

Torque Angular Momentum and Equilibrium

In earlier chapters, we have discussed the linear and angular motions of bodies in detail. In our daily life, we come across various types of motion, for example the motion of a train from one station to an other, the motion of a car along a road, the motion of a ceiling fan, the whirling of a stone which is tied to one end of a string, the other end being held in the hand, etc. The above motion can be divided into two groups (i) translatory motion and (ii) rotatory motion.

The motion of the train and the car belongs to the first group (translatory motion) whereas the motion of the fan and the stone belongs to the second group (rotatory motion).

Even a common man can differentiate these two types of motion according to his ability. As a student of Physics one can define these motion as follows:

Translatory motion:- Consider a frame of reference (x', y', z') which is imagined to be rigidly fixed to an object. For an observer the motion of the object is said to be purely translatory if the axes of the frame of reference of the object remains always parallel to the corresponding axes of observer's frame of reference (x, y, z). In a translatory motion the object may not be necessarily moving along a straight line. Fig.5.1, shows translational motion of an object moving from A to B and C. Observe that throughout the motion every point of the object undergoes same displacement as every other point. We can assume the body to be a particle because in

describing the motion of one point on the body we have described the motion as a whole.

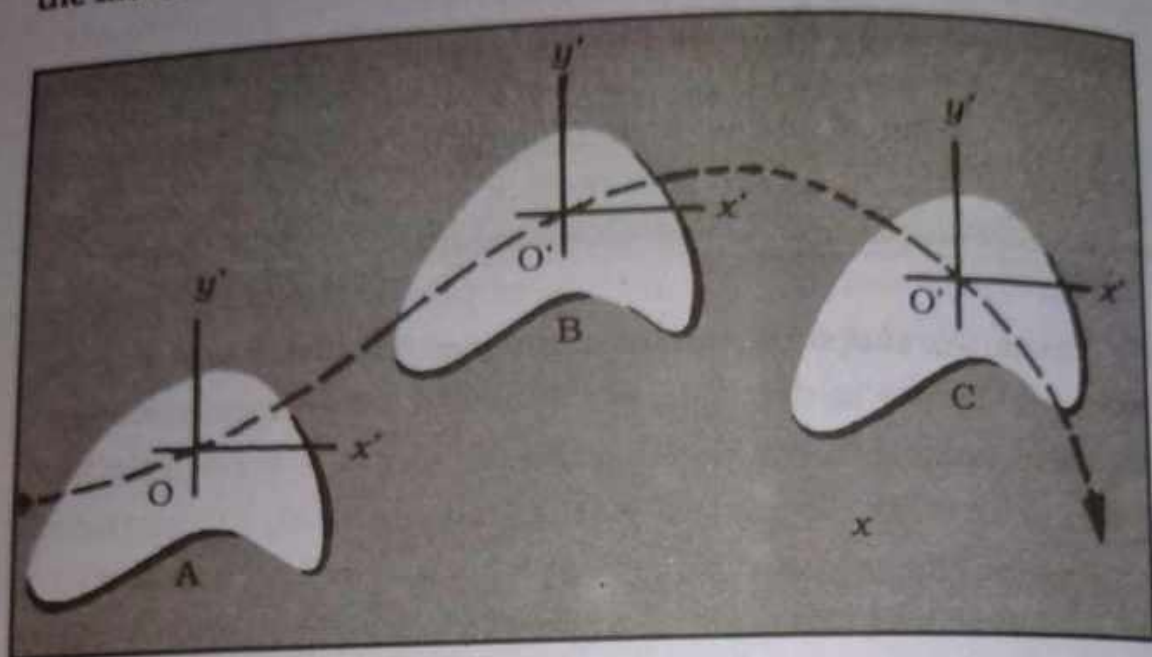


Fig. 5.1

Rotatory motion:- A body (a rigid body) is said to possess a purely rotatory motion if every constituent particle of the body moves in a circle, the centres of which are on a straight line called the axis of rotation. This motion is of two kinds (i) spin motion and (ii) orbital motion.

Before we define and give some examples of spin and orbital motions, we first define the axis of rotation.

By the axis of rotation, we mean the line about which a body rotates. If the line (the axis of rotation) passes through the body itself the corresponding motion is called the spin motion. Every point of the spinning object moves along an arc of a circle in a small interval of time and the centres of all these circles lie along a straight line. This straight line is called the axis of rotation. However, if the axis of rotation does not pass through the body, we call such a rotatory motion as the orbital motion.

Examples of spin and orbital motions:

The daily rotation of the earth about its own axis is an exam-

ple of spin motion. Rotation of fly wheel about its axle is another example of spin motion. A rotating top presents a familiar example of spin motion. If you happen to visit a non-electrified village, you would see that sugar-cane crushing machine is run by a camel or by a bullock that moves in a circular path around the machine. This motion of the camel or the bullock is an orbital motion. The axis of rotation passes through the machine and is perpendicular to the plane of rotation. The motion of planets round the sun is also an orbital motion. In this example the axis of rotation passes through the centre of the sun. The motion of electrons round the nucleus is an example of orbital motion. Here the axis of rotation passes through the centre of the atom.

5.1 TORQUE

Consider a particle of mass 'm' which is acted upon by a force $\vec{F}$. Let $\vec{r}$ be the position vector of the particle. This is also the position vector of the point of application of the force. We can resolve this force into two rectangular components (i) $\vec{F}_{\parallel}$ and (ii) $\vec{F}_{\perp}$. The component $\vec{F}_{\parallel}$ acts in the direction of $\vec{r}$ whereas $\vec{F}_{\perp}$ acts in a direction perpendicular to $\vec{r}$ as shown in the Fig.5.2.

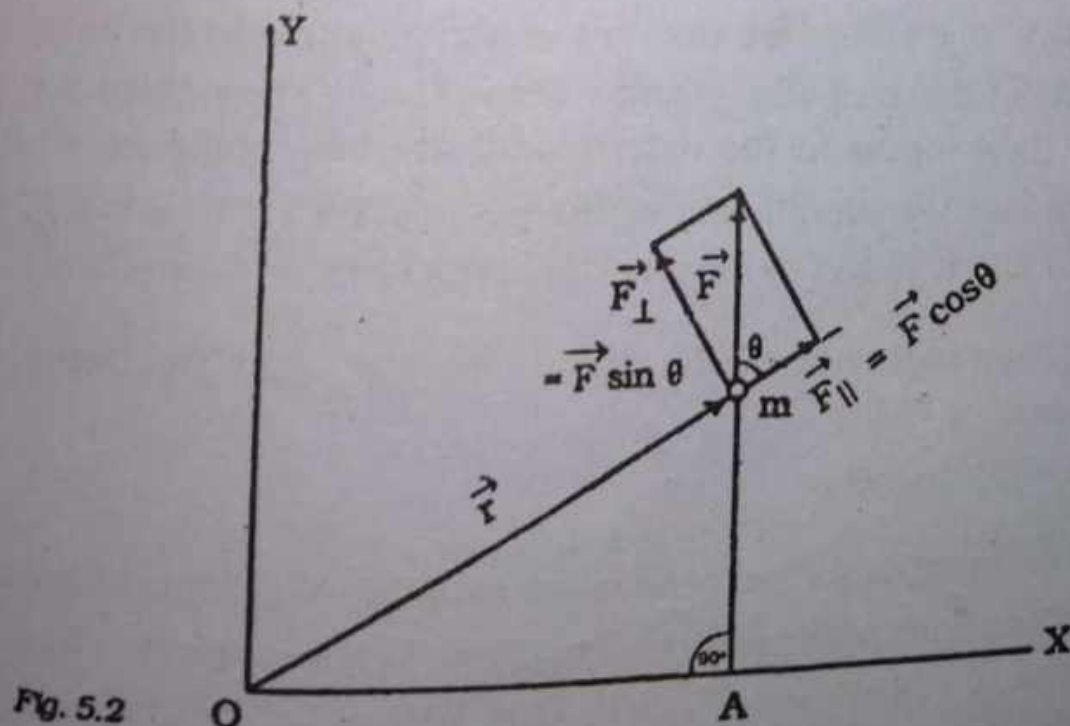


Fig. 5.2

The force $\vec{F}_{\parallel}$ can pull the mass but cannot rotate it. It is the force $\vec{F}_{\perp}$ which produces rotation. Let r and $F_{\perp}$ be the magnitudes of $\vec{r}$ and $\vec{F}_{\perp}$ respectively. The magnitude of the torque vector $\vec{\tau}$ produced by the force $\vec{F}$ about the centre 'O' is defined as

$$\tau = r F_{\perp} = r F \sin \theta \quad (5.1)$$

Where θ is the smaller angle between the positive directions of $\vec{r}$ and $\vec{F}$, i.e. $\theta \leq 2\pi - \theta$

