

INTRODUCTION

In earlier classes, we studied trigonometric functions and their real-life applications. Now, we extend that knowledge by exploring inverse trigonometric functions which help us to find angles when function values are known. This is essential in solving equations and modeling periodic behavior in calculus of inverse trigonometric functions. In this unit, we will define inverse trigonometric functions, their domains and ranges, examine their graphs within the given interval and understand their key properties and uses.

9.1 Inverse Trigonometric Functions and their Graph

We have studied about inverse of a function. We learned that only a one-to-one and onto function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

Recall!

Let f be a bijective function from a set X to a set Y . The **inverse function** of f , written as f^{-1} , is a bijective function from a set Y back to set X such that

$$x = f^{-1}(y), \forall y \in Y \Leftrightarrow y = f(x), \forall x \in X$$

Here domain of f^{-1} = range of f and range of f^{-1} = domain of f . This means that f^{-1} reverses the correspondence of f i.e.,

$f^{-1}(y) = x$, when $f(x) = y$, also their composition gives:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \forall x \in X$$

$$\text{and } (f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y, \forall y \in Y$$

The equation $\sin 90^\circ = 1$ clearly shows that 90° is the angle whose sine is 1. Sometimes, we know the value of sine, cosine or tangent, but not the angle itself. In such cases, we use the inverse trigonometric functions to find the angle. For example, the inverse of $\sin 90^\circ = 1$ can be written as $90^\circ = \sin^{-1}(1)$ or $90^\circ = \arcsin 1$, and read it as 90° is equal to sine inverse of 1 or arcsin of 1. Generally, the equation $x = \sin y$ gives value of x for a given angle of y . To find the angle y when x is known, we use the inverse function, that is,

$$y = \sin^{-1} x$$

This means y is an angle whose sine is x and the symbol $\sin^{-1} x$ is read as sine inverse of x or arcsine of x . In other words, $y = \sin^{-1} x$ iff $x = \sin y$. Similarly, we can write the other equations as $y = \cos^{-1} x$ iff $x = \cos y$ and $y = \tan^{-1} x$ iff $x = \tan y$, etc.

We know that trigonometric functions sine, cosine and tangent are periodic, meaning they repeat their values at regular intervals. Due to this periodicity, trigonometric

functions are generally not bijective (i.e., they are not both one-to-one and onto over their entire domain). As a result, their inverses do not naturally exist without restrictions. However, by carefully restricting their domains and codomains, we can make these functions bijective that allows us to define their inverses, known as inverse trigonometric functions. Next, we discuss these functions one by one in details.

9.1.1 Inverse of Sine Function

Consider the function $f: \mathbb{R} \rightarrow [-1, 1]$ define by $f(x) = \sin x$. The graph of the function is shown in Figure 9.1.

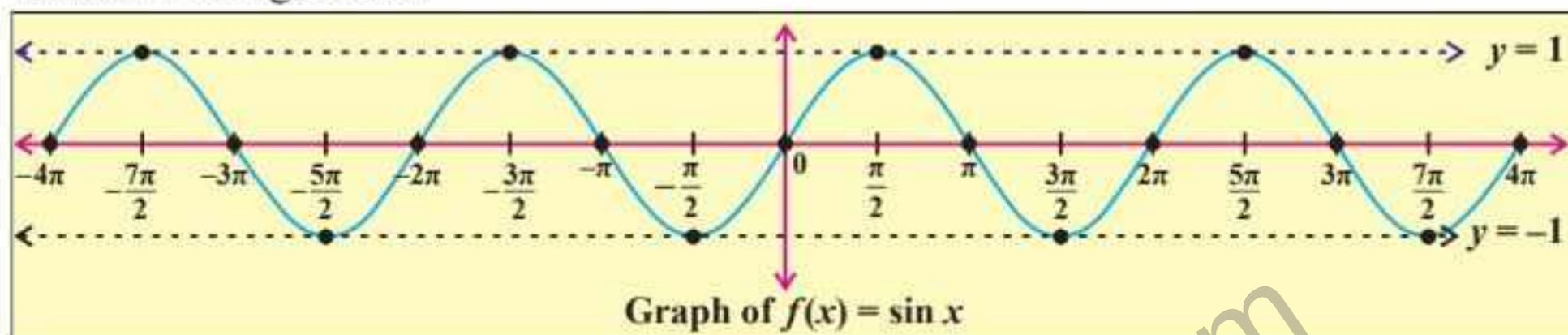


Figure 9.1

Clearly the graph of sine function is not bijective because it touches the line $y = 1$ and $y = -1$ at many points due to the domain of sine function which is the set of all real numbers. However, if we restrict the domain of $f(x) = \sin x$ to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function with restricted domain i.e., $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ becomes one-one and onto (bijective) that is invertible and is called **principal sine function**. Then the inverse sine function is defined by

$$y = \sin^{-1}x \text{ iff } x = \sin y$$

where the domain of $\sin^{-1}x$ is $[-1, 1]$ and range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The graph of the function $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ is shown in Figure 9.2(a). and graph

of $\sin^{-1}: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is shown in Figure 9.2(b).

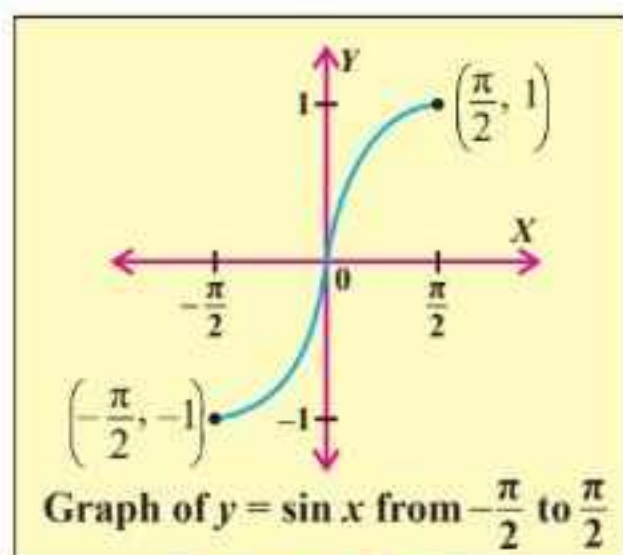


Figure 9.2(a)

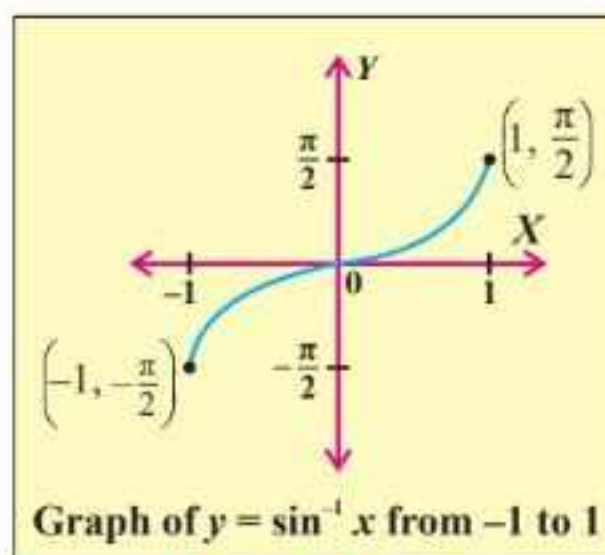


Figure 9.2(b)

The graph of $\sin^{-1}x$ is also obtain by reflecting the graph of principal sine function $y = \sin x$ along the line $y = x$ as a mirror image shown in Figure 9.3. Observe that the graph of $y = \sin x$ is symmetric about the line $y = x$.

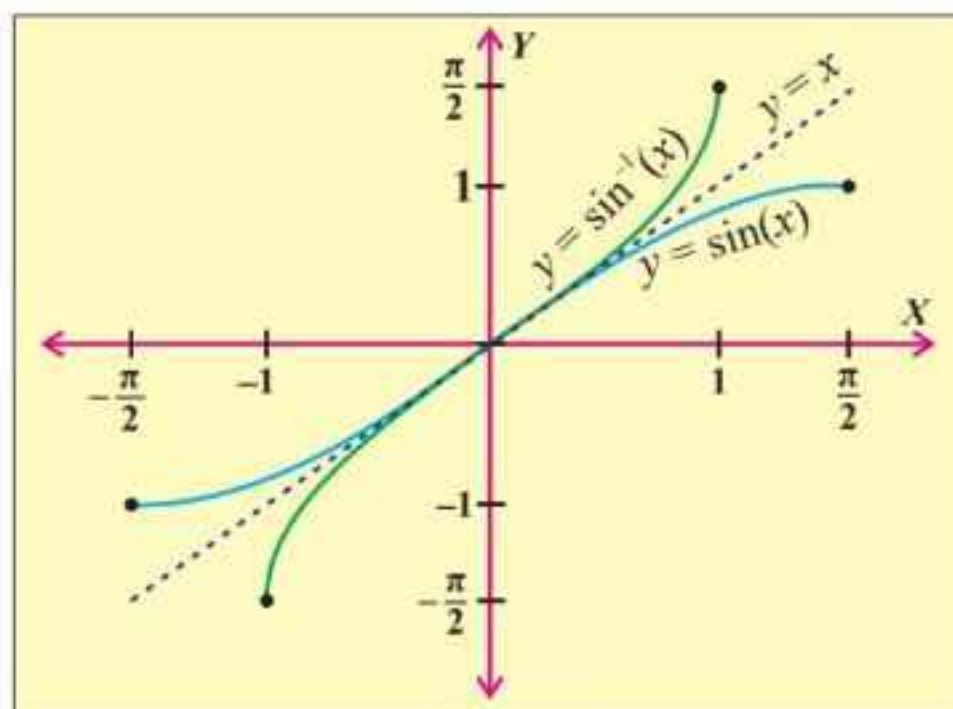


Figure 9.3

The sine function can be restricted to any one of the infinitely many intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ or generally $\left[n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right], n \in \mathbb{Z}$ with range $[-1, 1]$ and length of each interval π where it is strictly increasing or decreasing, to make the sine function one-to-one. If interval is not specified, then $\sin^{-1}x$ is a function with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note

$$\sin^{-1}x \neq (\sin x)^{-1} \text{ or } \frac{1}{\sin x}$$

9.1.2 Inverse of Cosine Function

Consider the function $f: \mathbb{R} \rightarrow [-1, 1]$ define by $f(x) = \cos x$. The graph of the function is shown in Figure 9.4.

