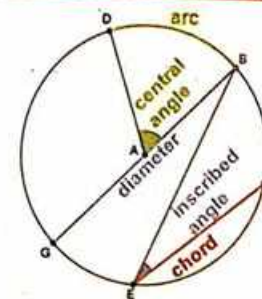
$$m\overline{AP} = m\overline{BP}$$

Construction

Join O with P . Write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof

Statements	Reasons
In $\triangle OPR$ and $\triangle OPS$	
$\overline{OP} = \overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	Point P is equidistant from radii.
	(Given)
$\therefore \triangle OPR \cong \triangle OPS$	(In $\triangle$ H.S $\cong$ H.S)
So $m\angle 1 \cong m\angle 2$	Central angles of a circle.
Chord $\overline{AP} \cong \overline{BP}$	
Hence $m\overline{AP} = m\overline{BP}$	Arcs corresponding to equal chords in a circle.



Theorem 11.3

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).

Case (a) For one circle**Given**

A circle with centre O . $\overline{AB}$ and $\overline{CD}$ are two chords of the circle (which are not diameters) such that $\overline{AB} \cong \overline{CD}$ or $m\overline{AB} = m\overline{CD}$.

Arcs subtend $\angle 1$ and $\angle 2$ at the centre.

To prove

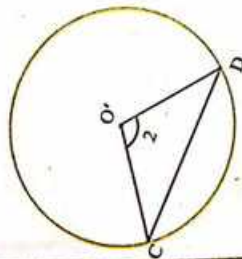
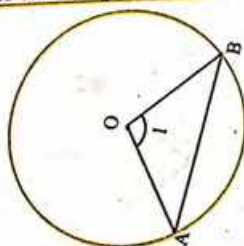
$$\angle 1 \cong \angle 2$$

Construction

We join O to A, B, C and D respectively so that $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD} = \text{radii}$ of a circle.

Proof

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore \triangle OAB \cong \triangle OCD$	S.S.S $\cong$ S.S.S
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of congruent triangles.

Case (b) For two congruent circles**Given**

Two congruent circles with centres O and O' having two equal chords $\overline{AB}$ and $\overline{CD}$ i.e. $\overline{AB} \cong \overline{CD}$.

To prove

These chords subtend equal angles at the centre i.e. $\angle 1 \cong \angle 2$.

Construction

Join O with A and B , and O' with C and D .

Proof

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore \triangle OAB \cong \triangle O'CD$	S.S.S $\cong$ S.S.S
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of congruent triangles.

Theorem 11.4

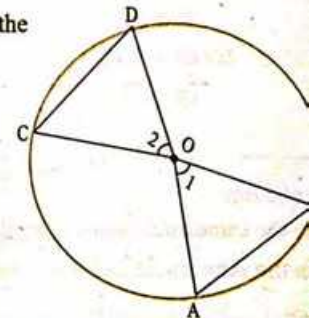
If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal in measures, then the chords are equal in measures.

Case (a) For one circle**Given**

A circle with centre O . $\overline{AB}$ and $\overline{CD}$ are two chords of the circle and $\angle 1 \cong \angle 2$.

To prove

$$\overline{AB} \cong \overline{CD}$$

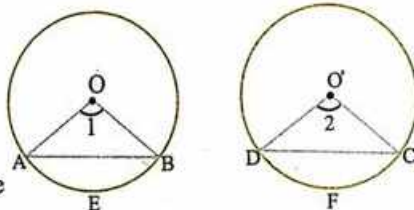


Proof

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\angle 1 \cong \angle 2$	Given
$\therefore \triangle OAB \cong \triangle OCD$	S.A.S Postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of congruent triangles.

Case (b) For two congruent circles**Given**

Two congruent circles with centres O and O' ,
 $\overline{AB}$ and $\overline{CD}$ are two chords of these circles
 such that they subtend equal angles at the centre
 i.e. $\angle 1 \cong \angle 2$.

**To prove**

$$\overline{AB} \cong \overline{CD}$$

Proof

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	
$\overline{OA} \cong \overline{OC}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{OC}$	Radii of two congruent circles.
$\angle 1 \cong \angle 2$	Given
$\therefore \triangle OAB \cong \triangle O'CD$	S.A.S Postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.

Corollaries

In the same circle equal central angles have equal arcs.
 In the same circle equal arcs have equal central angles.

Example 2 The internal bisector of a central angle in a circle bisects an arc on which it stands.

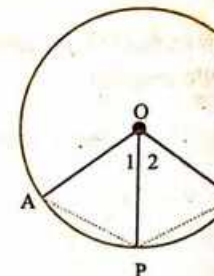
Given

In a circle with centre O , $\overline{OP}$ is an internal bisector of central angle AOB .

To prove $\widehat{AP} \cong \widehat{BP}$

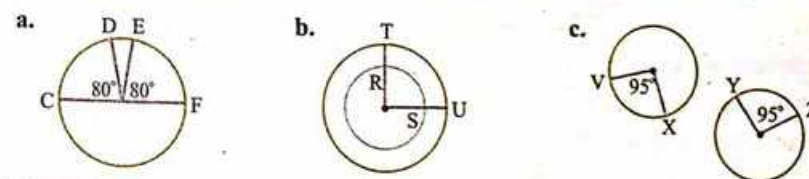
Construction

Draw AP and BP , then write $\angle 1$ and $\angle 2$ as shown in the figure.