Using vector notation, Eq. 5.1 can be re-written as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (5.2)$$

The torque vector $\vec{\tau}$ which is the vector product of $\vec{r}$ and $\vec{F}$ is directed along the normal to the plane defined by $\vec{r}$ and $\vec{F}$. The direction of $\vec{\tau}$ can also be given by the right hand rule. Right-hand rule:- Point your fingers of right hand towards $\vec{r}$ and curl them from $\vec{r}$ to $\vec{F}$. Then the direction of the thumb will give the direction of $\vec{r} \times \vec{F}$.

We can represent the torque vector $\vec{\tau}$ in the determinant form as shown below :

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad (5.3)$$

Here $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors pointing in the positive directions of the axes of x , y and z respectively. The elements x , y and z as they appear in the determinant are the components of $\vec{r}$. They are also the coordinates of the point mass 'm' whereas F_x , F_y and F_z are the x , y , and z components of the force $\vec{F}$ respectively.

A rearrangement of the factors on the right-hand side of Eq. 5.1 gives

$$\tau = (F) (r \sin \theta)$$

Here ' $r \sin \theta$ ' is the perpendicular distance of the line of action of the force $\vec{F}$ from the centre of rotation. It is represented by the line segment OA in the Fig. 5.1. This perpendicular distance is called the moment arm of the force. Thus the magnitude of the

torque is the ordinary product of the magnitude of the force and its moment arm. i.e.

$$\text{Magnitude of torque} = (\text{magnitude of force}) \times (\text{moment arm})$$

This means the greater the force, the larger will be the torque and larger the moment arm the greater will be the torque. It is, therefore, easier to rotate a body by applying a given force when its line of action is at a greater distance (moment arm) from the centre of the rotation than when it acts closer to the body. In the extreme case when the line of action passes through the centre of rotation, the body stops to rotate because the moment arm is zero. Thus a tangential push on the rim of a wheel will cause a rotation, while a similar push along the axle of the wheel is unable to produce any such rotation. In Fig 5.3 a wheel rotates about an axis passing

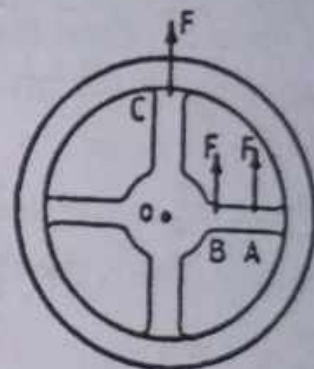


Fig. 5.3 The Turning effect of a force is greater, the farther is the line of action of the force from the axis of rotation.

through its centre 'O'. The turning effect on the wheel has been shown by applying a force at different points on the wheel.

A body can rotate clockwise or counter-clockwise. As a convention, counter-clockwise rotation is taken as positive while a clockwise rotation as negative. Hence a torque which produces a counter-clockwise rotation is considered to be positive and that producing a clockwise rotation is taken as negative.

Two forces which are equal in magnitude but opposite in direction and not acting along the same line constitute a couple as shown in Fig.5.4. The forces constituting the couple are represented by $\vec{F}$ and $-\vec{F}$.

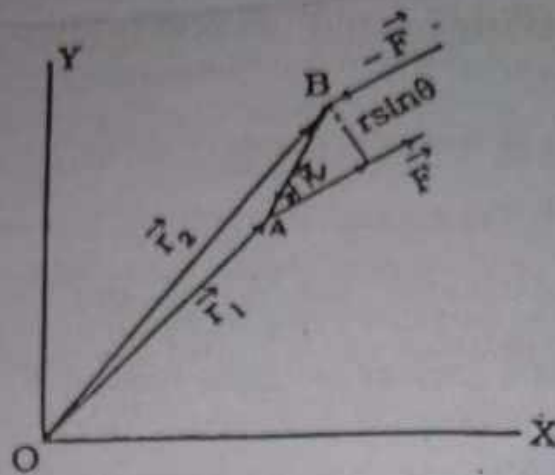


Fig: 5.4 Two forces equal in magnitude but opposite in direction and not acting along the same line constitute a couple.

Consider a couple composed of two forces $\vec{F}$ and $-\vec{F}$ acting at the points 'A' and 'B' respectively as shown in Fig. 5.4. The moment of force $\vec{F}$ about the origin 'O' is

$$\vec{r}_1 \times \vec{F}$$

where $\vec{r}_1$ is the position vector of the point A. The moment of the force $-\vec{F}$ about the same point is

$$\vec{r}_2 \times (-\vec{F})$$

Where $\vec{r}_2$ is the position vector of the point B. The total moment of the two forces is given as

$$\vec{\tau} = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F})$$

$$= \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$

$$= \vec{r} \times \vec{F}$$

5.4

Where $\vec{r}$ is the displacement vector from B to A. The vector $\vec{\tau}$ is called the moment of the couple ($\vec{F} - \vec{F}$). Since $\vec{\tau}$ is a vector product of two vectors $\vec{r}$ and $\vec{F}$, its direction is along the normal

to the plane containing $\vec{r}$ and $\vec{F}$ and is given by the right-hand rule. The magnitude of the vector $\vec{\tau}$ by definition

$$\tau = (r)(F) \sin(180^\circ - \theta)$$

$$\tau = r F \sin \theta$$

(5.5)

Where θ is the angle between $\vec{r}$ and $-\vec{F}$. It is evident from Fig 5.4 that $r \sin \theta$ is the perpendicular distance between the lines of action of the forces $\vec{F}$ and $-\vec{F}$. Let us denote this distance by 'd' ($d = r \sin \theta$). The Eq. 5.5 takes the form.

$$\tau = (F)(d) = Fd$$

(5.6)

The perpendicular distance 'd' is called the moment arm of the couple. Consequently,

Magnitude of the moment of a couple =

Magnitude of any of the forces forming the couple x

Area of the couple.

As $\vec{r}$ is the displacement vector from $-\vec{F}$ to $\vec{F}$, it is independent of the location of origin. Hence the moment of a given couple is independent of the location of origin.

5.2 CENTRE OF MASS

In translational motion each point on a body undergoes the same displacement as any other point as time goes on, so that motion of one particle represents the motion of the whole body. But even when a body rotates or vibrates as it moves, there is one point on the body, called the centre of mass, that moves in the same way that a single particle would move under the influence of the same external forces.

The centre of mass of a body or a system of particles is defined to be a point which moves as if the total mass of the body or the system of particles were concentrated there and all the applied forces were acting at that point. This means that we can describe the motion of the whole system or the body by the motion of

their centres of mass. For example, consider a rectangular block of wood lying on a smooth horizontal surface. Let the block be acted upon by a number of forces. In order to describe the motion of the block as a whole we assume that these forces were acting at the centre of mass which is the geometrical centre of the block and where the total mass is supposed to be concentrated. We then, find the resultant of these forces and apply Newton's second law of motion to determine the acceleration and hence the velocity of the centre of mass at any instant of time by using initial condition of motion. The motion of the block is same as the motion of the centre of mass.