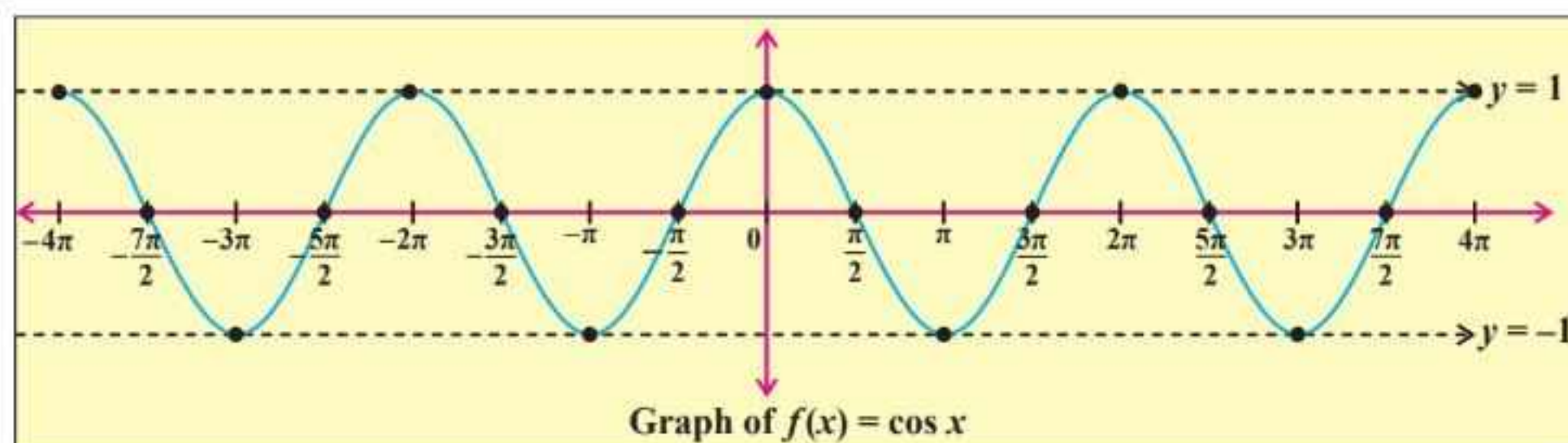


Figure 9.4

Like the graph of sine function, the graph of cosine function is not bijective because again it fails the horizontal line test. It touches the line $y = 1$ and $y = -1$ at multiple points. However, if we carefully restrict the domain of $f(x) = \cos x$ to the closed interval $[0, \pi]$, then the function with restricted domain i.e., $f: [0, \pi] \rightarrow [-1, 1]$ becomes one-one and onto that is invertible and is called **principal cosine function**. The inverse cosine function is defined by

$$y = \cos^{-1}x \text{ iff } x = \cos y$$

The domain of $\cos^{-1}x$ is $[-1, 1]$ and range of $\cos^{-1}x$ is $[0, \pi]$.

The graph of the function $\cos: [0, \pi] \rightarrow [-1, 1]$ is shown in Figure 9.5(a) and graph of $\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$ is shown in Figure 9.5(b).

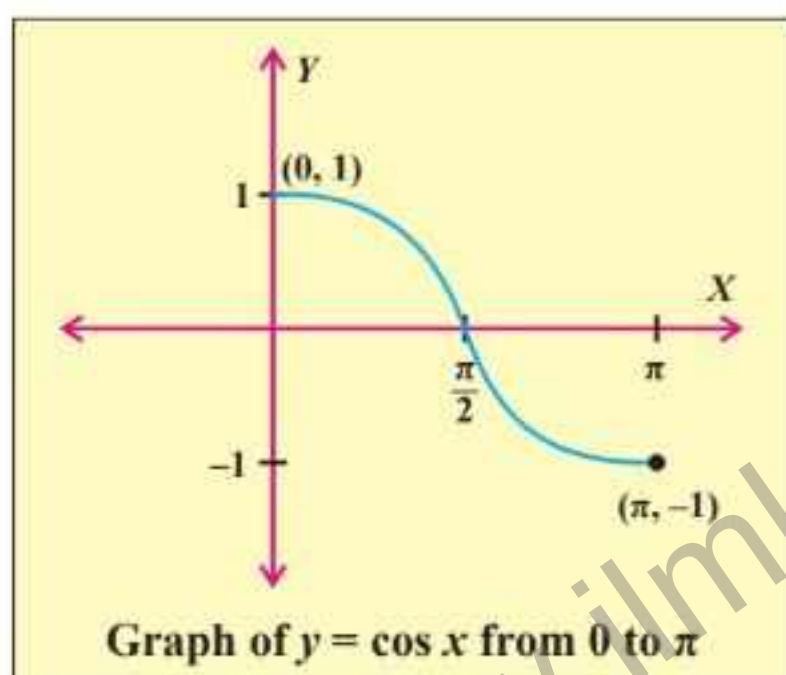


Figure 9.5(a)

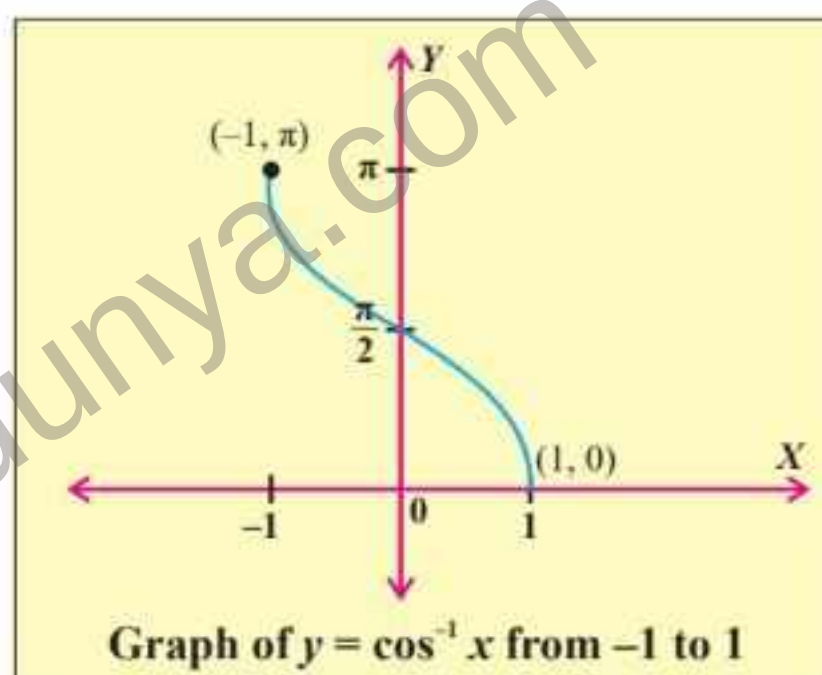


Figure 9.5(b)

The graph of $\cos^{-1}x$ is also obtain by reflecting the graph of principal cosine function $y = \cos x$ along the line $y = x$ as a mirror image shown in Figure 9.6. Observe that the graph of $y = \cos x$ is symmetric about the line $y = x$.

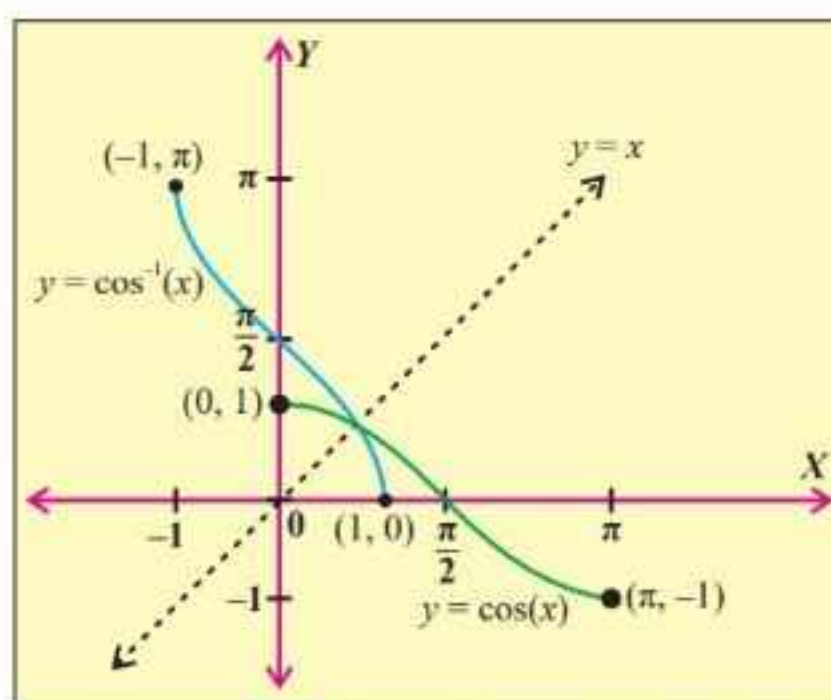


Figure 9.6

Like, sine function, the cosine function with range $[-1, 1]$ can be restricted to any one of the infinitely many intervals $[-2\pi, -\pi]$, $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ or generally $[n\pi, n\pi + \pi]$, $n \in \mathbb{Z}$ to make the cosine function one-to-one. If no interval is specified, then $\cos^{-1} x$ is a function with domain $[-1, 1]$ and range $[0, \pi]$.

