

A Textbook of

Statistics

For Grade XII

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Khyber Pakhtunkhwa Textbook Board
Peshawar

Statistics

for Grade XII



Khyber Pakhtunkhwa Textbook Board
Peshawar

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(سورۃ الحجرات: ۱۰)

Developed by University Books Printers & Publishers, Peshawar and approved by
The Directorate of Curriculum & Teacher Education (DCTE), Abbottabad vide NOC
No. 5400-5402/F.03/Vol-I/Statistics-XII dated 08-10-2018.

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Academic Year 2019-20
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PROBABILITY

After studying this unit, the students will be able to

- Know $n!$ (n factorial) as the notation to express the product of first n natural numbers.
- Describe fundamental principles of counting and illustrate it using tree diagram.
- Explain the meaning of permutation and interpret the number of permutations of n different objects taken r at a time.
- Explain the meaning of combination and interpret the number of combinations of n different objects taken r at a time.
- Define random experiment, sample space, sample point, event, simple and compound events, impossible and sure events, complementary events, equally likely events, exhaustive events, mutually exclusive events.
- Elaborate the term 'probability' through classical definition, relative frequency definition, Axiomatic definition.
- Recognize the formula for probability of occurrence of an event A
- Apply the formula and using Venn diagrams to find the probability in simple cases for the occurrence of an event.
- Describe probability of non-occurrence of an event, odds for the occurrence and odds against the occurrence for an event.
- Recognize the law of probability of complementation.
- State the laws of probability under addition and apply them to solve real life problems.
- Differentiate between dependent and independent events.
- Define the conditional probability and state the laws of probability under multiplication.
- Apply the laws of probability under multiplication to solve real life problems.
- Compute probabilities for real life problems involving counting techniques and probability trees.

1.1 Counting Techniques

Counting rules help us to know about the number of all possible results of an experiment without actually writing them. A few of the commonly used counting rules to solve the probability problems are:

♦ Factorial

The product of first n natural numbers is called factorial and is denoted by the symbol $n!$ (read n factorial). Thus

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$\vdots$$

$$n! = n (n-1) (n-2) \dots 3 \cdot 2 \cdot 1$$

This can simply be written as

$$n! = n (n-1) (n-2)!$$

$$= n (n-1)!$$

Remember that the quantity $0! = 1$ arbitrarily defined.

♦ Rule of permutation

The groups which can be made from a given number of objects by taking together some or all of the objects at a time with regard to the order of the objects are called permutations. For example, if there are three objects A , B and C , then permutations of size two will be AB , BA , AC , CA , BC , CB . Generally, if we have " n " objects, then total number of permutations of size " r " can be obtained by the formula

$${}_n P_r = \frac{n!}{(n-r)!}$$

Consider the above example, $n=3$ (number of objects), $r=2$ (group size), therefore the number of permutations is equal to
$${}_3P_2 = \frac{3!}{(3-2)!} = \frac{3 \cdot 2 \cdot 1}{1!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$$

Example 1.1

How many different three digit numbers can be formed from the digits 3, 4, 5, 6, 7, 8.

Solution:

Here $n=6$, $r=3$, the number of permutations is given by

$${}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

♦ **Permutations when all objects are taken at a time (without repetition)**

When all of the given objects are considered at a time in the formation of groups i.e. $r=n$ then the total number of permutations $= P = n!$

Example 1.2

How many different permutations can be made from the letters of the word "BOXER"?

Solution:

Here $r=n=5$, therefore, total number of permutations $= 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

♦ **Permutations when all objects are taken at a time (with repetition)**

When all of the given objects are considered at a time in the formation of groups and out of them n_1 are of one kind, n_2 are of second kind... n_k are of the k^{th} kind, then the total number of permutations is given by

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}, \text{ where } n_1 + n_2 + \dots + n_k = n$$

Example 1.3

How many different permutations can be made from the letters of the word "STATISTICS"?

Solution:

Here $n=10$ (total number of letters), $n_1=3$ (number of S's), $n_2=3$ (number of T's), $n_3=2$ (number of I's), $n_4=1$ (number of A's) and $n_5=1$ (number of C's). Therefore, the required number of permutations is given by

$$\binom{n}{n_1, n_2, n_3, n_4, n_5} = \frac{n!}{n_1! n_2! n_3! n_4! n_5!}$$

$$\binom{10}{3, 3, 2, 1, 1} = \frac{10!}{3! 3! 2! 1! 1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1} = 50400$$

♦ **Rule of combination**

The groups which can be made from a given number of objects by taking together some or all of the objects at a time without regard the order of the objects are called combinations. For example, if there are three objects A, B and C then combinations of size two will be AB, AC, and BC. Generally if there are " n " objects then the total number of combinations of size " r " is given by the

$$\text{formula } {}_n C_r = \frac{n!}{(n-r)! r!}$$

By formula the number of combinations for above example is equal to

$${}_3 C_2 = \frac{3!}{(3-2)! 2!} = \frac{3 \cdot 2 \cdot 1}{1! 2 \cdot 1} = 3$$

It can be deduced that:

- $\binom{n}{n} = \binom{n}{0} = 1$
- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

Example 1.4

In how many ways a committee of three men can be selected from seven men?