The centre of mass is often confused with the centre of gravity. The two terms are so similar in many respects that one can use the two terms interchangeably. The centre of gravity of an extended object coincides with its centre of mass if the object is in a completely uniform gravitational field. If this is not the case the centre of gravity does not coincide with the centre of mass.

Let x_c , y_c and z_c be the coordinates of the centre of mass, then in a completely uniform gravitational field, these are given by

$$x_c = \frac{\sum m_i x_i}{\sum m_i}$$

$$y_c = \frac{\sum m_i y_i}{\sum m_i}$$

$$z_c = \frac{\sum m_i z_i}{\sum m_i}$$

Where x_i , y_i , z_i are the coordinates of the particle of mass m_i .

Example 5.1

Locate the centre of mass of four particles which are placed at the four corners of a square of sides $2m$ each as shown in Fig.5.5.

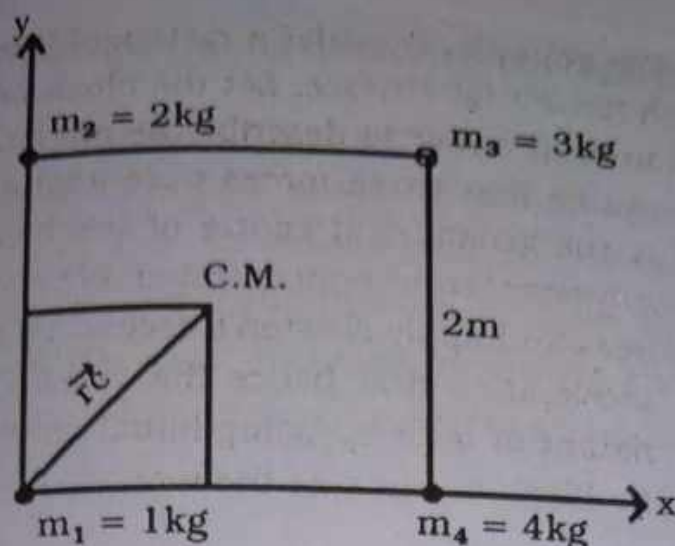


Fig. 5.5

Solution: This is a two dimensional problem. For the sake of simplicity we take the two adjacent sides of the square in the direction of the coordinate axes. Let x_c and y_c be the x and y coordinates of the centre of mass then

$$x_c = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{1 \times 0 + 2 \times 0 + 3 \times 2 + 4 \times 2}{1 + 2 + 3 + 4}$$

$$= \frac{14}{10} = 1.4$$

$$y_c = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{1 \times 0 + 2 \times 2 + 3 \times 2 + 4 \times 0}{1 + 2 + 3 + 4}$$

$$= \frac{10}{10} = 1$$

Hence the coordinates of the centre of mass are the point (1.4, 1.0)

5.3 EQUILIBRIUM AND CONDITIONS OF EQUILIBRIUM

A body is said to be in equilibrium if it is at rest or is moving with a uniform velocity. A body at rest is said to be in static equilibrium, while a body in uniform motion along a straight line is said to be in dynamic equilibrium. In both the cases the body does not possess any acceleration neither linear nor angular. Hence all the bodies in equilibrium do not possess any acceleration.

(A) Static equilibrium

A book lying on a horizontal table is in static equilibrium. Building and bridges are also in static equilibrium. There are many more examples of static equilibrium around us.

(B) Dynamic equilibrium

Consider a vertically downward motion of a small steel ball through a viscous liquid like mustard oil contained in a vertical tube which is clamped with a stand. The ball is dropped gently into the oil. Initially the ball has accelerated motion. But after covering a certain distance it attains a uniform vertical velocity. This uniform vertical motion of the steel ball is an example of dynamic equilibrium. In this example the force of gravity acting vertically downward on the ball is counter balanced by the viscous force acting vertically upward on it. Hence a net external force acting on the ball is zero. As a result of this the ball acquires uniform vertical motion.

Another example of dynamic equilibrium can be presented by considering the jumping of a paratrooper from an helicopter. In this case the force of gravity acts on the paratrooper in a vertically downward direction whereas the reaction of air on the parachute acts in the vertically upward direction. These two forces balance each other as shown in Fig. 5.6. Thus the net force acting on

the system (the parachute and the paratrooper) is zero. Hence the system moves downward with a uniform velocity.



Fig. 5.6 After a parachute opens and falls a certain distance, it moves downward thereafter with a uniform velocity.

5.4 FIRST CONDITION OF EQUILIBRIUM

The condition states that a body will be in equilibrium if the resultant of all the forces acting on it is zero. This condition is referred to as the first condition of equilibrium.

Let $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ be the n external forces acting on a body. The first condition of equilibrium states that

$$\vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = 0 \quad 5.7(a)$$

Using the summation sign, the above Eq 5.7 can be written as

$$\sum_{i=1}^n \vec{F}_i = 0 \quad 5.7(b)$$

If we restrict to forces in one plane, say, the x - y plane, then we write

$$\vec{F}_1 = F_{1x} \hat{i} + F_{1y} \hat{j}$$

Where F_{1x} is x - component of the force $\vec{F}_1$;

F_{1y} is y - component of the force $\vec{F}_1$;

and $\hat{i}, \hat{j}$ are the unit vectors in the direction of x and y axes respectively. Thus the Eq. 5.7 can be written as

$$(F_{1x} \hat{i} + F_{1y} \hat{j}) + (F_{2x} \hat{i} + F_{2y} \hat{j}) + \dots + (F_{nx} \hat{i} + F_{ny} \hat{j}) = 0$$

or

$$(F_{1x} + F_{2x} + \dots + F_{nx}) \hat{i} + (F_{1y} + F_{2y} + \dots + F_{ny}) \hat{j} = 0$$

Let $\vec{F}$ be the resultant of the forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$

$$\therefore \vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$

or

$$F_x \hat{i} + F_y \hat{j} = (F_{1x} + F_{2x} + \dots + F_{nx}) \hat{i} + (F_{1y} + F_{2y} + \dots + F_{ny}) \hat{j}$$

Equating the x and y components of the forces on both the sides of the above equation we get

$$F_x = F_{1x} + F_{2x} + \dots + F_{nx}$$

and

$$F_y = F_{1y} + F_{2y} + \dots + F_{ny}$$

$$\text{Since } \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = 0$$

$$\text{So } \vec{F} = 0$$

Since $\vec{F}$ is zero.

$$F_x = 0 \text{ and } F_y = 0$$

$$\text{or } F_{1x} + F_{2x} + \dots + F_{nx} = 0, F_{1y} + F_{2y} + \dots + F_{ny} = 0$$

Using the summation convention

$$\left. \begin{aligned} \sum_{i=1}^n F_{ix} &= 0 \\ \text{and} \quad \sum_{i=1}^n F_{iy} &= 0 \end{aligned} \right\}$$

5.8(a)

For the sake of simplicity and convenience we omit the index 'i' from the summation sign in the above equation and write the condition of equilibrium in the form

$$\left. \begin{aligned} \Sigma F_x &= 0 \\ \text{and} \quad \Sigma F_y &= 0 \end{aligned} \right\}$$

5.8(b)

One important point to be noted here is that all the component forces should be written with their proper sign.