Note

$$\cos^{-1} x \neq (\cos x)^{-1} \text{ or } \frac{1}{\cos x}$$

9.1.3 Inverse of Tangent Function

Consider the function define by $f(x) = \tan x$ with domain is the set $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range is the set of all real numbers. The graph of the tangent function is shown in Figure 9.7.

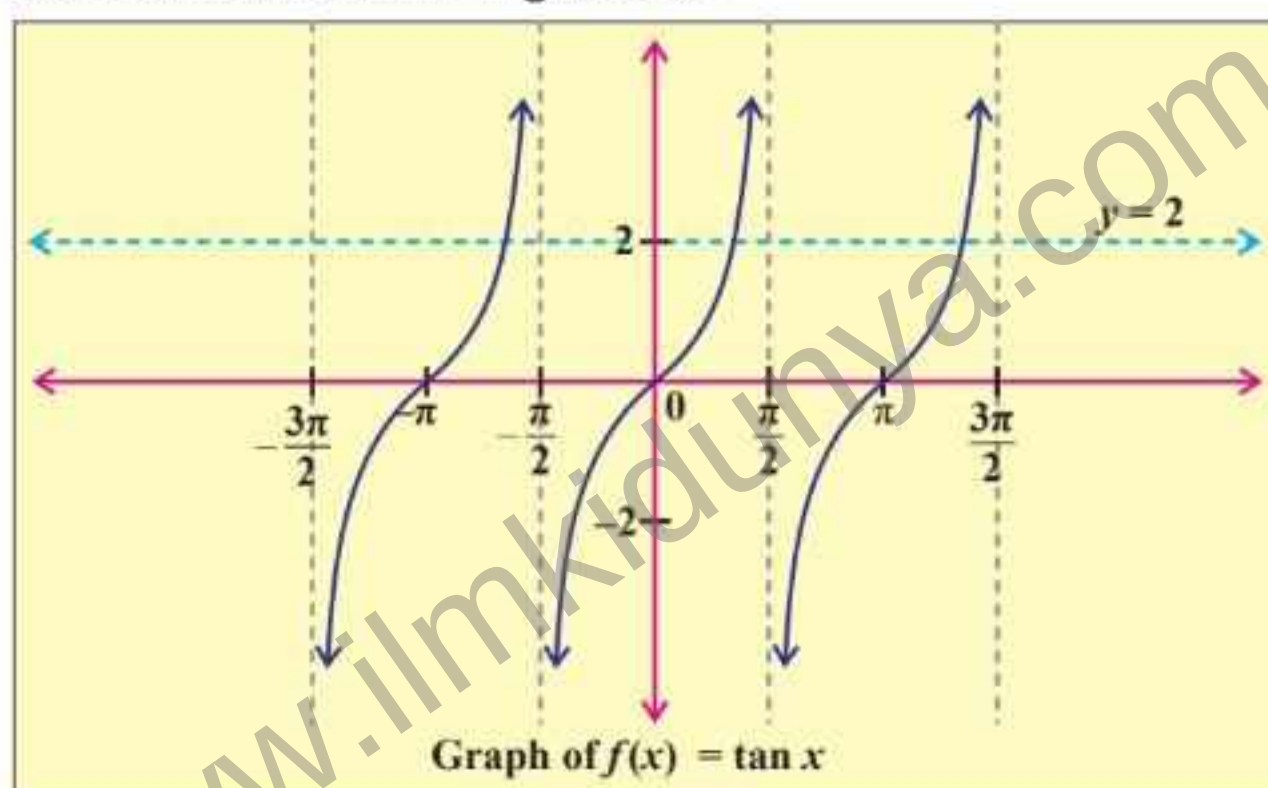


Figure 9.7

Clearly the graph of tangent function is not bijective because the line $y = 2$ intersect the graph at multiple points. However, if we restrict the domain of $f(x) = \tan x$ to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the function with restricted domain i.e. $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ becomes one-one and onto that is invertible. This function with domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and range $(-\infty, \infty)$ is called **principal tangent function**. The inverse tangent function is defined by $y = \tan^{-1} x$ iff $x = \tan y$.

The domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The graph of the function $\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ is shown in Figure 9.8(a) and graph of

$\tan^{-1} : (-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in Figure 9.8(b).

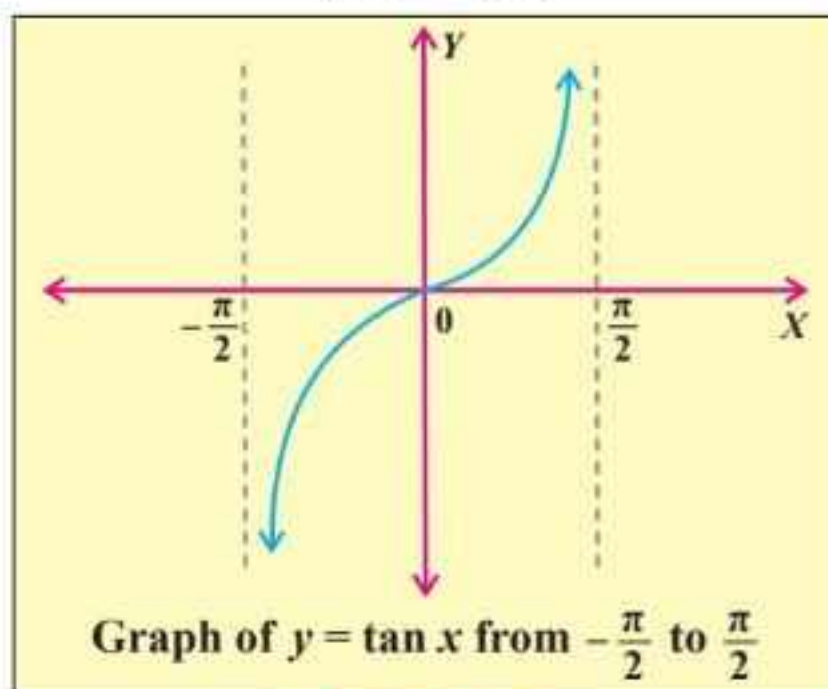


Figure 9.8(a)

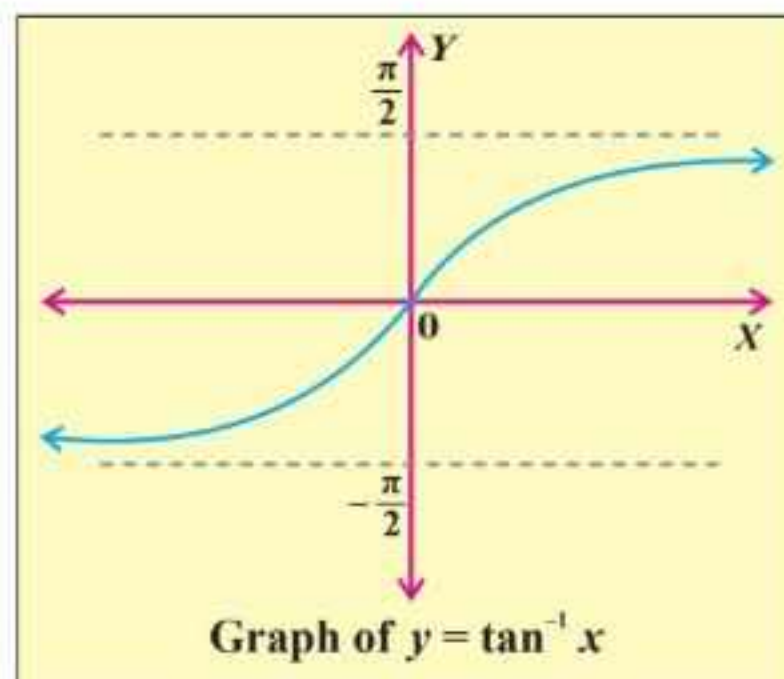


Figure 9.8(b)

The graph of $\tan^{-1} x$ is also obtain by reflecting the graph of principal tangent function $y = \tan x$ along the line $y = x$ as a mirror image shown in Figure 9.9. Observe that the graph of $y = \tan x$ is symmetric about the line $y = x$.