Solution:

Here $n = 7$, $r = 3$, therefore, the total number of combinations is

$${}^7C_3 = \frac{7!}{(7-3)!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2 \cdot 1} = 35$$

Rule of addition

If there are two mutually exclusive operations having m and n results respectively, then the two operations combined have $(m + n)$ results.

Rule of multiplication

If an experiment has m results and another independent experiment has n results, then the compound experiment has exactly (mn) results.

1.2 Introduction to probability

In everyday life, we face two situations either we may be sure about the occurrence of an event or not sure (uncertain). If sure, then there is no need of probability but if not sure, then probability is used, which is defined as "the numerical evaluation of occurrence of an event is called probability".

Originally, it was known as "science of gamblers" because its foundation was laid by two French mathematicians Pascal and Fermat in connection with gambling problems but nowadays probability theory has wider applications in almost all areas of learning. It is a base for inferential statistics.

For learning the probability technique, it is necessary to understand the following key terms.

1.2.1 Random experiment

The dictionary meaning of random is "unexpected" or "unpredictable", chosen by chance rather than according to a plan.

An experiment whose results cannot be predicted in advance is called random experiment. For example; toss of a coin, roll of a die, etc.

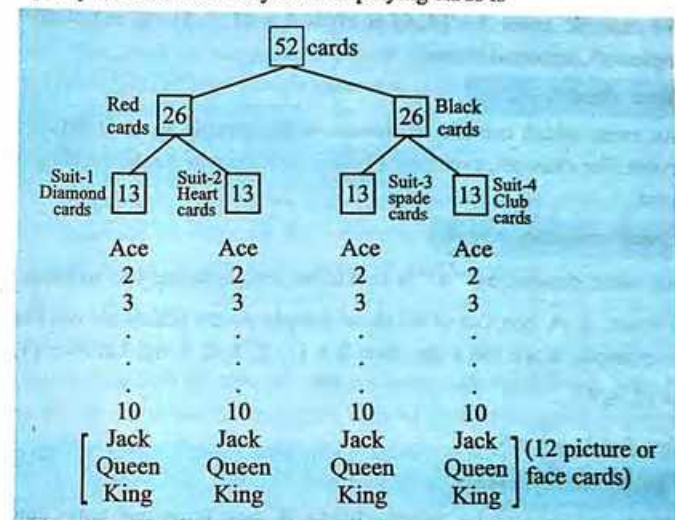
1.2.2 Sample space

The collection of all possible results of a random experiment is called sample space and is denoted by S .

The sample space for tossing a coin is $S = \{H, T\}$.

The sample space for rolling a die is $S = \{1, 2, 3, 4, 5, 6\}$.

The sample space for an ordinary deck of playing cards is

**Definition of outcome**

Each result of the sample space is called an outcome or sample point.

1.2.3 Event

The collection of favourable outcomes to a happening from the sample space is called an event or a subset of the sample space is called an event. It is denoted by $E_1, E_2, E_3, \dots$ or $A, B, C, \dots$

Impossible event

An event which has no favourable outcome is called an impossible or null event and is denoted by ϕ . For example, $A =$ observing "7" when a die is rolled once. Surely " A " is impossible event because the number 7 cannot happen when an ordinary die is rolled once.

Simple event

An event having exactly one sample point is called simple or elementary event. For example, event $A = \{5\}$ when a die is rolled, is a simple event.

Compound event

An event which contains more than one outcome is called compound event. For example, event $A = \{4, 6\}$ or event $A = \{1, 3, 5\}$ etc. in rolling of a die are examples of compound event.

Sure event

An event which contains all results of the sample space is called sure or certain event. For example, event $A = \{1, 2, 3, 4, 5, 6\}$ when a die is rolled once, is a sure event.

Complementary event

An event, denoted by " $\bar{A}$ " is said to be complementary to an event " A " in a sample space, if $\bar{A}$ consists of all those sample points which are not contained in A . For example, if we roll a die, then $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 3, 5\}$, then $\bar{A} = \{2, 4, 6\}$.

Remember that $A \cup \bar{A} = S$.

Equally likely events

Events are said to be equally likely if they have the same chance of occurrence. For example, if we toss a fair coin, then head (H) and tail (T) have same chance of occurrence. So head and tail are equally likely events.