Let $\theta_1, \theta_2, \dots, \theta_n$ be the angles which the forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ make with x-axis respectively as shown in Fig. 5.7

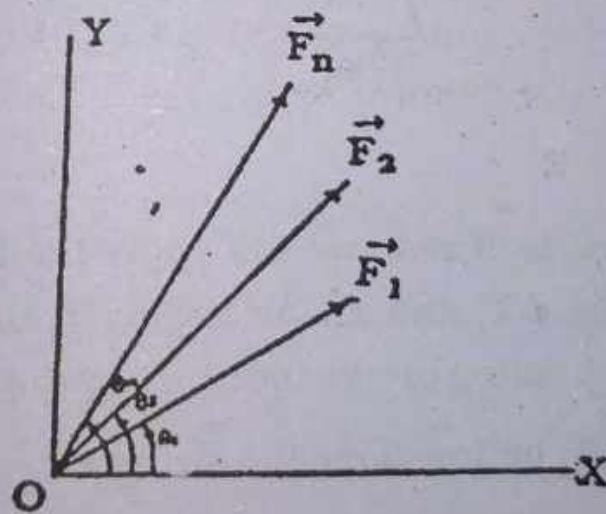


Fig: 5.7 Vectors $\vec{F}_1, \vec{F}_2$ and $\vec{F}_n$ are drawn in x, y - Plane

Then we have

$$F_{1x} = F_1 \cos \theta_1, F_{2x} = F_2 \cos \theta_2, \dots, F_{nx} = F_n \cos \theta_n$$

and

$$F_{1y} = F_1 \sin \theta_1, F_{2y} = F_2 \sin \theta_2, \dots, F_{ny} = F_n \sin \theta_n$$

The first condition of equilibrium is then written as

$$\left. \begin{aligned} \Sigma F_x &= \sum_{i=1}^n F_{ix} = \sum_{j=1}^n F_j \cos \theta_j = 0 \\ \text{and} \quad \Sigma F_y &= \sum_{i=1}^n F_{iy} = \sum_{j=1}^n F_j \sin \theta_j = 0 \end{aligned} \right\} \quad 5.8(c)$$

Example 5.2:

A block of mass 20 kg is suspended as shown in Fig. 5.8. Find the tensions T_1 and T_2 in the two strings. Take the value of $g = 9.8 \text{ m/s}^2$.

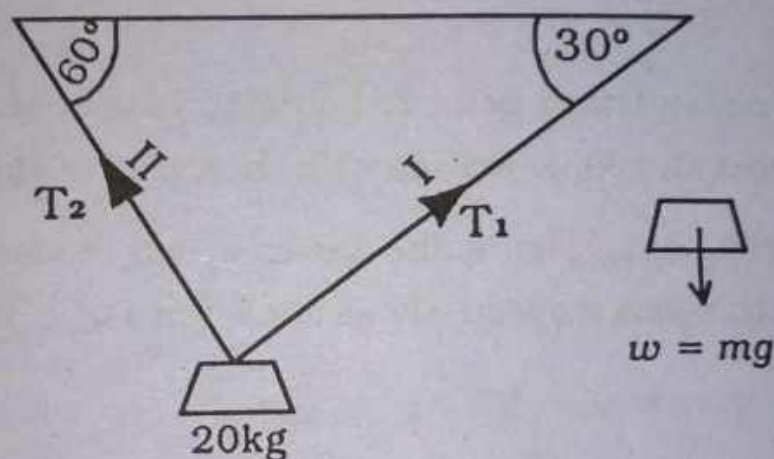


Fig: 5.8

Solution:

Since the block is at rest, we will apply the first condition of equilibrium to evaluate T_1 and T_2 . As shown in the force diagram there are three forces acting on the block. They are

- (i) the tension T_1 acting along the string I.
- (ii) the tension T_2 acting along the string II
- (iii) the weight W of the block acting vertically downward.

We resolve these forces into their respective x and y components. Applying first condition of equilibrium for the x-components of the forces we have

$$\Sigma F_x = 0$$

Since the weight of the block "W" has no component along the horizontal direction, the above summation reduces to

$$T_1 \cos 30^\circ - T_2 \cos 60^\circ = 0$$

$$T_1 \times \frac{\sqrt{3}}{2} - T_2 \times \frac{1}{2} = 0$$

$$T_2 = \sqrt{3} T_1$$

The first condition of equilibrium for the y-components of the forces i.e.

$$\Sigma F_y = 0$$

$\therefore$

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - w = 0$$

$$T_1 \times \frac{1}{2} + T_2 \times \frac{\sqrt{3}}{2} - w = 0$$

$$T_1 + \sqrt{3} T_2 = 2w$$

$$T_1 + \sqrt{3} \times \sqrt{3} T_1 = 2w$$

$$4T_1 = 2w$$

$$T_1 = \frac{w}{2} = \frac{20(\text{kg}) \times 9.8(\text{m/s}^2)}{2} = 98\text{N}$$

solving for T_2 , we get

$$T_2 = 98\sqrt{3}\text{N} = 169.74\text{N}$$

5.5 SECOND CONDITION OF EQUILIBRIUM

Earlier we have studied that whenever, a net torque acts on a body, rotation is produced in it. The body is said to be in rotational equilibrium, if the vector sum of the torques acting on it is zero. Then if $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3, \dots, \vec{\tau}_n$ are the torques on the body, then

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 + \dots + \vec{\tau}_n = 0$$

$$\sum_{i=1}^n \vec{\tau}_i = 0$$

Where $\vec{r}_i$ is the moment of the i th force. For convenience we omit the subscript from the summation sign. Thus we have

$$\sum \vec{r}_i = 0 \quad 5.9(a)$$

Thus the two conditions of equilibrium written together are

$$(i) \quad \sum \vec{F} = 0, \quad \sum F_x = 0, \quad \sum F_y = 0$$

$$(ii) \quad \sum \vec{r} = 0, \text{ about any axis} \quad 5.9(b)$$

Example 5.3

A 15m ladder weighing 600N rests against a smooth wall at a point 12m above the ground. The centre of gravity of the ladder is one third the way up. A man weighing 400N climbs half way up the ladder. Assuming that the wall is smooth, find the reaction of the ground and the wall.

Solution:

By the term smooth we mean that there is no friction and that the reaction of the wall on the ladder is perpendicular to the wall. Here we isolate the ladder as the body under consideration. The forces acting on the ladder are shown in Fig. 5.9 (b). The weight W of the ladder acts vertically downward at the point C so that $m \overline{AC} = \frac{1}{3} m \overline{AB}$. The weight W of the man acts vertically downward at the mid point D of the ladder and F the reaction force of the wall acts perpendicular to the wall. The unknown force of reaction of the ground on the ladder is represented in terms of its horizontal and vertical component H and V respectively. From the geometry of the Fig. 5.9. we get

$$m \overline{AE} = 9m, \quad m \overline{Ak} = 3m \text{ and } m \overline{AJ} = 4.5m$$

Applying the first condition of equilibrium to the ladder we get

$$\sum F_x = 0$$

$$\therefore H - F = 0$$

$$H = F$$

$$\sum F_y = 0$$

$$5.10(a)$$

$$5.10(b)$$

$$\begin{aligned}\therefore V - w - W &= 0 \\ V - 600 - 400 &= 0 \\ V &= 1000 \text{ N}\end{aligned}$$

To write the torque equation we choose an axis passing through the point A where the unknown force (with unknown direction) of reaction acts. By doing so the unknown force will not appear in this equation.

Now taking the moment about the point A we get

$$\Sigma \tau = 0$$

$$F \times (\overline{mAG}) - wx(\overline{mAK}) - Wx(\overline{mAJ}) = 0$$

$$\text{or } F \times 12 - 600 \times 3 - 400 \times 4.5 = 0$$

$$12F - 1800 - 1800 = 0$$

$$12F = 3600$$

$$F = 300 \text{ N}$$

$$\text{Since } H = F$$

$$\therefore H = 300 \text{ N}$$

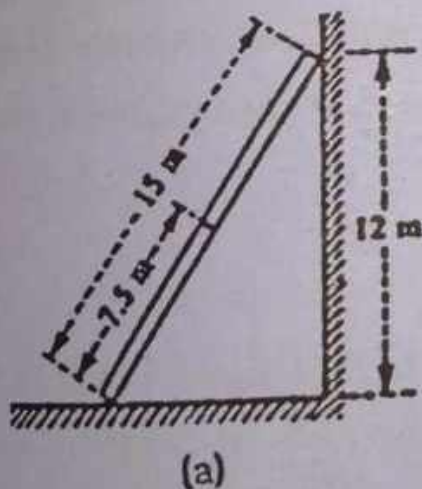
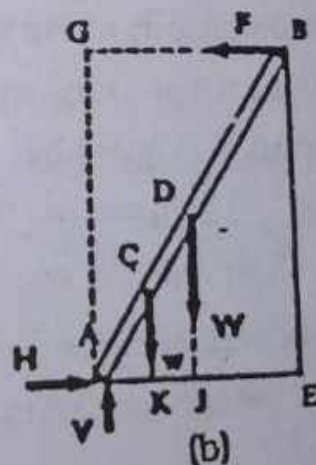