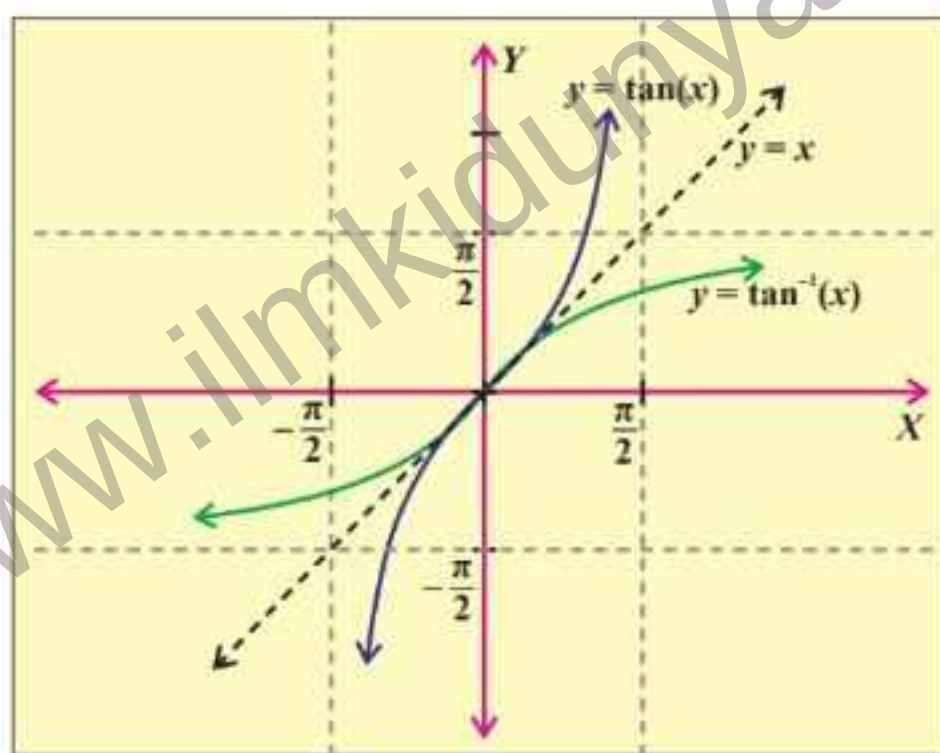


Figure 9.9

Like sine and cosine functions, the domain of tangent function can be restricted to any one of the infinitely many intervals $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ or generally $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right), n \in \mathbb{Z}$ with range $\mathbb{R}$ to make the tangent function one-to-one. If no interval is specified then $\tan^{-1} x$ is a function with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Remember!

$$\tan^{-1} x \neq (\tan x)^{-1}, \text{ or } \frac{1}{\tan x}$$

Example 1 Find the principal value of

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution:

(i) Let $\sin^{-1}\left(\frac{1}{2}\right) = y$

We want to find the angle y , whose sine is $\frac{1}{2}$

$\Rightarrow \sin y = \frac{1}{2}$, where the principal

value of $\sin^{-1}x$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\sin y = \frac{1}{2} = \sin \frac{\pi}{6}$

$\Rightarrow y = \frac{\pi}{6}$

Thus, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y$

Then, $\tan y = -\frac{1}{\sqrt{3}}$, where the principal value of $\tan^{-1}x$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\tan y = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow y = -\frac{\pi}{6}$

Hence, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(ii) Let $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y$

We want to find the angle y , whose cosine is $-\frac{\sqrt{3}}{2}$

$\Rightarrow \cos y = -\frac{\sqrt{3}}{2}$, where the principal

value of $\cos^{-1}x$ lies in $[0, \pi]$

As $\cos y = -\frac{\sqrt{3}}{2} = \cos \frac{5\pi}{6}$

$\Rightarrow y = \frac{5\pi}{6}$

Thus, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

9.1.4 Inverse of Cosecant Function

The graph of the function $f(x) = \csc x$ with domain $\{x \mid x \in \mathbb{R} \wedge x \neq n\pi, n \in \mathbb{Z}\}$ and range is the set $\mathbb{R} - (-1, 1)$ is shown in Figure 9.10.

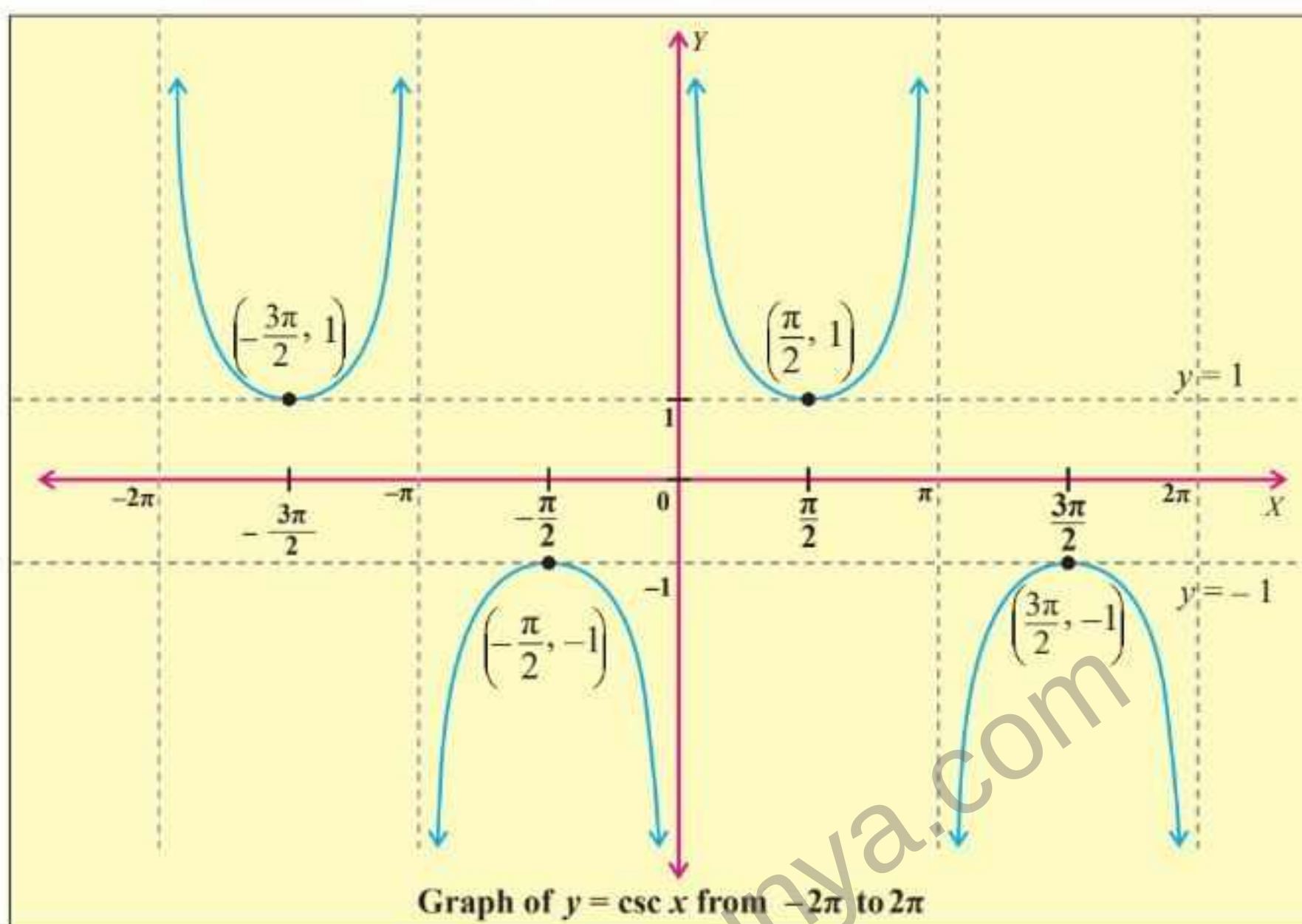


Figure 9.10

Clearly from the graph it follows that cosecant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \csc x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, then the function with restricted domain i.e. $f: \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \rightarrow (-\infty, -1) \cup (1, \infty)$ becomes one-one and onto that is invertible and is called **principal cosecant function**. The inverse cosecant function is defined by

$$y = \csc^{-1} x \text{ iff } x = \csc y$$

The domain of $\csc^{-1} x = \mathbb{R} - (-1, 1)$ and range of $\csc^{-1} x = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

The graph of the function $\csc: \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \rightarrow (-\infty, -1) \cup (1, \infty)$ is shown in

Figure 9.11(a) and graph of $\csc^{-1}: (-\infty, -1) \cup (1, \infty) \rightarrow \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is shown in Figure 9.11(b).

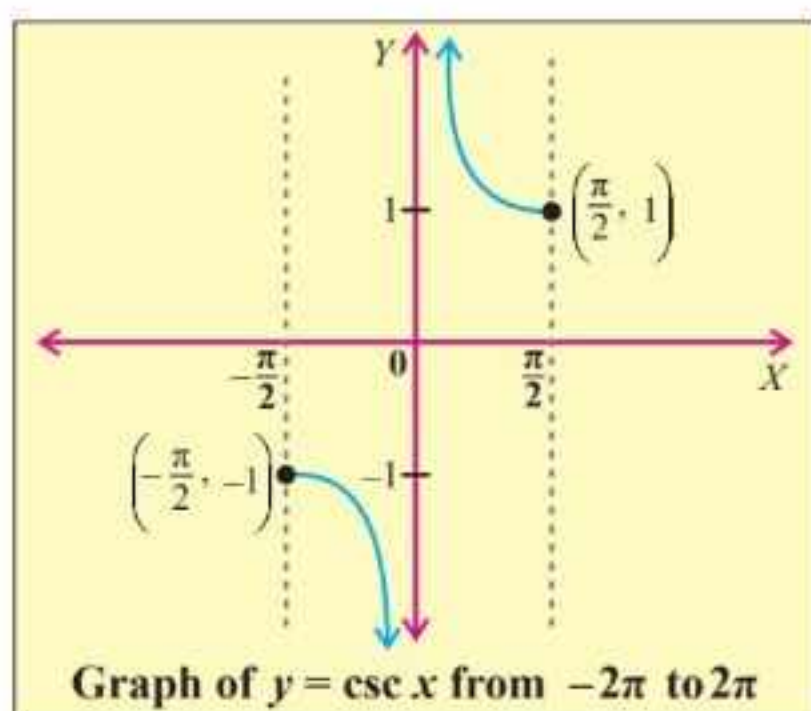


Figure 9.11(a)