Mutually exclusive events

If two events cannot occur simultaneously then they are called mutually exclusive events. For example, if we toss a coin, H and T cannot occur together so they are mutually exclusive events. Similarly, success and failure, male and female births etc. are mutually exclusive events.

Exhaustive events

Events are called exhaustive if (i) they are mutually exclusive events and (ii) their union makes again the entire sample space.

Dependent events

If the occurrence of an event affects the probability of occurrence of another event, they are called dependent events. Without replacement sampling is an example of dependent events.

Independent events

If the occurrence of an event does not affect the probability of occurrence of any other event, then they are said to be independent events. With replacement sampling is an example of independent events. Similarly, results of two fair coins or examination results of students are independent of each other.

1.2.4 Definitions of probability

The Classical definition of probability

If a sample space has " n " equally likely and mutually exclusive outcomes and if " m " outcomes of them are favourable to the occurrence of an event " A ", then the probability of the event A , denoted by $P(A)$, is given by

$$P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}$$

Since events in practical life may not always be equally likely that is why this definition has the shortcoming that it can only be applied to pattern experiments like tossing of a coin, rolling of a die, drawing of a playing card. This definition was given by Laplace. It is also called mathematical or priori definition of probability.

The Relative frequency definition of probability

This definition uses the relative frequency of the occurrence of an event A , over a very large number of trials, that is,

$$P(A) = \lim_{n \rightarrow \infty} \frac{m}{n}$$

This definition, given by Von Mises, has the shortcoming that the experiment needs to be performed a large number of times which is practically time consuming and expensive. It is also called statistical or posteriori definition of probability because it is calculated after conducting the actual experiment.

Axiomatic definition of probability

Axiom means a rule or principle that many people accept as true. To avoid different shortcoming and compute probability, Russian mathematician Kolmogorov imposed some axioms on the probability of an event given below:

Axiom (I) $0 \leq P(E) \leq 1$, where E is any event.

Note that:

- (i) If $P(E) = 0$, the event E is said to be null event.
 (ii) If $P(E) = 1$, the event E is said to be sure event.

Axiom (II) $P(S) = 1$

Axiom (III) If E_1 and E_2 are mutually exclusive events, then
 $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

1.2.5 Calculation of probability in case of simple events

Example 1.5

If a coin is tossed, what is the chance of a head?

Solution:

$$n(S) = 2^n = 2^1 = 2$$

$$S = \{H, T\}$$

Let A: head occurs

$$A = \{H\}$$

$$n(A) = 1$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{m}{n} = \frac{1}{2} = 0.5$$

It will be more suitable for interpretation to express the answer in percentage.

Example 1.6

A fair coin is tossed twice. What is the probability that exactly one head occurs?

Solution:

$$n(S) = 2^n = 2^2 = 4$$

$$S = \{HH, HT, TH, TT\}$$

Let A: exactly one head

$$A = \{HT, TH\}$$

$$n(A) = 2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2} = 0.5$$

Example 1.7

Three fair coins are tossed once. Find the probability of

- (i) exactly two tails (ii) at least two tails (iii) at most 2 heads.

Solution:

$$n(S) = 2^n = 2^3 = 8$$

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- (i) Let A: exactly two tails

$$A = \{HTT, THT, TTH\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8} = 0.375$$

- (ii) Let B: at least two tails

$$B = \{HTT, THT, TTH, TTT\}$$

$$n(B) = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{8} = \frac{1}{2} = 0.5$$

- (iii) Let C: at most 2 heads

$$C = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$n(C) = 7$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{7}{8} = 0.875$$

Example 1.8

If a die is rolled, what is the probability that the number appearing on top is (i) an odd number (ii) an even number less than 5.

Solution:

$$\text{For die } n(S) = 6^n$$

$$\text{Here } n(S) = 6^1 = 6$$

$$S = \{1, 2, 3, 4, 5, 6\}$$

- (i) Let A: an odd number

$$A = \{1, 3, 5\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = 0.5$$

- (ii) Let B: an even number less than 5

$$B = \{2, 4\}$$

$$n(B) = 2$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

Example 1.9

Two dice are thrown once, what is the probability that (i) the total score is 8 (ii) score is at least 10 (iii) 5 occurs on the second die.

Solution:

$$\text{For two dice } n(S) = 6^2 = 36$$

$$S = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

- (i) Let A: the total score is 8

$$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

$$n(A) = 5$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

- (ii) Let B: score is at least 10

$$B = \{(4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)\}$$

$$n(B) = 6$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

- (iii) Let C: 5 occurs on the second die

$$C = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5)\}$$

$$n(C) = 6$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Example 1.10

From an ordinary pack of 52 cards, one card is drawn at random. Find the chance of drawing (i) a heart (ii) a red card.