Fig 5.9



The magnitude of the force of reaction of the ground is R.

$$\begin{aligned}R &= \sqrt{H^2 + V^2} \\ &= \sqrt{(300)^2 + (1000)^2} \\ &= 1044 \text{ N}\end{aligned}$$

The direction of the reaction force may be found by evaluating the angle which it makes with the horizontal. Let θ be such an angle, as shown in Fig. 5.9 (a)

$$\tan \theta = \frac{V}{H} = \frac{1000}{300} = \frac{10}{3} = 3.33$$

$$\theta = 73.3^\circ$$

Example 5.4

A cylinder of weight $\vec{W}$ and radius R is to be raised on to a step of height 'h' as shown in Fig. 5.10. A rope is wrapped around the cylinder and pulled horizontally. Assuming that the cylinder does not slip on the step, find the minimum force $\vec{F}$ necessary to raise the cylinder and the reaction force at the point P.

Solution:

When the cylinder is just ready to be raised, the reaction force at Q goes to zero. Hence only three forces are acting on the cylinder as shown in Fig. 5.10(b). From the dotted triangle drawn in Fig. 5.10(a), we see that the moment arm d of the weight $\vec{W}$ relative to the point P is given by

$$d = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

The moment arm of $\vec{F}$ relative to the same point is $2R-h$. Hence the net torque acting on the cylinder about the point P is

$$Wd - F(2R-h).$$

Since the system is at rest, we apply the second condition of equilibrium to get

$$\Sigma \tau = 0$$

$$\text{or } Wd - F(2R-h) = 0$$

$$F = \frac{Wd}{2R-h} = \frac{W \sqrt{2Rh-h^2}}{2R-h}$$

$$= W \sqrt{\frac{h}{2R-h}}$$

Thus the second condition of equilibrium is sufficient to obtain the magnitude of $\vec{F}$. We can determine the reaction force $\vec{N}$ at P by using the first condition of equilibrium. Thus we have

$$\Sigma F_x = 0$$

$$\text{or } F - N \cos \theta = 0$$

$$F = N \cos \theta$$

$$\text{and } \Sigma F_y = 0$$

$$N \sin \theta - W = 0$$

$$N \sin \theta = W$$

From these equations we can easily get

$$(i) \tan \theta = \frac{W}{F} \text{ and } (ii) N = |\vec{N}| = \sqrt{F^2 + W^2}$$

As an example if we take $W = 500\text{N}$, $h = 0.3\text{m}$ and $R = 0.8\text{m}$, then

$$F = 385\text{ N}, \theta = 52.4^\circ \text{ and } |\vec{N}| = 631\text{ N}$$

This problem can also be solved by noting that since only three forces act on the body, they must be concurrent. Hence the reaction force $\vec{N}$ must pass through the point C where $\vec{F}$ and $\vec{W}$ intersect. The three forces form the sides of the triangle as shown in fig. 5.10(c). Expressions (i) and (ii) are obtained directly from this.

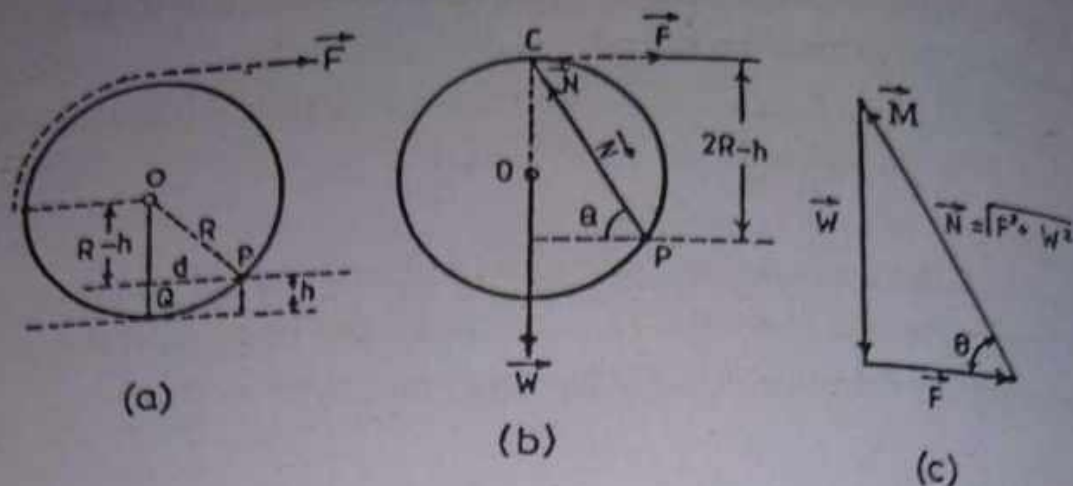


Fig. 5.10. (a) A cylinder of weight W pulled by a force F over a step.
 (b) The free body diagram of the cylinder when it is just ready to be raised.
 (c) The vector sum of the three forces is zero.

5.6 LOCATION OF AXIS

We will now show that if a body is in translational equilibrium and the net torque is zero about a given point, it must be zero at any other point.

In order to prove the above statement, we consider a simple example of a uniform bar of length ' l ' resting on two pegs at its ends as shown in the Fig.5.11. The bar is in equilibrium under the action of three forces that is the weight of the bar $\vec{W}$ and the forces $\vec{F}_1$ and $\vec{F}_2$ exerted by the pegs in the upward direction as shown in the figure.

Applying the first condition of equilibrium we get

$$\Sigma F_x = 0 \quad (\text{No forces acting along horizontal})$$

$$\Sigma F_y = 0$$

$$F_1 + F_2 - W = 0$$

To write the torque equation we must choose an axis. Let this axis pass through the point ' O ' the centre of the bar. Since the bar is at rest we apply the second condition of equilibrium, taking

the moment about the point O. Thus we have

$$\Sigma \tau = 0$$

$$F_1 \times \frac{l}{2} - F_2 \times \frac{l}{2} + W \times 0 = 0$$

$$F_1 - F_2 = 0$$

Now we take the moment about an axis passing through a point P lying to the right of 'O' at a distance x from it.

Now the sum of the moments of all the three forces is

$$\Sigma \tau = F_1 \left(\frac{l}{2} - x \right) - F_2 \left(\frac{l}{2} + x \right) + W \times x$$

$$= (F_1 - F_2) \frac{l}{2} - (F_1 - F_2 - W) x$$

$$\text{since } F_1 - F_2 = 0 \text{ and } F_1 + F_2 - W = 0$$

$$\Sigma \tau = 0$$

which proves the statement. We are thus independent in choosing the axis about which the moment is to be taken. This is an important result because this provides a suitable choice for the location of the axis. By this we mean that one should try to select such an axis for which

(i) The unknown forces, if any, may be inactive while taking the moment. This is possible only if the moment arms of these forces are zero.

(ii) The forces involved may be minimum in number so as to ease the calculation work.

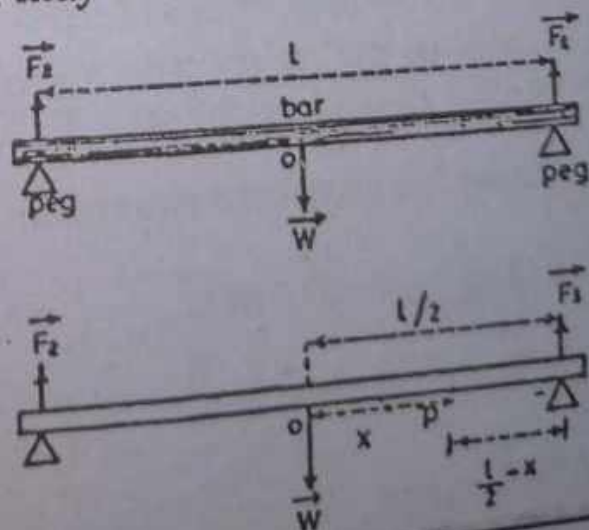


Fig. 5.11 Location of an axis.