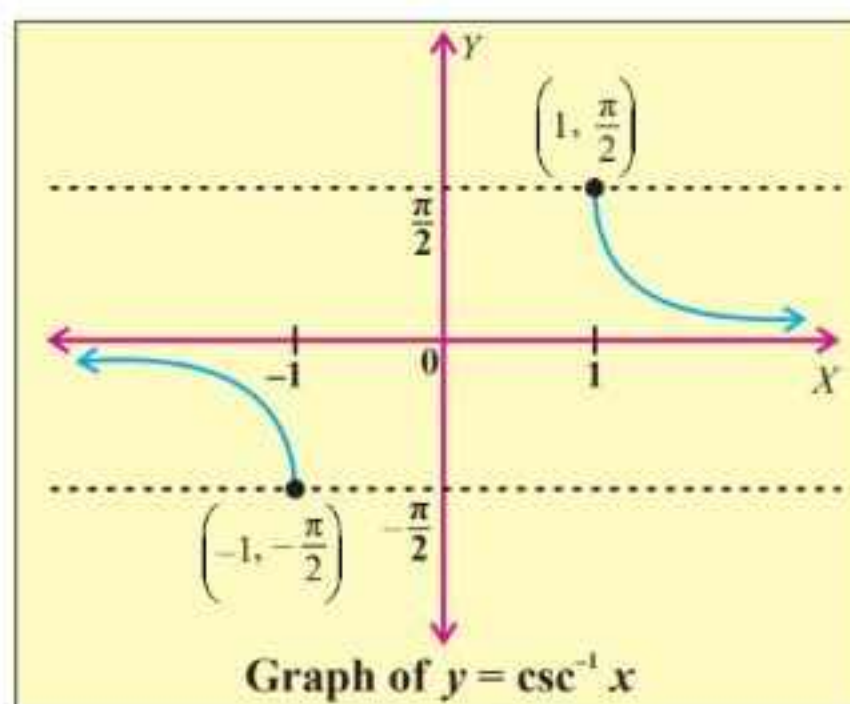


Figure 9.11(b)

9.1.5 Inverse of Secant Function

The graph of the function $f(x) = \sec x$ with domain $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range $\mathbb{R} - (-1, 1)$ is shown in Figure 9.12.

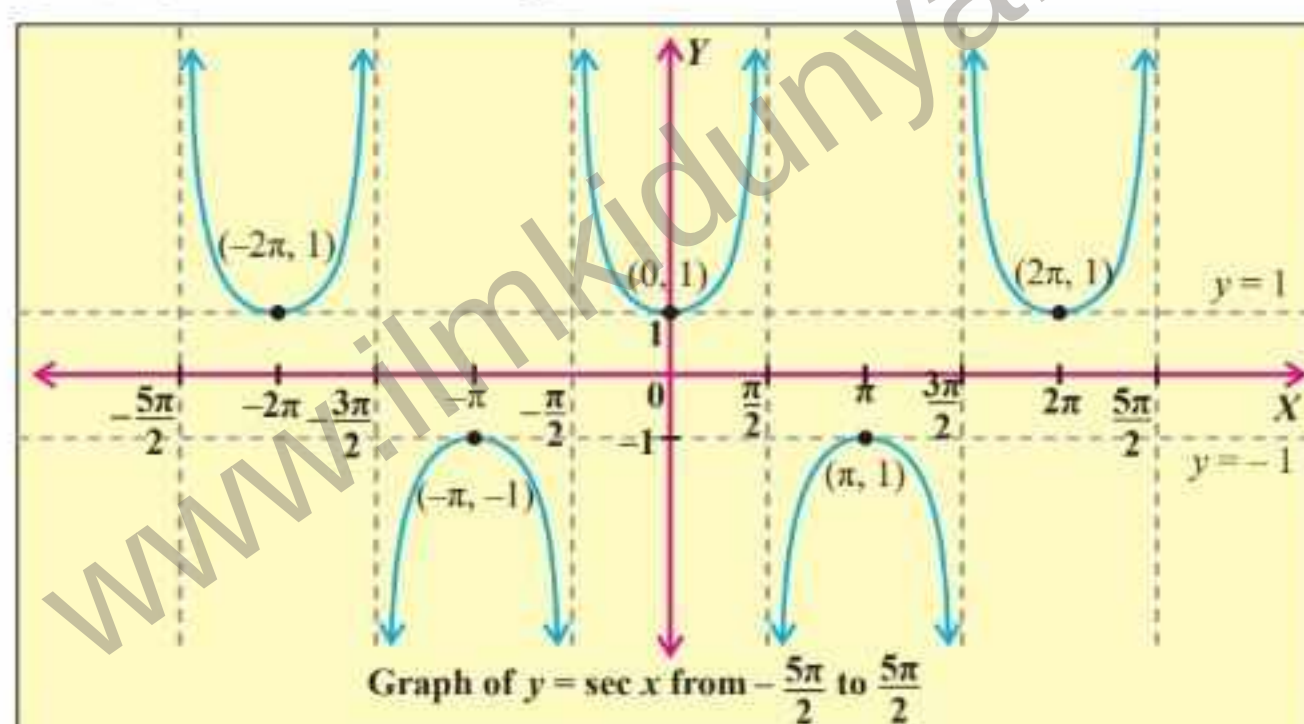


Figure 9.12

Clearly from the graph it follows that secant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \sec x$ to the interval $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, then the function with restricted domain

i.e. $f: \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \rightarrow (-\infty, -1] \cup [1, \infty)$ becomes one-one and onto (bijective)

that is invertible and is called **principal secant function**. The inverse secant function is defined by

$$y = \sec^{-1} x \text{ if } x = \sec y$$

The domain of $\sec^{-1}x$ is $\mathbb{R} - (-1, 1)$ and range of $\sec^{-1}x$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

The graph of the function $\sec: \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \rightarrow (-\infty, -1] \cup [1, \infty)$ is shown in

Figure 9.13(a) and graph of $\sec^{-1}: (-\infty, -1] \cup [1, \infty) \rightarrow \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is shown in

Figure 9.13(b).

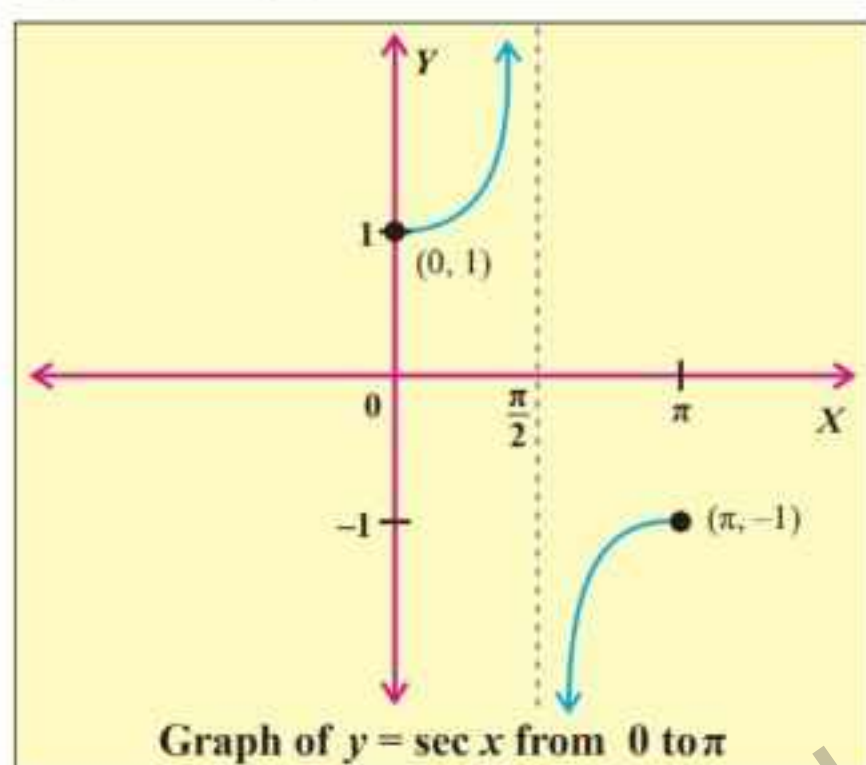


Figure 9.13(a)

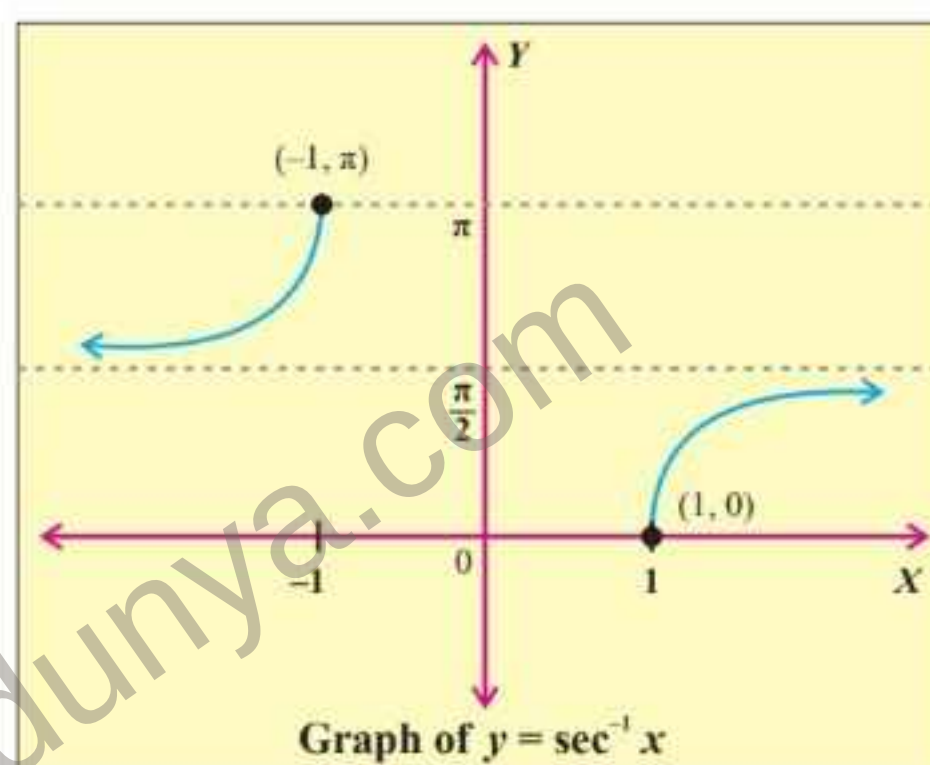


Figure 9.13(b)

9.1.6 Inverse of Cotangent Function

The graph of the function $f(x) = \cot x$ with domain $\{x \mid x \in \mathbb{R} \wedge x \neq n\pi, n \in \mathbb{Z}\}$ and range $(-\infty, \infty)$ is shown in Figure 9.14.

