

Based on National Curriculum of Pakistan 2022-23

Model Textbook of
Mathematics
Science Group
Grade
11

National Curriculum Council
Ministry of Federal Education and Professional Training



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A Model Textbook of Mathematics for Grade 11
based on National Curriculum of Pakistan(NCP) 2022-23

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Note

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Preface

This Model Textbook for Mathematics Grade 11 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building the foundation of learning from the previous grades. A key emphasis of the present textbook is creating real life linkage of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they grow up in the learning curve and also to fully grasp the conceptual basis that will be built in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its textbooks. The present textbook features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement, the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this textbook.

May Allah guide and help us (Ameen).

Dr. Kamran Jahangir
Managing Director

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ
اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

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COMPLEX NUMBERS

After studying this unit, students will be able to:

- Recall complex number z represented by an expression of the form $z = a + ib$ or of the form (a, b) where a and b are real numbers and $i = \sqrt{-1}$
- Recognize a as a real part of z and b as an imaginary part of z .
- Know the condition for equality of complex numbers.
- Carry out basic operations on complex numbers
- Define $z = a - ib$ as the complex conjugate of $z = a + ib$.
- Define $z = \sqrt{a^2 + b^2}$ as the absolute value or modulus of a complex number $z = a + ib$.
- Solve the simultaneous linear equations with complex coefficients. For example,

$$\begin{cases} 5z - (3 + i)w = 7 - i \\ (2 - i)z + 2iw = -1 + i \end{cases}$$
- Write the polynomial $P(z)$ as a product of linear factors. For example,
 $z^2 + a^2 = (z - ia)(z + ia)$ and $z^3 - 3z^2 + z + 5 = (z + 1)(z - 2 - i)(z - 2 + i)$
- Solve quadratic equation of the form $pz^2 + qz + r = 0$ by completing squares where p, q, r are real numbers and z is a complex number. For example solve
 $z^2 - 2z + 5 = 0 \Rightarrow (z - 1 - 2i)(z - 1 + 2i) = 0 \Rightarrow z = 1 + 2i, 1 - 2i$
- Explain the polar coordinates system
- Describe the polar representation of a complex number.
- Apply the operations with complex numbers in polar representation
- Demonstrate simple equations and in-equations involving complex numbers in polar form
- Apply concepts of complex numbers to real world problems (such as cryptography, wave phenomena, calculate voltage, current, circuits, the velocity and pressure of the fluid).

Complex numbers are used in many branches of science; especially quantum mechanics (a branch of Physics) heavily depends upon complex numbers.

In mathematics the need of complex numbers is to solve the polynomials which do not have the solution in the set of real numbers. e.g., The polynomial $x^2 - 1 = 0$ has the solutions $x = \pm 1$, which are the real numbers. But the polynomial $x^2 + 1 = 0$ do not have any solution in the set of real numbers, since there is no real number, whose square is -1 . To overcome this difficulty, we extended the set of real numbers to the set of complex numbers by introducing a number i such that $i^2 = -1$ or $i = \sqrt{-1}$. Remember that $i^2 = -1$ is the Euler's notation for the imaginary unit number.

1.1 Complex Number

A complex number is an expression of the form $x + iy$ where $x, y \in \mathbb{R}$. A complex number is denoted by z , i.e., $z = x + iy$ and the set of all complex numbers is denoted by $\mathbb{C}$. The complex number $x + iy$ is also denoted by the ordered pair (x, y) . The reason for this notation is justified since there is one to one corresponding between $x + iy$ and (x, y) .

Clearly $i = 0 + i = (0, 1)$ and $1 = 1 + 0i = (1, 0)$

1.1.1 Real and Imaginary Parts of a Complex Number

Every complex number $z = x + iy$ has two parts x and y . x is called the real part and y is called the imaginary part i.e., $Re(z) = x$ and $Im(z) = y$.

If the real part of a complex number is zero then it is called pure imaginary number and if the imaginary part of the complex number is zero then it is called real number.