5.7 ANGULAR MOMENTUM

A body having translatory motion possesses linear velocity and linear momentum. In the same way a body having rotatory motion possesses angular velocity and angular momentum. Thus the angular momentum is associated with angular (rotatory) motion. This means, in rotatory motion angular momentum plays a role which is completely analogous to that played by linear momentum in translatory motion. Angular momentum, like linear momentum obeys the law of conservation.

In order to have a tangible concept of angular momentum, we first discuss the angular momentum of a particle. Consider a particle of mass 'm'. Let $\vec{r}$ be its position vector and $\vec{p}$ be the line-

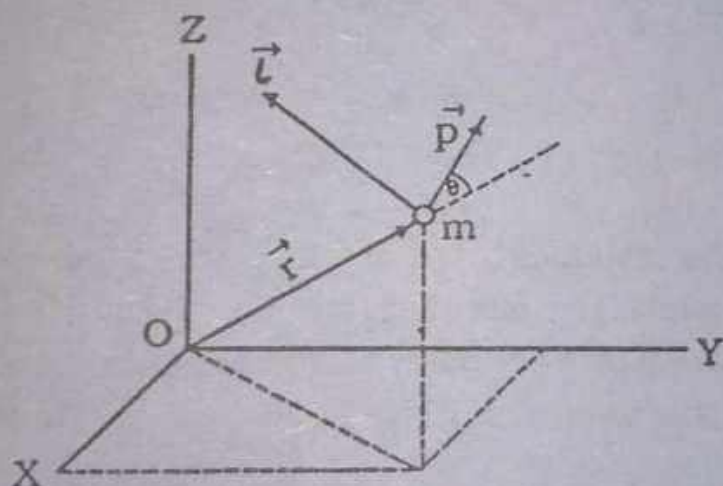


Fig. 5.12 Illustrating the definition of the angular momentum $\vec{l}$ of a particle about the origin O . With $\vec{l} = \vec{r} \times \vec{p}$ we see that the vector $\vec{l}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

ar momentum measured in an inertial frame of reference with origin 'O' as shown in Fig. 5.12. The angular momentum of the particle about the origin 'O' is defined as the vector product of $\vec{r}$ and $\vec{p}$. Thus if $\vec{l}$ stands for angular momentum, then

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m \vec{v} \quad (5.11)$$

$$\text{where} \quad = m \vec{r} \times \vec{v}$$

$$\vec{p} = m \vec{v}$$

and

$\vec{v}$ represents the velocity of the particle.

Since the vector product of two vectors is itself a vector, so the angular momentum is a vector. Its direction lies along the normal to the plane formed by the vectors $\vec{r}$ and $\vec{p}$ as given by the right-hand rule. The magnitude of the angular momentum is given by

$$|\vec{l}| = l = (r) (p) \sin\theta$$

where r and p represent magnitude of $\vec{r}$ and $\vec{p}$ respectively

$$l = m\vec{v} r \sin\theta \quad 5.12(a)$$

Where θ is the angle between $\vec{r}$ and $\vec{p}$. In circular motion $\vec{r}$ and $\vec{p}$ are perpendicular to each other, so the measure of θ is 90° and $\sin 90^\circ = 1$

Thus for circular motion we have

$$|\vec{l}| = l = (r) (p) (1) = rp \quad 5.12(b)$$

If x, y, z are the components of $\vec{r}$ and p_x, p_y, p_z are the components of $\vec{p}$, then using the definition of vector product we write

$$\begin{aligned} \vec{l} &= \vec{r} \times \vec{p} \\ &= (xi + yj + zk) \times (p_x i + p_y j + p_z k) \\ &= (y p_z - z p_y) i + (z p_x - x p_z) j + (x p_y - y p_x) k \end{aligned} \quad 5.13$$

The above expression for $\vec{l}$ can be written in the determinant form as

$$\vec{l} = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

The scalar components of angular momentum $\vec{l}$ are

$$l_x = y p_z - z p_y$$

$$l_y = z p_x - x p_z$$

$$l_z = x p_y - y p_x$$

The dimensions of angular momentum are given below

$$[l] = [r] [p]$$

$$= [r] [m] [v]$$

$$= L \cdot M \cdot \frac{L}{T}$$

$$= L^2 M T^{-1}$$

In the system international (SI), the unit of length is the metre (m), the unit of mass is the kilogram (kg) and the unit of time is the second (s). Hence the units of angular momentum are

$$\frac{(m^2) (kg)}{s} = \frac{(kg) (m)}{s^2} \times (m) (s)$$

$$= (N) (m) (s)$$

$$= Js$$

Thus the units of angular momentum in the S I system are Joule-second (Js).

Example 5.5

A particle of mass 400 gram rotates in a circular orbit of radius 20 cm at a constant rate of 1.5 revolutions per second. Evaluate the angular momentum of the particle with respect to the centre of the orbit.

Solution:

As is clear from Fig. 5.13, the angular momentum $\vec{l}$ which is normal to the plane of the orbit is parallel to the angular velocity

$\vec{\omega}$ which lies along the axis of rotation. Using Eq. (5.12) the magnitude of the angular momentum $\vec{l}$ is

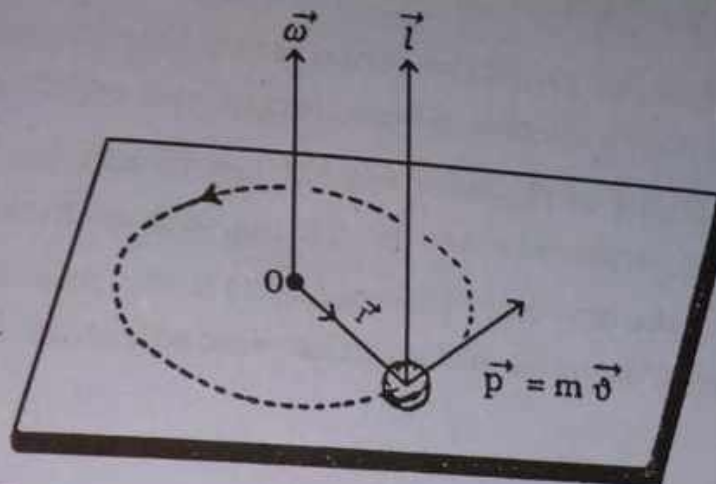


Fig: 5.13

$$l = m v r \sin\theta$$

$$= m r^2 \omega \sin\theta$$

Where

$$v = r \omega$$

$$m = 400 \text{ gm} = 0.4 \text{ kg}$$

$$r = 20 \text{ cm} = 0.2 \text{ metre}$$

$$\theta = 90^\circ$$

$$l = 0.4 \times (0.2)^2 \times \omega \sin 90^\circ$$

$$= 0.4 \times 0.04 \times \omega \times 1$$

$$= 0.016 \omega$$

$$\text{But } \omega = 2\pi f$$

Where

$$f = \text{no of revolutions/sec} = 1.5 \text{ rev /sec}$$

$$l = 0.016 \times 2\pi \times 1.5$$

$$= 0.151 \text{ J.S}$$

5.9 CONSERVATION OF ANGULAR MOMENTUM OF A PARTICLE

In order to derive the conservation law for angular momentum, we obtain a relation between torque and angular momentum.