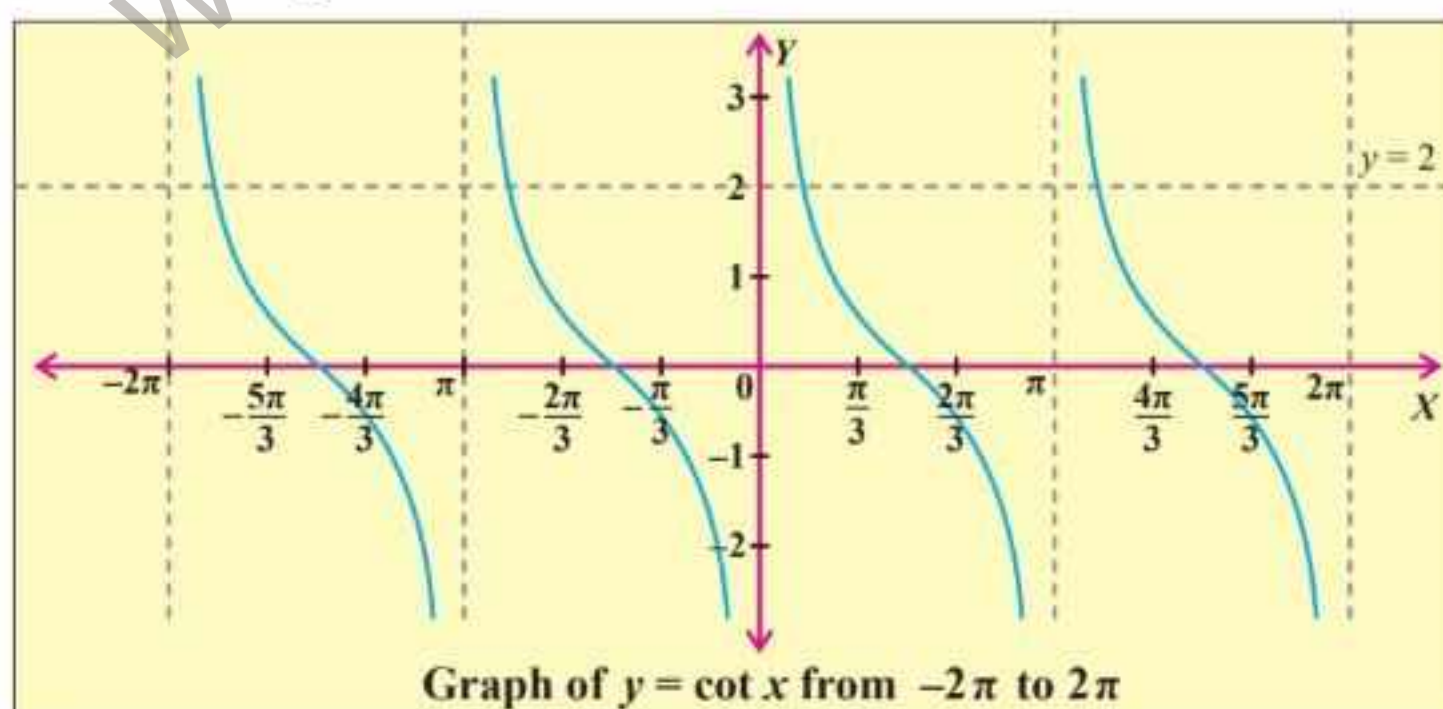


Figure 9.14

Clearly from the graph it follows that cotangent function is not bijective because the line $y = 2$ intersects the graph at many points. So, we restrict the domain of $f(x) = \cot x$ to the

interval $(0, \pi)$, then the function with restricted domain i.e., $f: (0, \pi) \rightarrow (-\infty, \infty)$ becomes one-one and onto that is invertible and is called **principal cotangent function**. The inverse cot x function is defined by

$$y = \cot^{-1} x \text{ iff } x = \cot y$$

The domain of $\cot^{-1} x$ is $(-\infty, \infty)$ and range of $\cot^{-1} x$ is $(0, \pi)$.

The graph of the function $\cot: (0, \pi) \rightarrow (-\infty, \infty)$ is shown in Figure 15(a) and graph of $\cot^{-1}: (-\infty, \infty) \rightarrow (0, \pi)$ is shown in Figure 9.15(b).

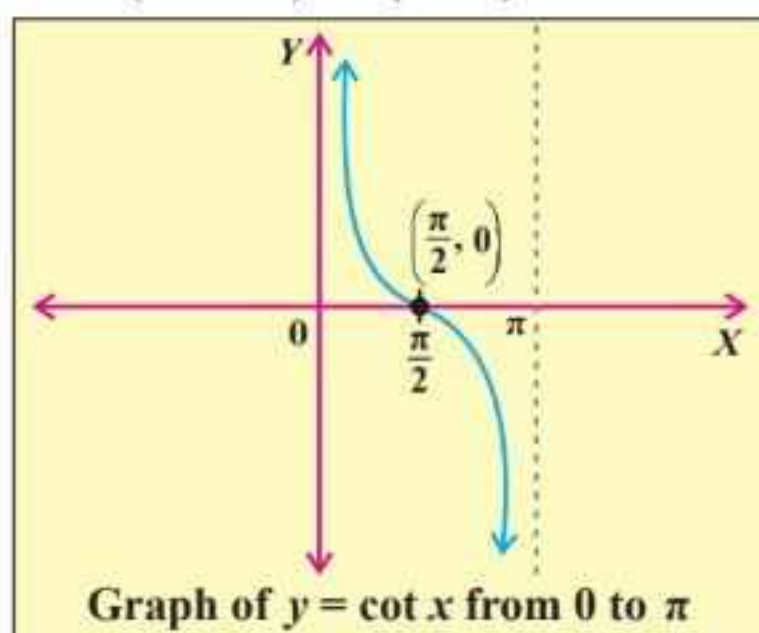


Figure 9.15(a)

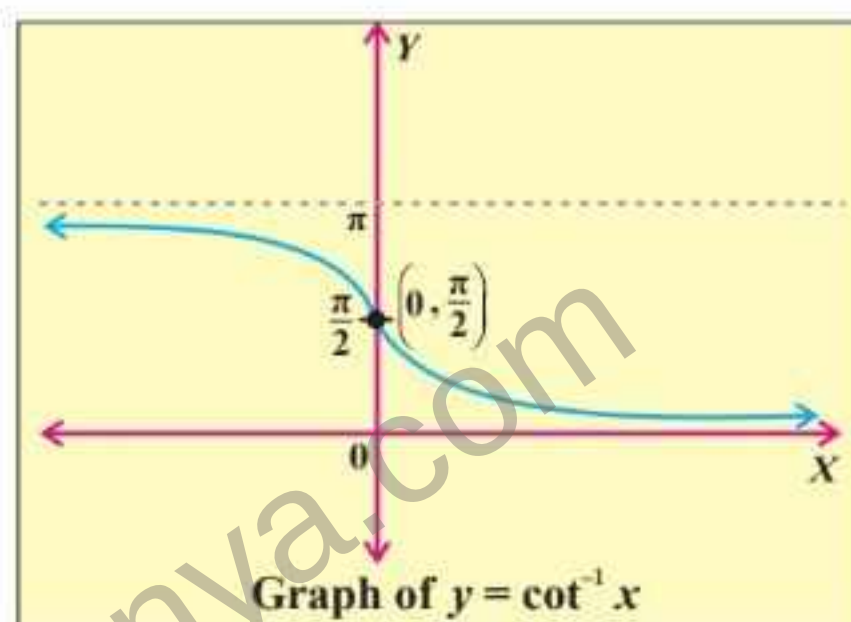


Figure 9.15(b)

Example 2 Find value of:

(i) $\sec\left[\tan^{-1}(-\sqrt{3})\right]$ (ii) $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ (iii) $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

Solution: (i) $\sec\left[\tan^{-1}(-\sqrt{3})\right]$
Let $\tan^{-1}(-\sqrt{3}) = y$

First, we find the angle y , whose tangent is $-\sqrt{3}$

$$\tan y = -\sqrt{3}, \text{ where } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{Now } \tan y = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right) \Rightarrow y = -\frac{\pi}{3}$$

$$\text{So, } \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$$\text{Now, } \sec\left[\tan^{-1}(-\sqrt{3})\right] = \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\text{Therefore } \sec\left[\tan^{-1}(-\sqrt{3})\right] = 2$$

$$(ii) \cot \left[\cos^{-1} \left(-\frac{1}{2} \right) \right]$$

$$\text{Let } \cos^{-1} \left(-\frac{1}{2} \right) = y$$

First, we find the angle I , whose cosine is $-\frac{1}{2}$.

$$\cos y = -\frac{1}{2}, \text{ where } y \in [0, \pi]$$

$$\text{Now, } \cos y = -\frac{1}{2} = \cos \left(\frac{2\pi}{3} \right) \Rightarrow y = \frac{2\pi}{3}$$

$$\text{So, } \cos^{-1} \left(-\frac{1}{2} \right) = \frac{2\pi}{3}$$

$$\text{Now, } \cot \left[\cos^{-1} \left(-\frac{1}{2} \right) \right] = \cot \left(\frac{2\pi}{3} \right) = -\frac{1}{\sqrt{3}}$$

$$\text{Therefore } \cot \left[\cos^{-1} \left(-\frac{1}{2} \right) \right] = -\frac{1}{\sqrt{3}}$$

$$(iii) \csc \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right]$$

$$\text{Let } \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = y$$

$$\Rightarrow \sin y = \frac{\sqrt{3}}{2}, \text{ where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\text{Now, } \sin y = \frac{\sqrt{3}}{2} = \sin \left(\frac{\pi}{3} \right) \Rightarrow y = \frac{\pi}{3}$$

$$\text{So, } \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{3}$$

$$\text{Now, } \csc \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right] = \csc \left(\frac{\pi}{3} \right) = \frac{2}{\sqrt{3}}$$

$$\text{Therefore, } \csc \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right] = \frac{2}{\sqrt{3}}$$

The table below summarizes the domains and ranges of the principal trigonometric and inverse trigonometric functions.