**Proof**

Statements	Reasons
In $\triangle OAP \leftrightarrow \triangle OBP$	
$\overline{OA} = \overline{OB}$	Radii of the same circle.
$\angle 1 = \angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and $\overline{OP} = \overline{OP}$	Common
$\triangle OAP \cong \triangle OBP$	(S.A.S $\cong$ S.A.S)
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow \widehat{AP} \cong \widehat{BP}$	Areas corresponding to equal chords in a circle.

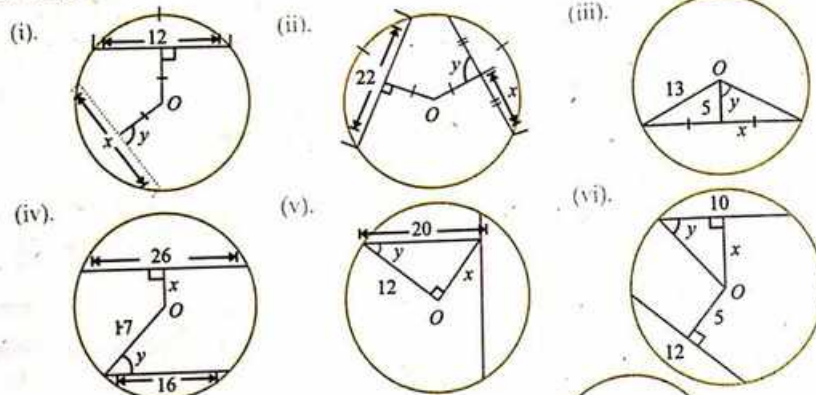
Example 3 Tell whether the red arcs are congruent. Explain why or why not.

**Solution**

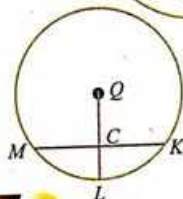
- a. $\widehat{CD} \cong \widehat{EF}$ because they are in the same circle and $m\widehat{CD} = m\widehat{EF}$.
 b. $\widehat{RS}$ and $\widehat{TU}$ have the same measure, but are not congruent because they are arcs of circles that are not congruent.
 c. $\widehat{VX} = \widehat{YZ}$ because they are in congruent circles and $m\widehat{VX} = m\widehat{YZ}$

Exercise 11

1. Given that O is the centre of each of the following circles, find the value of x and y in the following cases.



2. In $\odot Q$, $\widehat{KL} = \widehat{LM}$. If $CK = 2x + 3$ and $CM = 4x$, find x .



Review Exercise 11

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i) $\odot P$ has a radius of 3 and $\widehat{AB}$ has a measure of 90° .

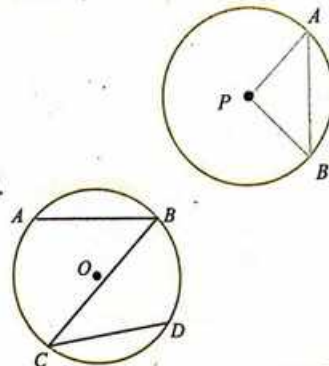
What is the length of $\widehat{AB}$?

- ☐ $3\sqrt{2}$ ☐ $3\sqrt{3}$
☐ 6 ☐ 9

(ii) In the accompanying diagram of circles O, $\widehat{AB} \cong \widehat{CD}$.

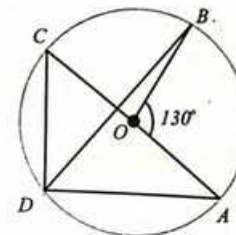
Which statement is true?

- ☐ $\widehat{AB} \cong \widehat{CD}$ ☐ $\widehat{AB} \parallel \widehat{CD}$
☐ $\widehat{AC} \cong \widehat{BD}$ ☐ $\angle ABC \cong \angle BCD$



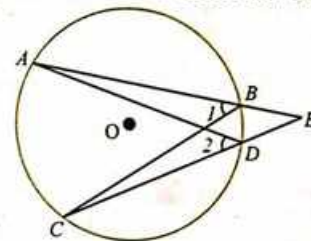
2. In a circle if any pair of diameters are $\perp$ each other then the lines joining its ends in order, form a square.

3. In a circle with centre O, AOC is diameter. $\angle AOB = 130^\circ$, find $\angle ADB$, $\angle BDC$ and $m\widehat{BC}$.



4. Chords AB and CD of a circle meet at the point E outside the circle. Prove that:

- (a) $\angle A = \angle C$ (b) $\angle 1 = \angle 2$
 (c) $\triangle ADE$ and $\triangle CBE$ are equiangular.



Summary

- ⦿ A central angle of a circle is an angle whose vertex is the centre of the circle.
- ⦿ The measure of a minor arc is the measure of its central angle.
- ⦿ The measure of the entire circle is 360.
- ⦿ The measure of major arc is the difference between 360 and the measure of the related minor arc.
- ⦿ The measure of the semi circle is 180° .
- ⦿ Two arcs of the same circle are adjacent if they have a common end points.
- ⦿ The measure of an arc formed by two adjacent arcs is the sum of the measure of the two arcs.
- ⦿ Two circles are congruent circles if they have same radius.
- ⦿ Two arcs are congruent arcs if they have the same measure and they are arcs of the same circle or of congruent circles.
- ⦿ In the same circle, or in congruent circles, two minor arcs are congruent if and only if their corresponding chords are congruent.
- ⦿ In the same circle, or in congruent circles, two chords are congruent if and only if they are equidistant from the center.