Solution:

$$n(S) = {}^n C_r = {}^{52}_1 = \frac{52!}{(52-1)!1!} = \frac{52 \cdot 51!}{51!1!} = 52$$

- (i) Let A: card is heart

$$n(A) = {}^{13}_1 = 13 \text{ (As there are 13 heart cards in an ordinary pack)}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{13}{52} = \frac{1}{4} = 0.25$$

- (ii) Let B: card is red

$$n(B) = {}^{26}_1 = 26 \text{ (As there are 26 red cards in a pack)}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{26}{52} = \frac{1}{2} = 0.5 = 50\%$$

Example 1.11

A bag contains 12 identical balls of which 5 white, 4 red and 3 black balls. Two balls are drawn. What is the probability that (i) both are red (ii) one white ball and one black ball?

Solution:

The bag contains $(5W+4R+3B) = 12$ balls

$$n(S) = {}^{12}_2C = \frac{12!}{(12-2)!2!}$$

$$= \frac{12 \cdot 11 \cdot 10!}{10! \cdot 2 \cdot 1} = 66$$

(i) Let A: 2 balls drawn are red

$$n(A) = {}^4_2C = 6,$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{66} = \frac{1}{11}$$

(ii) Let B: one white ball and one black ball

$$n(B) = {}^5_1C \times {}^3_1C = 5 \times 3 = 15$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{15}{66} = \frac{5}{22}$$

Example 1.12

A retailer wishes to buy two mobile sets from a shop having only 10 Samsung and 5 i-Phone mobiles sets. Find the probability that he will buy (i) two Samsung sets (ii) at least one Samsung set (iii) one Samsung and one i-Phone.

Solution:

There are $(10 \text{ Samsung} + 5 \text{ i-Phone}) = 15$ sets

$$n(S) = {}^{15}_2C = 105$$

(i) Let A: 2 Samsung sets

$$n(A) = {}^{10}_2C = 45,$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{45}{105} = 0.43$$

(ii) Let B: at least one Samsung set

B = 1 Samsung or 2 Samsung sets
= (1 Samsung and 1 i-Phone) or 2 Samsung sets

$$n(B) = \left\{ {}^{10}_1C \times {}^5_1C \right\} + {}^{10}_2C = (10 \times 5) + 45 = 50 + 45 = 95$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{95}{105} = 0.90$$

(iii) Let C: one Samsung and one i-Phone set

$$n(C) = {}^{10}_1C \times {}^5_1C = 10 \times 5 = 50$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{50}{105} = 0.48$$

1.3 Complementation law of probability**Statement:**

If $\bar{A}$ is the complement of an event A in the sample space then
 $P(\bar{A}) = 1 - P(A)$

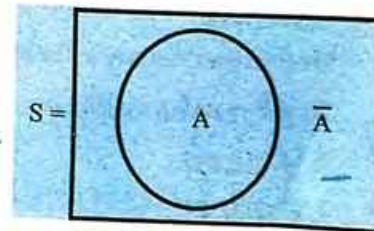
Proof:

As we see from the Venn diagram that event A and $\bar{A}$ are mutually exclusive and exhaustive so $A \cup \bar{A} = S$. Taking probability on both sides

$$P(A \cup \bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1 \text{ (Using axiom II and III)}$$

$$\text{or } P(\bar{A}) = 1 - P(A).$$

**Example 1.13**

Five fair coins were tossed once. What is the probability that at least one head occurs?

Solution:

$$n(S) = 2^n = 2^5 = 32$$

Let A: at least one head.

It will be tedious to list the sample space and pick out the favourable results for at least one head. Alternatively,

$\bar{A}$: No head

$$\bar{A} = (TTTTT)$$

$$n(\bar{A}) = 1,$$

$$P(\bar{A}) = \frac{n(\bar{A})}{n(S)} = \frac{1}{32}$$

Finally, we can obtain the required probability of A by using law of complementation

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{32} = \frac{31}{32} = 0.97$$

Example 1.14

What is the probability that a randomly selected family of four children will have at least one boy?

Solution:

$$n(S) = 2^4 = 16$$

Let E: at least one boy in the family

Then $\bar{E}$ = no boy i.e. all girls

$$\bar{E} = \{g g g g\}$$

$$P(\bar{E}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

The required probability of E by complementation law is given by

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{1}{16} = \frac{15}{16} = 0.94$$

1.3.1 Addition law of probability for mutually exclusive events**Statement:**

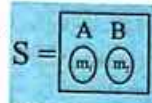
If A and B are two mutually exclusive events, then the probability that any one of them happens is equal to the sum of individual probabilities of A and B. In symbol, $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

Proof:

Let n be the total number of sample points in the sample space. Let m_1 be the favourable cases to the occurrence of an event A, then $P(A) = \frac{m_1}{n}$. Let m_2 be

the favourable cases to B, so $P(B) = \frac{m_2}{n}$. Since A and B are mutually exclusive events, therefore, favourable cases to (A or B) are equal to $m_1 + m_2$.