Function	Domain	Range	Function	Domain	Range
$\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$	$[-1, 1]$	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$	$(-\infty, \infty)$	$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
$\csc x$	$\left[-\frac{\pi}{2}, 0 \right) \cup \left(0, \frac{\pi}{2} \right]$	$(-\infty, -1] \cup [1, \infty)$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0 \right) \cup \left(0, \frac{\pi}{2} \right]$
$\sec x$	$\left[0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$	$(-\infty, -1] \cup [1, \infty)$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

Example 3 Find the domain of the following functions

(i) $f(x) = \cos^{-1}(3x-1)$ (ii) $g(x) = \sin^{-1}\left(\frac{x^2-5}{4}\right)$ (iii) $h(x) = \sec^{-1}(\ln x)$

Solution: (i) The inverse cosine function, $f(x) = \cos^{-1}x$ is defined for all x such that $-1 \leq x \leq 1$.

Thus, the domain of $f(x) = \cos^{-1}(3x-1)$ is defined for all x that satisfy

$$-1 \leq 3x-1 \leq 1 \Rightarrow 0 \leq 3x \leq 2 \Rightarrow 0 \leq x \leq \frac{2}{3} \Rightarrow x \in \left[0, \frac{2}{3}\right]$$

Hence, the domain of $f(x) = \cos^{-1}(3x-1)$ is $\left[0, \frac{2}{3}\right]$ such that $-1 \leq x \leq 1$.

(ii) The inverse sine function, $f(x) = \sin^{-1}x$ is defined for all x in the interval $[-1, 1]$.

Thus, the domain of $g(x) = \sin^{-1}\left(\frac{x^2-5}{4}\right)$ is defined for all x that satisfy

$$\begin{aligned} -1 \leq \frac{x^2-5}{4} \leq 1 &\Rightarrow -4 \leq x^2-5 \leq 4 \Rightarrow 1 \leq x^2 \leq 9 \\ \Rightarrow x^2 \geq 1 &\quad \text{and} \quad x^2 \leq 9 \\ \Rightarrow x \geq 1 \text{ or } x \leq -1 &\quad \text{and} \quad (x-3)(x+3) \leq 0 \\ \Rightarrow x \in (-\infty, -1] \cup [1, \infty) &\quad \text{and} \quad x \in [-3, 3] \end{aligned}$$

The intersection of two result gives $[-3, -1] \cup [1, 3] \Rightarrow x \in [-3, -1] \cup [1, 3]$

Thus, the domain of $f(x) = \sin^{-1}\left(\frac{x^2-5}{4}\right)$ is $[-3, -1] \cup [1, 3]$

(iii) The inverse secant function $f(x) = \sec^{-1}x$ is defined for all x in the interval $(-\infty, -1] \cup [1, \infty)$.

Thus, the domain of $h(x) = \sec^{-1}(\ln x)$ is defined for all x that satisfy

$$\begin{aligned} \Rightarrow \ln x \geq 1 \text{ or } \ln x \leq -1 &\Rightarrow x \geq e \text{ or } x \leq e^{-1} \Rightarrow x \geq e \text{ or } x \leq \frac{1}{e} \\ \Rightarrow x \in \left(-\infty, \frac{1}{e}\right] \cup [e, \infty) \end{aligned}$$

Hence, the domain of $h(x) = \sec^{-1}(\ln x)$ is $\left(-\infty, \frac{1}{e}\right] \cup [e, \infty)$

Example 4 Prove that:

(i) $\csc^{-1}x = \sin^{-1}\frac{1}{x}, x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}, x \in (-\infty, -1] \cup [1, \infty)$ (iii) $\cot^{-1}x = \tan^{-1}\frac{1}{x}, x \in (0, \infty)$

Solution:**Proof: (i)** Let $\csc^{-1}x = y$ then $x = \csc y$

$$\Rightarrow \sin y = \frac{1}{x}$$

$$\Rightarrow y = \sin^{-1}\left(\frac{1}{x}\right)$$

Hence, $\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$

Proof: (ii) Let $\sec^{-1}x = y$ then $x = \sec y$

$$\Rightarrow \cos y = \frac{1}{x}$$

$$\Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right)$$

Hence, $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

Proof: (iii) Let $\cot^{-1}x = y$ then $x = \cot y$

$$\Rightarrow \tan y = \frac{1}{x} \Rightarrow y = \tan^{-1}\left(\frac{1}{x}\right)$$

Hence, $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

Example 5 Prove that

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$ (ii) $\csc^{-1}x + \sec^{-1}x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in (-\infty, \infty)$

Solution: (i) Let $\sin^{-1}x = y$

$$\Rightarrow x = \sin y = \cos\left(\frac{\pi}{2} - y\right) \quad \left(\because \cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha\right)$$

$$\Rightarrow x = \cos\left(\frac{\pi}{2} - y\right) \Rightarrow \cos^{-1}x = \left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \cos^{-1}x + y = \frac{\pi}{2} \Rightarrow \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$$

Hence, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Similarly, we can prove (ii) and (iii)

Example 6 Show that $\cos^{-1}\frac{7}{25} = \csc^{-1}\frac{25}{24}$ **Solution:** Let $\cos^{-1}\frac{7}{25} = \alpha \Rightarrow \cos \alpha = \frac{7}{25}, \alpha \in \left[0, \frac{\pi}{2}\right]$ and

$$\sin \alpha = \pm \sqrt{1 - \left(\frac{7}{25}\right)^2} = \pm \sqrt{\frac{625 - 49}{625}} \quad \left(\because \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha}\right)$$

$$= \pm \sqrt{\frac{576}{625}} = \pm \frac{24}{25}$$

$$\Rightarrow \sin \alpha = \frac{24}{25} \quad \left(\because \alpha \in \left[0, \frac{\pi}{2}\right] \right)$$

$$\Rightarrow \csc \alpha = \frac{25}{24} \Rightarrow \alpha = \csc^{-1} \frac{25}{24}$$

$$\text{Thus, } \cos^{-1} \frac{7}{25} = \csc^{-1} \frac{25}{24}$$

Example 7: Without using calculator find the value of $\sin \left[\frac{1}{2} \tan^{-1} \frac{4}{3} \right]$.

Solution: Let $\tan^{-1} \frac{4}{3} = \alpha \Rightarrow \tan \alpha = \frac{4}{3}$

Here, $x = 3, y = 4$

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

So, $\sin \alpha = \frac{3}{5}$ and $\cos \alpha = \frac{4}{5}$

Using half angle identity

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - \frac{4}{5}}{2}} = \frac{1}{\sqrt{5}}$$

$$\text{Thus, } \sin \left(\frac{1}{2} \tan^{-1} \frac{4}{3} \right) = \frac{1}{\sqrt{5}}$$

EXERCISE 9.1

1. Find the principal value of the following:

(i) $\sin^{-1} \left(\frac{\sqrt{3}}{2} \right)$

(ii) $\cos^{-1} \left(-\frac{\sqrt{2}}{2} \right)$

(iii) $\tan^{-1} (-\sqrt{3})$

(iv) $\sec^{-1} \left(\frac{2}{\sqrt{3}} \right)$

(v) $\csc^{-1} (-\sqrt{2})$

(vi) $\sin^{-1} \left[\cos \left(\frac{3\pi}{4} \right) \right]$

(vii) $\tan^{-1} \left[\cot \left(\frac{\pi}{6} \right) \right]$

(viii) $\sec^{-1} \left[2 \cos \left(\frac{3\pi}{4} \right) \right]$

(ix) $\csc^{-1} \left(\sqrt{2} \cos \frac{\pi}{4} \right)$

$$(x) \tan^{-1} \left[3 \cos \left\{ 3 \cos^{-1} \left(\frac{\sqrt{3}}{2} \right) \right\} \right]$$

$$(xi) \csc^{-1} \left[\frac{4}{3} \cos \left\{ \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \right\} \right]$$