According to Newton's second law of motion, the net force acting on a particle of mass 'm' moving with an instantaneous velocity $\vec{v}$ is the time rate of change of its linear momentum. Thus if $\vec{F}$ stands for force and $\vec{p}$ for the linear momentum, then

$$\vec{F} = \frac{d}{dt} (\vec{p}) \quad 5.14$$

Taking the vector product of both the sides of the above equation with $\vec{r}$ from the left we get

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \quad 5.15$$

But by definition $\vec{r} \times \vec{F}$ is the torque acting on the particle.

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Also by definition, the angular momentum is

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating the above equation with respect to time, we get

$$\begin{aligned} \frac{d\vec{l}}{dt} &= \frac{d}{dt} (\vec{r} \times \vec{p}) \\ &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \end{aligned}$$

where

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\text{and } \vec{p} = m \vec{v}$$

$$\frac{d\vec{l}}{dt} = m \vec{v} \times \vec{v} + \vec{\tau}$$

$$= 0 + \vec{\tau}$$

Since the vector product of a vector with itself is zero
(i.e. $\vec{v} \times \vec{v} = 0$)

$$\therefore \vec{\tau} = \frac{d\vec{l}}{dt} \quad 5.16$$

This is the required relation. This equation states that the torque acting on a particle is the time rate of change of its angular momentum.

If the net external torque acting on the particle is zero, then Eq. 5.16, reduces to,

$$\frac{d\vec{l}}{dt} = 0 \quad 5.17$$

$$\vec{l} = \text{constant} \quad 5.18$$

Thus the angular momentum of a particle is conserved (constant) if the net torque acting on it is zero.

The conservation law obtained for single particle Eq.(5.18) can be extended to demonstrate conservation law for the angular momentum of a system of particles.

Consider a system of n particles which is acted upon by external as well as internal forces. We assume that the internal forces obey the law of action and reaction. Hence they cancel out as a whole and the system is purely under the action of external (ap-

plied) forces. The total angular momentum of the system is given as:

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_n \quad 5.19(a)$$

$$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \dots + \vec{r}_n \times \vec{p}_n \quad 5.19(b)$$

Taking the time derivative of both the sides of the above equation, we get

$$\therefore \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}_1 \times \vec{p}_1) + \frac{d}{dt}(\vec{r}_2 \times \vec{p}_2) + \frac{d}{dt}(\vec{r}_n \times \vec{p}_n)$$

$$= \vec{r}_1 \times \frac{d\vec{p}_1}{dt} + \vec{r}_2 \times \frac{d\vec{p}_2}{dt} + \dots + \vec{r}_n \times \frac{d\vec{p}_n}{dt}$$

$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \dots + \vec{r}_n \times \vec{F}_n$$

$$= \vec{\tau}_1 + \vec{\tau}_2 + \dots + \vec{\tau}_n \quad 5.20(a)$$

$$= \sum_{i=1}^n \vec{\tau}_i = \vec{\tau} \quad 5.20(b)$$

Where $\vec{F}_i$ and $\vec{\tau}_i$ are the external force and the external torque respectively acting on the i th particle.

If the net external torque $\vec{\tau}$ acting on the system is zero, then the total angular momentum $\vec{L}$ (Eq. 5.19) is given by

$$\frac{d\vec{L}}{dt} = 0 \quad 5.21$$

$$\therefore \vec{L} = \text{constant} \quad 5.22$$

Thus the total angular momentum of a system of particles is conserved (constant) if the net external (applied) torque acting on the system is zero.

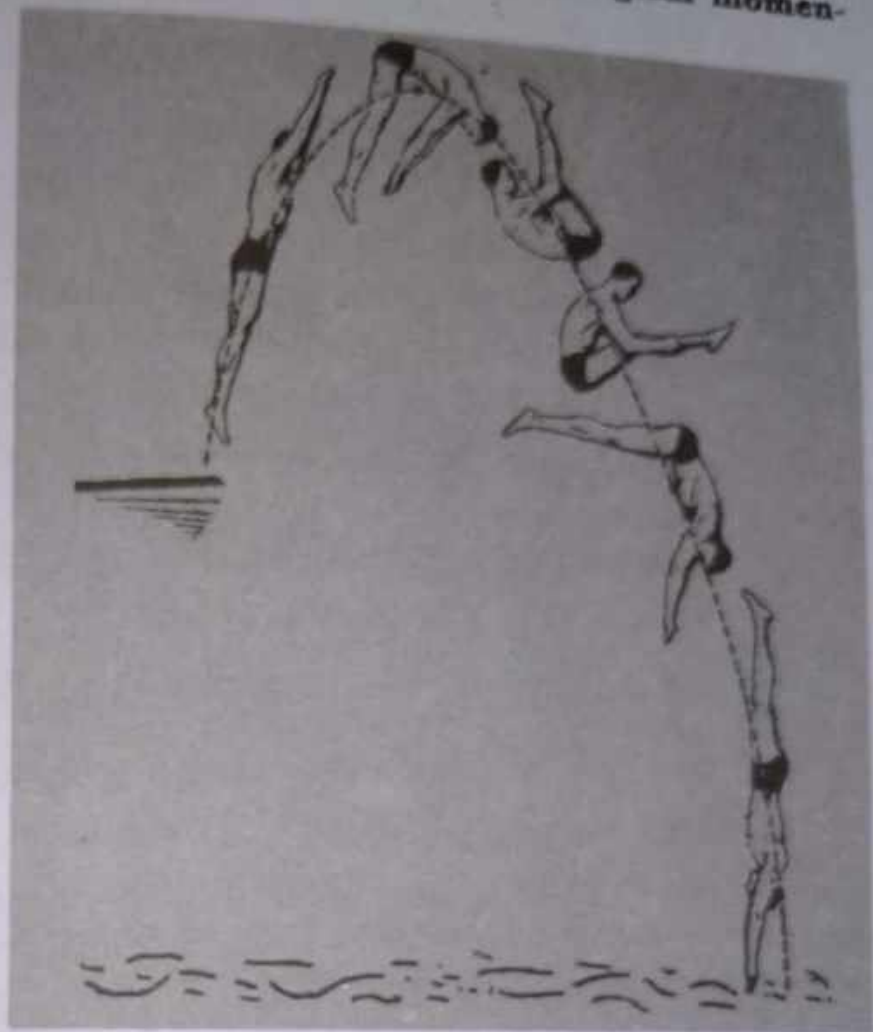


Fig. 5.14

(i) You might have seen a figure skater under going a spin motion in the finale of an act. We assume that there is no friction between the skater and the ice and hence there are no external torques acting on the skater. Thus the angular momentum $I \omega$ of the skater which is the product of moment of inertia, I , of the skater and the angular velocity, ω , of the skater, is constant, since the moment of inertia depends on the distribution of mass, the skater can decrease his or her moment of inertia by pulling his or her hands and feet close to the body. As a result the angular velocity of the skater increases and so the spinning takes place at a rapid rate.

(ii) When divers and acrobats who wish to make several somer saults, they pull their hands and feet close to their bodies in

order to rotate at a higher rate. Due to the close distribution of mass the moment of inertia decreases. This causes an increase in the angular velocity enabling them to make several somer saults.

In these examples the external force due to gravity acts through the centre of gravity and hence exerts no external torque about this point. Therefore, the angular momentum about the centre of gravity must be constant. Thus when divers and figure skaters wish to double their angular velocity of their spin motion, they must reduce their moment of inertia to the half its initial value by pulling their hands and feet close to their bodies.

Objective questions

1.

A) Fill in the blanks with appropriate words:

- (i) The physical quantity which tends to rotate a body is called
- (ii) In the system international, the units of torque are
- (iii) Conventionally the torque, producing clock wise rotation is taken as ...
- (iv) The angular momentum of a body is conserved if the net ... on it is zero.
- (v) If the net torque on a body is zero, it is said to be in ... equilibrium.
- (vi) Two forces which cannot be replaced by a single equivalent force are said to form a
- (vii) is defined as the time rate of change of angular momentum.
- (viii) In the system international the units of angular momentum are
- (ix) The angular momentum of an isolated system is ...
- (x) A body is said to be in equilibrium if the net force on it is zero.