Hence $P(A \text{ or } B) = P(A \cup B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$

**Example 1.15**

A card is drawn from an ordinary deck of playing cards. What is the probability that the card will be either a king or a jack?

Solution:

Let the event king be denoted by A and the event jack be denoted by B. These are mutually exclusive events as both cannot occur at a time. Thus we use addition law for mutually exclusive events i.e. $P(A \text{ or } B) = P(A) + P(B)$

$$\text{Now } n(S) = \frac{52}{1} = 52$$

Let A: card is king

$$n(A) = \frac{4}{1} = 4$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Let B: card is a jack

$$n(B) = \frac{4}{1} = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{52}$$

$$\begin{aligned}\therefore P(A \text{ or } B) &= P(A) + P(B) \\ &= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}\end{aligned}$$

Example 1.16

A pair of fair dice is rolled once. Find the probability that the sum of the upper dots is either 6 or 9.

Solution:

Let A: Sum of dots is 6 and B: sum of dots is 9. The events are mutually exclusive because they cannot occur together. Hence we use addition rule for mutually exclusive events i.e. $P(A \text{ or } B) = P(A) + P(B)$

$$\text{Now } n(S) = 6^2 = 36$$

$$S = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

$$\therefore A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

$$n(A) = 5,$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

$$B = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$$

$$n(B) = 4,$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{36}$$

$$\therefore P(A \text{ or } B) = P(A \cup B) = \frac{5}{36} + \frac{4}{36} = \frac{9}{36} = \frac{1}{4}$$

1.3.2 Addition law of probability for not-mutually exclusive events**Statement:**

If A and B are two not-mutually exclusive events, then the probability that at least one of the two events A and B occurs is equal to the probability that A occurs plus the probability that B occurs minus the probability that both events A and B occur together. In symbol, $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

From the Venn diagram we see that

$$A \cup B = \{A \cup (\bar{A} \cap B)\}$$

$$P(A \cup B) = P\{A \cup (\bar{A} \cap B)\}$$

$$= P(A) + P(\bar{A} \cap B), \text{ as } A \text{ and } (\bar{A} \cap B)$$

are mutually exclusive.....(i)

Again see the diagram, the set B is given by

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$= P(A \cap B) + P(\bar{A} \cap B), \text{ (as } (A \cap B) \text{ and } (\bar{A} \cap B) \text{ are mutually exclusive events)}$$

or $P(\bar{A} \cap B) = P(B) - P(A \cap B)$, put in equation (i) we get,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

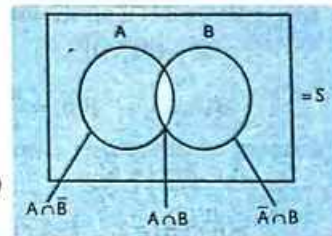
Example 1.17

A fair die is thrown once. What is the chance that either an even number or a number greater than 3 will turn up?

Solution:

Let A: an even number occur and B: a number greater than 3 occurs. The events are not mutually exclusive because A may happen, B may happen or both A and B happen. So we use addition law for not-mutually exclusive events:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$



Now $n(S) = 6^n = 6^1 = 6$

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A : an even number occurs

$$A = \{2, 4, 6\}$$

$$n(A) = 3, \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$

Let B : number is greater than 3

$$B = \{4, 5, 6\}$$

$$n(B) = 3, \quad P(B) = \frac{n(B)}{n(S)} = \frac{3}{6}$$

$$A \cap B = \{4, 6\}$$

$$n(A \cap B) = 2, \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{6}$$

$$\text{Hence } P(A \text{ or } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{3+3-2}{6} = \frac{4}{6} = \frac{2}{3}$$

Example 1.18

Ahsan appeared in the annual examination. The probability that he will pass (i) Mathematics is 0.60 (ii) Statistics is 0.50 and (iii) both Mathematics and Statistics is 0.30. What is the probability that Ahsan will pass either Mathematics or Statistics?

Solution:

Let M : Ahsan will pass Mathematics

S : Ahsan will pass Statistics

Here the events are not mutually exclusive as Ahsan may pass Mathematics or Statistics or both. So we use addition law for not-mutually exclusive events.

Given that $P(M) = 0.60$

$$P(S) = 0.50$$

$$P(M \cap S) = 0.30$$

$$\therefore P(M \text{ or } S) = P(M) + P(S) - P(M \cap S)$$

$$= 0.60 + 0.50 - 0.30 = 0.80$$

1.3.3 Multiplication law for independent events

Statement:

If A and B are two independent events, then the probability of their simultaneous happening is equal to the product of their separate probabilities. Symbolically, $P(A \text{ and } B) = P(A \cap B) = P(A) P(B)$

Proof:

Let " m " and " n " be the favourable and possible outcomes respectively for an event " A ", then $P(A) = \frac{m}{n}$.