2. Find the domain of the following:

$$(i) f(x) = \sin^{-1}(5x-3)$$

$$(ii) g(x) = \cos^{-1} \sqrt{x-3}$$

$$(iii) h(x) = \csc^{-1}(x^2-3)$$

$$(iv) k(x) = \sin^{-1}(\ln x)$$

$$(v) p(x) = \tan^{-1} \sqrt{4-x^2}$$

$$(vi) q(x) = \sec^{-1}(x+2)$$

$$(vii) u(x) = \sec^{-1} \left(\frac{3x-1}{2} \right)$$

$$(viii) v(x) = \sin^{-1}(\ln x) + \csc^{-1}(e^x)$$

3. Show that:

$$(i) \cos^{-1} \frac{8}{17} = \sin^{-1} \frac{15}{17}$$

$$(ii) \csc^{-1} \frac{5}{4} = \cos^{-1} \frac{3}{5}$$

$$(iii) 2 \tan^{-1} \frac{3}{4} = \tan^{-1} \frac{24}{7}$$

$$(iv) 2 \sin^{-1} \frac{7}{25} = \sin^{-1} \frac{336}{625}$$

4. Without using calculator find the value of the following:

$$(i) \cos \left(\frac{1}{2} \cot^{-1} \frac{5}{12} \right)$$

$$(ii) \sec \left(\frac{1}{2} \sin^{-1} \frac{7}{25} \right)$$

$$(iii) \tan \left(\frac{1}{2} \cot^{-1} \frac{24}{7} \right)$$

$$(iv) \tan \left(\frac{1}{2} \sin^{-1} \frac{\sqrt{3}}{2} \right)$$

$$(v) \cot \left(\frac{1}{2} \sec^{-1} \frac{2\sqrt{6}}{3} \right)$$

$$(vi) \csc \left(\frac{1}{2} \cot^{-1} \frac{4}{3} \right)$$

9.2 Sum and Difference Formulas for Inverse Trigonometric Functions

In this section, we will derive the addition and subtraction formulas for inverse trigonometric functions. These identities can be derived by applying the fundamental trigonometric identities along with algebraic manipulation. These identities are valid only within the principal value of the respective inverse trigonometric functions. We will now establish the sum and difference formulas for the inverse sine, cosine and tangent functions that are given below.

$$(i) \sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

$$(ii) \sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2})$$

$$(iii) \cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{(1-x^2)(1-y^2)})$$

$$(iv) \cos^{-1} x - \cos^{-1} y = \cos^{-1} (xy + \sqrt{(1-x^2)(1-y^2)})$$

$$(v) \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$$

$$(vi) \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right)$$

Remember!

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \pm \tan \alpha \tan \beta}$$

To prove (i), we proceed as follow:

$$\text{Let } \sin^{-1} x = \alpha \Rightarrow \sin \alpha = x \text{ and } \sin^{-1} y = \beta \Rightarrow \sin \beta = y$$

$$\text{Now } \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha} = \pm \sqrt{1 - x^2}$$

$$\text{Since } \alpha, \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{ so } \cos \alpha, \cos \beta \geq 0$$

$$\therefore \cos \alpha = \sqrt{1 - x^2}$$

$$\text{Similarly, } \cos \beta = \sqrt{1 - y^2}$$

$$\text{Now } \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2} \Rightarrow \alpha + \beta = \sin^{-1} \left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2} \right)$$

$$\text{Thus, } \sin^{-1} x + \sin^{-1} y = \sin^{-1} \left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2} \right)$$

Similarly, we can prove (ii), (iii) and (iv).

Next, we proceed to prove formula (v) as follows:

$$\text{Let } \tan^{-1} x = \alpha \Rightarrow \tan \alpha = x \text{ and } \tan^{-1} y = \beta \Rightarrow \tan \beta = y$$

$$\text{Now } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha + \beta) = \frac{x + y}{1 - xy} \Rightarrow \alpha + \beta = \tan^{-1} \frac{x + y}{1 - xy}$$

$$\text{Thus, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

Further by putting $y = x$ in (v), we get

$$\tan^{-1} x + \tan^{-1} x = \tan^{-1} \frac{x + x}{1 - xx} \Rightarrow 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2}$$

Similarly, we can prove (vi).

In a similar way, the following results can be derived from (i) and (ii)

$$2 \sin^{-1} x = \sin^{-1} \left(2x\sqrt{1 - x^2} \right) \quad \text{and} \quad 2 \cos^{-1} x = \cos^{-1} \left(2x^2 - 1 \right)$$

Example 8 Prove that $\cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}$.

Solution: Let $\alpha = \cos^{-1} x$, then $\cos \alpha = x$

$$\text{Now } \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - x^2}$$

$$\Rightarrow \alpha = \sin^{-1} \left(\sqrt{1 - x^2} \right) \quad \left(\therefore \cos^{-1} x = \sin^{-1} \left(\sqrt{1 - x^2} \right) \right)$$

Example 9 Prove that $\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{12}{13}\right) = \frac{\pi}{2}$.

Solution: Let $x = \frac{5}{13}$ and $y = \frac{12}{13}$

Using the identity $\sin^{-1} x + \sin^{-1} y = \sin^{-1} \left(x\sqrt{1-y^2} + y\sqrt{1-x^2} \right)$, we have

$$\begin{aligned} \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{12}{13} &= \sin^{-1} \left(\frac{5}{13} \sqrt{1 - \left(\frac{12}{13}\right)^2} + \frac{12}{13} \sqrt{1 - \left(\frac{5}{13}\right)^2} \right) \\ &= \sin^{-1} \left(\frac{5}{13} \sqrt{1 - \frac{144}{169}} + \frac{12}{13} \sqrt{1 - \frac{25}{169}} \right) \\ &= \sin^{-1} \left(\frac{5}{13} \sqrt{\frac{25}{169}} + \frac{12}{13} \sqrt{\frac{144}{169}} \right) = \sin^{-1} \left(\frac{5}{13} \cdot \frac{5}{13} + \frac{12}{13} \cdot \frac{12}{13} \right) \\ &= \sin^{-1} \left(\frac{25}{169} + \frac{144}{169} \right) = \sin^{-1} \left(\frac{169}{169} \right) = \sin^{-1}(1) = \frac{\pi}{2} \end{aligned}$$

Therefore, $\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{12}{13}\right) = \frac{\pi}{2}$

Example 10 Prove that $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \frac{\pi}{4}$.

Solution: L.H.S = $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$\begin{aligned} &= \tan^{-1} \left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y} \cdot \frac{x-y}{x+y}} \right) = \tan^{-1} \left(\frac{\frac{x^2 + xy - yx + y^2}{y(x+y)}}{\frac{yx + y^2 + x^2 + xy}{y(x+y)}} \right) \\ &= \tan^{-1} \left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 + xy} \right) = \tan^{-1} \left(\frac{x^2 + y^2}{x^2 + y^2} \right) \\ &= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S} \end{aligned}$$

Thus, $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \frac{\pi}{4}$

Example 11 Solve the equation $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x}{x+1}\right) = \frac{\pi}{4}$.

Solution: Given that $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x}{x+1}\right) = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x}{x+1}}{1 - \frac{x-1}{x+1} \cdot \frac{2x}{x+1}}\right) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{\frac{3x-1}{x+1}}{1 - \frac{2x(x-1)}{(x+1)^2}}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{3x-1}{x+1}}{\frac{-x^2+4x+1}{(x+1)^2}}\right) = \frac{\pi}{4} \Rightarrow \left(\frac{(3x-1)}{(x+1)} \cdot \frac{(x+1)^2}{(-x^2+4x+1)}\right) = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{(3x-1)}{(x+1)} \cdot \frac{(x+1)^2}{(-x^2+4x+1)} = 1$$

$$\Rightarrow 3x^2 + 2x - 1 = -x^2 + 4x + 1 \Rightarrow 2x^2 - x - 1 = 0$$

$$\Rightarrow (x-1)(2x+1) = 0 \Rightarrow x-1 = 0 \text{ or } 2x+1 = 0$$

$$\Rightarrow x = 1 \text{ or } x = -\frac{1}{2}$$

Here, $x = -\frac{1}{2}$ does not satisfy the equation.

Thus, $x = 1$ is the only solution.

EXERCISE 9.2

Prove the following:

- $\sin^{-1} \frac{7}{25} + \sin^{-1} \frac{15}{17} = \cos^{-1} \frac{87}{425}$
- $\cos^{-1} \frac{7}{25} - \cos^{-1} \frac{5}{13} = \cos^{-1} \frac{323}{325}$
- $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19} = \frac{\pi}{4}$
- $\sin^{-1} \frac{12}{13} - \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{33}{65}$
- $\sin^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \sin^{-1} \frac{63}{65}$
- $\sin^{-1} \frac{3}{5} + 2 \tan^{-1} \frac{1}{4} = \cos^{-1} \frac{36}{85}$
- $\sin^{-1} \frac{1}{\sqrt{2}} + \sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{10}} = \frac{\pi}{2}$
- $\tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$

9. $\tan^{-1} \frac{10}{3} - \tan^{-1} \frac{7}{13} + \tan^{-1} \frac{9}{5} - \tan^{-1} \frac{2}{7} = \frac{\pi}{2}$
10. $\sec^2 \left(\tan^{-1} \frac{1}{4} \right) + \csc^2 \left(\cot^{-1} \frac{2}{5} \right) = \frac{889}{400}$
11. $\tan^2 (\sec^{-1} 3) + \cot^2 (\csc^{-1} 4) = 23$
12. $2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right), -1 \leq x \leq 1$
13. $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$
14. $2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$
14. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$
15. $\tan^{-1} x + \cot^{-1} (x+1) = \tan^{-1} (x^2 + x + 1)$
16. $2 \tan^{-1} \left(\tan \frac{\theta}{2} \tan \frac{\phi}{2} \right) = \cos^{-1} \left(\frac{\cos \theta + \cos \phi}{1 + \cos \theta \cos \phi} \right)$
17. $\cot^{-1} \left(\frac{\cos x}{1 - \sin x} \right) = \frac{\pi}{4} - \frac{x}{2}$
18. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then show that $x + y + z = xyz$
19. Find the value of x if, $\tan^{-1} \left(\frac{2x-1}{3x-1} \right) + \tan^{-1} \left(\frac{2x+1}{3x+1} \right) = \frac{\pi}{4}$
20. If $\sin^{-1} \frac{2p}{1+p^2} + \cos^{-1} \frac{1-q^2}{1+q^2} = \tan^{-1} \frac{2y}{1-y^2}$, show that $y = \frac{p+q}{1-pq}$