(B) Write "True" or "False" against each statement.

- (i) Torque is a vector product of two appropriate vectors.
- (ii) Angular momentum is a scalar product of two appropriate vectors.
- (iii) The magnitude and direction of angular momentum depend on the choice of origin.
- (iv) A particle moving along a line that passes through the origin has zero angular momentum about that origin.
- (v) The angular momentum of a body which is under the action of central forces is constant.
- (vi) The direction of angular velocity is taken to be the same as that of angular momentum.
- (vii) The torque due to action - reaction force is not zero.
- (viii) Angular acceleration is in the same direction as the angular velocity if the latter is decreasing with time.
- (ix) An electron of an atom possesses only orbital angular momentum.
- (x) The orbital angular momentum of a body is its angular momentum about an axis passing through the body itself.
- (xi) The centre of mass of a body coincides with its centre of gravity irrespective of its mass and size.

(C) Tick (✓) the correct answer.

- (i) Torque is defined as
 - (a) time rate of change of angular momentum.
 - (b) time rate of change of linear momentum.
 - (c) time rate of change of angular velocity.
- (ii) The vector quantity torque
 - (a) depends on the choice of origin.
 - (b) does not depend on the choice of origin.
 - (c) rotates a body always in clockwise.

- (iii) Every point of a rotating rigid body has
 - (a) the same angular velocity.
 - (b) the same linear velocity.
 - (c) the same linear acceleration.
- (iv) The right hand rule is applied to find
 - (a) the direction of a vector obtained by the vector product of two vectors.
 - (b) the magnitude of a vector obtained in the above manner.
 - (c) neither the direction nor the magnitude
- (v) Two forces which form a couple
 - (a) can be replaced by a single equivalent force.
 - (b) cannot be replaced by a single equivalent force.
 - (c) are perpendicular to each other.
- (vi) The direction of torque is
 - (a) the same as the direction of the corresponding applied force.
 - (b) opposite to the direction of the applied force.
 - (c) perpendicular to the direction of the applied force.
- (vii) The centre of mass of a system of particles
 - (a) coincides always with the centre of gravity.
 - (b) never coincides with the centre of gravity.
 - (c) coincides with the centre of gravity only in a uniform gravitational field.

Questions

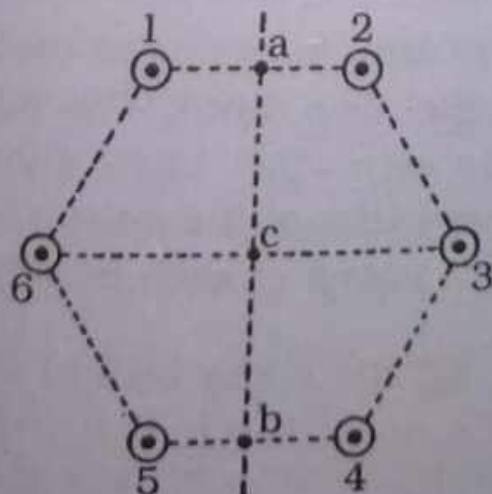
1. Is it possible to calculate the torque acting on rigid body without specifying the axis ?
- (2) Is torque independent of location of axis ?
- (3) In the expression for torque $\vec{\tau} = \vec{r} \times \vec{F}$, is $\vec{r}$ equal to the moment arm? Explain.
- (4) If a particle moves in a straight line, is its angular momentum zero with respect to an arbitrary axis ? Is its

angular momentum zero with respect to any specified origin? Explain.

- (5) If a torque on a particle about an arbitrary axis is zero, what can you say about its angular momentum, about that origin.
- (6) If the linear velocity of a particle is constant in time, can its angular momentum be constant in time about an arbitrary axis?
- (7) A particle moves in a straight line and you are told that the torque acting on it is zero about some specified origin. Does this necessarily mean that the net force acting on the particle is zero. Can you conclude that its linear velocity is constant? Explain
- (8) Suppose that the velocity of a particle is completely specified. What can you say about the direction of its angular momentum vector with respect to the direction of motion?
- (9) Why is it easier to keep your balance on a moving bicycle than on a bicycle at rest?
- (10) A projectile is fired into air and suddenly explodes into several fragments. What can be said about the motion of the fragments after explosion?

Problems:

1. Locate the centre of mass of a system of particles each of mass ' m ', arranged to correspond in position to the six corner of a regular (planar) hexagon.



(Ans) Centre of the hexagon

2. Find the position of centre of mass of five equal-mass particles located at the five corners of a square-based right pyramid with sides of length ' l ' and altitude ' h '.
(Ans $\frac{1}{5}h$ from base along the line perpendicular to the base and passing through peak).
3. The mass of the sun is 329.390 times the earth's mass and the mean distance from the centre of the sun to the centre of the earth is 1.496×10^8 km. Treating the earth and sun as particles with each mass concentrated at the respective geometric centre, how far from the centre of the sun is the C.M (centre of mass) of the earth-sun system? Compare this distance with the mean radius of the sun (6.9960×10^5 km)
(Ans. 4.54×10^2 km; 6.48×10^{-4}).
4. A particle with mass 4 kg moves along the x-axis with a velocity $v = 15t$ m/sec, where $t = 0$ is the instant that the particle is at the origin.
(a) At $t = 2$ sec, what is the angular momentum of the particle about a point P located on the + y-axis, 6m from the origin? (b) What torque about P acts on the particle?
(Ans. (a) $720 \text{ kg m}^2/\text{sec}$; (b) 360 N.m).
5. A particle of mass ' m ' is located at the vector position $\vec{r}$ and has a linear momentum vector $\vec{p}$. The vectors $\vec{r}$ and $\vec{p}$ are non zero. If the particle moves only in the x, y plane, prove that
$$L_x = L_y = 0 \text{ and } L_z \neq 0$$
6. A light rigid rod 1 m in length rotates in the xy-plane about a pivot through the rod's centre. Two particles of mass 2kg and 3kg are connected to its ends. Determine the angular momentum of the system about the origin at the instant the speed of each particle is 5m/sec.
(Ans: $12.5 \text{ kg m}^2 / \text{sec}$ out of the plane)

7. A uniform beam of mass 'M' supports two masses m_1 and m_2 . If the knife edge of the support is under the beam's centre of gravity and m_1 is at a distance 'd' from the centre, determine the position of m_2 such that the system is balanced.
(Ans. $m_1/m_2 d$)
8. A uniform ladder of length l and weight $W=50\text{N}$ rests against a smooth vertical wall. If the coefficient of friction between the ladder and the ground is 0.40, find the minimum angle $\theta_{\min}$ such that the ladder may not slip.
(Ans. $\theta_{\min} = 51.3^\circ$)
9. A ladder with a uniform density and a mass 'm' rests against a frictionless vertical wall at an angle of 60° . The lower end rests on a flat surface where the coefficient of friction (static) is 0.40. A student with a mass ($M = 2m$) attempts to climb the ladder. What fraction of the length 'L' of the ladder will the student have reached when the ladder begins to slip?
(Ans. 0.789)
10. A particle of mass 0.3 kg moves in the xy-plane. At the instant its coordinates are (2,4) m, its velocity is $(3\mathbf{i} + 4\mathbf{j})$ m/sec. At this instant determine the angular momentum of the particle relative to the origin.
(Ans. $-1.2\mathbf{k} \text{ J.S}$)
11. A uniform horizontal beam of length 8 m and weighing 200N is pivoted at the wall with its far end supported by a cable that makes an angle of 53° with the horizontal. If a person weighing 600N stands 2m from the wall, find the tension and the reaction force at the pivot.

$$\text{Ans. } \left\{ \begin{array}{l} T = \text{tension} = 313\text{N} \\ R = \text{reaction force} = 581\text{N} \\ \theta = \text{angle made by the reaction force with the horizontal} \\ \quad = 71.1^\circ \end{array} \right\}$$