Let M and N be the favourable and possible outcomes respectively for an event " B ", then $P(B) = \frac{M}{N}$.

Since A and B are independent events, therefore, mM , nN be the favourable and possible outcomes respectively for event $(A \text{ and } B)$. Thus,

$$P(A \text{ and } B) = P(A \cap B) = \frac{mM}{nN} = \frac{m}{n} \cdot \frac{M}{N} = P(A) P(B)$$

Example 1.19

Suppose that a bag contains 10 balls of which 4 are red balls and 6 are green balls. Find the probability of drawing two green balls in succession if the ball that is drawn first is replaced.

Solution:

Let A : first ball drawn is green, B : second ball drawn is green. The events are independent because the ball drawn first is replaced before the next draw, so probability of both green balls will remain the same.

The bag contains $(4R + 6G) = 10$ balls

$$P(A) = \frac{{}^6C_1}{{}^{10}C_1} = \frac{6}{10}$$

$P(B) = \frac{\frac{6}{10}}{\frac{6}{10}} = \frac{6}{10}$. Therefore required probability of both green balls is

$$P(A \text{ and } B) = P(A) P(B) \\ = \frac{6}{10} \times \frac{6}{10} = \frac{36}{100} = 0.36$$

1.3.4 Conditional Probability

Many times probability of an event is asked that is conditioned on some available information. For example (i) what is the probability that a person selected at random has diabetes given that he has a family history of diabetes. (ii) What is the probability that 3 occurs on a die given that an odd number has occurred. The given information reduces the original sample space by excluding some outcomes as being impossible which before receiving the information were believed possible. This reduced sample space is called conditional sample space and probabilities associated to it are called conditional probabilities.

♦ Definition of conditional probability

The conditional probability of an event A given that another event B has already occurred is denoted by $P(A/B)$ and is defined as;

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Example 1.20

A coin is tossed twice. Find the conditional probability that two tails result, given that there is at least one tail?

Solution:

$$n(S) = 2^n = 2^2 = 4$$

$$S = \{HH, HT, TH, TT\}$$

Let A: 2 tail appear

$$A = \{TT\}$$

Let B: at least one tail appears

$$B = \{HT, TH, TT\}$$

$$n(B) = 3$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{4}$$

$$A \cap B = \{TT\}$$

$$n(A \cap B) = 1$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4}$$

$$\therefore P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

1.3.5 Multiplication law for dependent events

Statement:

If A and B are two dependent events then the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability of B given that A has already occurred. Symbolically,

$$P(A \text{ and } B) = P(A \cap B) = P(A) P(B/A)$$

Proof:

The conditional probability of B given that A has already occurred is

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

Multiplying both sides by $P(A)$, we get

$$P(A) P(B/A) = P(B \cap A)$$

$$\text{or } P(A \cap B) = P(A) P(B/A) \quad (\because P(A \cap B) = P(B \cap A))$$

Example 1.21

Suppose that a bag contains 10 balls of which 3 are white balls and 7 are green balls. If two balls are drawn at random one after another without replacement, find the probability that both balls drawn are green.

Solution:

Let A: first ball is green, B: second ball is green

Since the first ball drawn is not replaced, therefore, the events are dependent and so we use multiplication law of probability for dependent events i.e.

$$P(A \text{ and } B) = P(A) P(B/A)$$

The bag contains $(3W + 7G) = 10$ balls

$$P(A) = \frac{{}^7C_1}{{}^{10}C_1} = \frac{7}{10}$$

Since green ball drawn is not replaced, so new position of the bag is $(3W + 6G) = 9$ balls, so

$$P(B/A) = \frac{{}^6C_1}{{}^9C_1} = \frac{6}{9}$$

$$\text{Hence } P(A \text{ and } B) = P(A \cap B) = P(A) P(B/A) = \frac{7}{10} \times \frac{6}{9} = \frac{42}{90} = \frac{7}{15}$$

Example 1.22

If $P(A) = 0.60$, $P(B) = 0.40$, $P(A \cap B) = 0.24$. (i) What is the relation between A and B? Also find (ii) $P(\bar{A})$ (iii) $P(A \cup B)$ (iv) $P(A/B)$.

Solution:

- (i) Here $P(A) P(B) = (0.60)(0.40) = 0.24 = P(A \cap B)$. Multiplication law for independent events is satisfied. Thus, A and B are independent events.
- (ii) By complementation law

$$P(\bar{A}) = 1 - P(A) = 1 - 0.60 = 0.40$$

$$\begin{aligned} \text{(iii) } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.60 + 0.40 - 0.24 = 0.76 \end{aligned}$$

$$\text{(iii) } P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.24}{0.40} = 0.60 = P(A)$$

(This statement also shows that events A and B are independent).

1.3.6 Odds for the occurrence of an event

The term "odds" refers to the ratio of two probabilities; the ratio of probability of success to the probability of failure. Any kind of probability can be expressed as odds, defined as $d = \frac{p}{1-p}$ where d = odds and p = Probability of success. For example; in throwing a die the probability of 3 is one in six that is $p = \frac{1}{6}$. The odds of getting a 3 are $d = \frac{p}{1-p} = \frac{1/6}{1-1/6} = \frac{1/6}{5/6} = \frac{1}{5}$. That is, odds for

3 are 1 to 5. Odds are used by researchers in different fields.

Example 1.23

- A card is drawn from a deck of 52 cards. Find (i) the probability and (ii) the odds that the card drawn will be (a) a diamond? (b) a red card? (c) a king?

Solution:

$$n(S) = {}^{52}C_1 = 52$$

$$\text{(a) } p = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52}$$

$$d = \frac{p}{1-p} = \frac{13/52}{1-13/52} = \frac{13/52}{39/52} = \frac{13}{39}$$

$$\Rightarrow 13:39 \text{ or } 1:3$$

$$(b) \quad p = \frac{{}^{26}C_1}{{}^{52}C_1} = \frac{26}{52}$$

$$d = \frac{p}{1-p} = \frac{26/52}{1-26/52} = \frac{26/52}{26/52} = 1$$

$$(c) \quad p = \frac{{}^4C_1}{{}^{52}C_1} = \frac{4}{52}$$

$$d = \frac{p}{1-p}$$

$$= \frac{4/52}{1-4/52} = \frac{4/52}{48/52} = \frac{4}{48}$$

$$\Rightarrow 1:12$$

Key points

- Remember that the quantity $0! = 1$ arbitrarily defined and $1! = 1$
- ${}_r^n C = \frac{n!}{(n-r)!r!}$
- An experiment whose results cannot be predicted in advance is called random experiment
- The collection of all possible results of a random experiment is called sample space
- The collection of favourable outcomes to a happening from the sample space is called an event.
- If two events cannot occur together, then they are called mutually exclusive events.
- If the occurrence of an event affects the probability of occurrence of another event they are called dependent events.
- If the occurrence of an event does not affect the probability of occurrence of any other event, they are said to be independent events.
- $P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}$
- $0 \leq P(E) \leq 1$, where E is any event.
- If $\bar{A}$ is the complement of an event A in the sample space then $P(\bar{A}) = 1 - P(A)$
- $P(A \text{ or } B) = P(A) + P(B)$ if A and B are mutually exclusive events.
- $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$, if A and B are not- mutually exclusive events.
- $P(A \text{ and } B) = P(A \cap B) = P(A) P(B)$, if A and B are independent
- $P(A/B) = \frac{P(A \cap B)}{P(B)}$, If A and B are dependent.

Exercise

1.1 Write T for true and F for false statement.

- The range of probability is from zero to one.
- An event which contains only one sample point is called compound event.
- Events which cannot occur at a time are called not mutually exclusive events.
- When a die is rolled four time, the number of sample points in the sample space will be 1296.
- Two events are mutually exclusive if they have no outcomes in common.
- A and B are mutually exclusive events if $P(A \text{ and } B) = P(A) P(B)$
- When two events are independent, the occurrence of one event will not change the probability of the second event.
- The probability of drawing a red card from pack of 52 cards is $\frac{26}{52}$.
- Probability of an event will never be negative.
- The complementary events are always not-mutually exclusive events.

1.2 Fill in the suitable word in the blanks.

- An event which cannot occur is known as _____ event.
- Totality of all possible outcomes of a random experiment is called _____
- An orderly arrangements of r distinct things out of n are called _____
- The limits of probability are from _____
- The need for probability was originally felt in _____
- $P(A \cup B)$ can be expressed by the _____ law of probability.
- If two events A and B are disjoint, the $P(A \cup B) =$ _____
- If $P(A \cap B) = P(A) P(B)$, then events A and B are _____
- If $\bar{A}$ is the compliment of A, the probability of $\bar{A}$ is equal to _____
- The probability of obtaining a total of 7 in single throw of two dice is _____

1.3 Choose the correct answer.

- The probability of an event A lies between
 - 1 and +1
 - 1 and 0
 - 0 and 1
 - +1 and -1

- Probability is expressed as
 - ratio
 - proportion
 - percentage
 - all of the above
- When two events cannot happen simultaneously in a single trial, the events are said to be
 - dependent
 - equally likely
 - mutually Exclusive
 - independent
- An event consisting of those elements, which are not in A is called
 - primary event
 - derived event
 - simple event
 - complementary event
- If A is an event, the conditional probability of A given A is equal to
 - zero
 - one
 - ∞
 - 0.8
- If a coin is tossed three times, then the probability of getting at most one head is equal to
 - $\frac{3}{8}$
 - $\frac{7}{8}$
 - $\frac{1}{2}$
 - $\frac{1}{8}$
- The probability of throwing an even sum with two fair dice is
 - $\frac{1}{4}$
 - $\frac{1}{16}$
 - 1
 - $\frac{1}{2}$
- The probability of six on a fair die is $\frac{1}{6}$. The probability of not six is
 - $\frac{2}{6}$
 - $\frac{5}{6}$
 - $\frac{1}{2}$
 - $\frac{6}{6}$
- $P(A) + P(\bar{A})$ is equal to
 - 0
 - ∞
 - 1
 - 0.5
- A fair die and a fair coin are thrown at a time. The number of combined outcomes is
 - 6
 - 12
 - 2
 - 8

1.4 Write short notes on factorials, permutations and combinations.

1.5 Find the values of (i) ${}^8P_7$, (ii) ${}^{25}P_5$, (iii) ${}^{24}C_4$, (iv) ${}^{19}C_4$, (v) ${}^{20}C_{11}$

1.6 Find all possible (i) permutations and (ii) combinations of 2 letters chosen from the four letters A, B, C, D.

1.7 How many permutations can be made of the letter of the word "TRIANGLE"?
How many of these will begin with T and end with E?

1.8 In how many ways can 5 people be seated on a sofa if there are only three seats available?

1.9 How many permutations can be formed from the letters of the words?

(i) MATHEMATICS (ii) MISSISSIPPIANS (iii) INTERMEDIATE

(iv) EXAMINATION (v) ABBOTTABAD

1.10 Out of 12 books in how many ways can a selection of 5 are made when one specified book is always included.

1.11 Give in brief the concept of probability.

1.12 Explain the following terms.

(i) Random experiment (ii) sample space (iii) outcomes (iv) event (v) impossible event (vi) sure event.

1.13 Differentiate between;

(i) Mutually exclusive events and not mutually exclusive events.

(ii) Independent events and dependent events.

(iii) Probability and conditional probability.

1.14 Define equally likely events, compound events, and exhaustive events.

1.15 Find the probability that an even number appears when a perfect cubical die is rolled.

1.16 When a pair of fair dice is thrown. Find the probability that the sum of 8 appears.

1.17 A fair coin is tossed three times. Find the probability that (i) no head occurs
(ii) exactly two head occurs (iii) at least one head occurs.

1.18 From a pack of 52 cards, two cards are drawn randomly. What is the probability that they are king?

1.19 From a deck of 52 cards you are dealt one face drawn what is the chance that the card will turn out to be (i) a club? (ii) a black card? (iii) an ace?

1.20 Of 10 eggs in a refrigerator, 2 are bad. From these 4 eggs are chosen at random. Find the probabilities that (i) all are good (ii) 2 are bad.

1.21 (a) State and prove addition law of probability for not mutually exclusive events.

(b) Abid can solve 60% problems and Ali can solve 80% problems in a book. A problem is chosen at random from this book. What is the probability that Abid or Ali can solve?

1.22 A pair of fair dice is rolled. Find the probability of a sum of either 7 or 11.

1.23 State and prove multiplication laws of probability for independent and dependent events.

1.24 Two coins are tossed at a time. What is the probability of getting a head on the first coin and a tail on second coin?

1.25 From a well-shuffled deck of 52 playing cards, two cards are drawn randomly. What is the probability that both are queens if the first card is (i) replaced, (ii) not replaced.

1.26 (a) What is conditional probability?

(b) If two balanced dice are rolled. Find the conditional probability that sum of dots will be 7, given that it is odd.

1.27 Suppose that A and B are independent events, with $P(A) = 0.6$ and $P(B) = 0.2$
Find (i) $P(A \text{ and } B)$ (ii) $P(A/B)$ (iii) $P(A \text{ or } B)$.

1.28 A problem of statistics is given to 3 students A, B and C whose chances of solving it are $1/2$, $1/3$ and $1/4$ respectively. What is the probability that the problem will be solved?

1.29 In an interview 3 persons A, B and C attended. The chances of being selected for that post is, for $A = 1/6$, for $B = 1/5$ and for $C = 1/7$.

(i) What is the probability of being selected all the three persons A, B and C.

(ii) What is the probability of being not selected A, B and C respectively.

1.30 Two coins are tossed. What are the probability and the odds that (a) exactly one head occur? (b) Tail occurs on both coins?