Since every real number x can be written as $x + i0$ thus every real number is a complex number but note that every complex number need not be a real number. Only the complex numbers with zero imaginary part are real numbers. Thus, the set of real number is a subset of set of complex numbers, i.e., $\mathbb{R} \subset \mathbb{C}$.

Example 1:

Let $z = 2 + 3i$ where z is a complex number having 2 as real part and 3 as the imaginary part.

1.1.2 Condition for the Equality of Two Complex Numbers

Like real numbers any two complex numbers are not comparable. i.e., We cannot say that one complex number is greater than or less than the other complex number. Two complex numbers are said to be equal if both has same real and imaginary parts.

Example 2:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers they will be equal i.e.; $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$

1.2 Basic Algebraic Operations on Complex Numbers

1.2.1 Addition of Two Complex Numbers

Suppose we have two complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$. Then their sum is:

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) = x_1 + iy_1 + x_2 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2) = (x_1 + x_2, y_1 + y_2) \end{aligned}$$

Example 3: Find the sum of $z_1 = 2 + 3i$ and $z_2 = 6 + 8i$.

Solution:

$$\begin{aligned} z_1 + z_2 &= (2 + 3i) + (6 + 8i) = (2 + 6) + (3 + 8)i \\ &= 8 + 11i = (8, 11) \end{aligned}$$

1.2.2 Subtraction of Two Complex Numbers

Suppose we have two complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$.

The difference of the two complex numbers is given by:

$$\begin{aligned} z_1 - z_2 &= (x_1, y_1) - (x_2, y_2) = (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) = (x_1 - x_2, y_1 - y_2) \end{aligned}$$

Example 4: If $z_1 = 4 - 3i$ and $z_2 = 7 + 6i$, then find $z_1 - z_2$.

Solution:

$$\begin{aligned} z_1 - z_2 &= (4 - 3i) - (7 + 6i) = (4 - 7) + (-3 - 6)i \\ &= -3 + (-9i) = -3 - 9i \end{aligned}$$

1.2.3 Product of Two Complex Numbers

If $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ are any two complex numbers, then their product is given as:

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) = (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2 = x_1 x_2 + i(x_1 y_2 + x_2 y_1) - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$

Example 5: Find the product of the complex numbers $z_1 = (2, -6)$ and $z_2 = (4, 9)$

Solution:

$$\begin{aligned} z_1 z_2 &= (2, -6)(4, 9) = (2 - 6i)(4 + 9i) \\ &= 8 + 18i - 24i - 54i^2 = 8 - 6i - (-54) \\ &= 8 + 54 - 6i = 62 - 6i = (62, -6) \end{aligned}$$

1.2.4 Division of Complex Numbers

The division of the two complex numbers is not simple. Since the number in the denominator has two independent parts. To make the denominator a single term we rationalize (multiply and divide) the given complex number by the conjugate of the denominator. After rationalization the denominator will be converted into a single real number and thus division can be done easily.

If $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$, are any two complex numbers, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right) \times \left(\frac{x_2 - iy_2}{x_2 - iy_2} \right) = \frac{x_1 x_2 - ix_1 y_2 + ix_2 y_1 - i^2 y_1 y_2}{x_2^2 - ix_2 y_2 + ix_2 y_2 - i^2 y_2^2} \\ &= \frac{x_1 x_2 + i(x_2 y_1 - x_1 y_2) + y_1 y_2}{x_2^2 + y_2^2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \end{aligned}$$

Example 6: If $z_1 = 3 + 7i$ and $z_2 = -4 + 6i$, then find the sum, difference, product and quotient of the two complex numbers.

Solution:

$$\begin{aligned} z_1 + z_2 &= (3 + 7i) + (-4 + 6i) = (3 - 4) + (7 + 6)i = -1 + 13i \\ z_1 - z_2 &= (3 + 7i) - (-4 + 6i) = 3 + 7i + 4 - 6i = (3 + 4) + (7 - 6)i = 7 + i \\ z_1 z_2 &= (3 + 7i)(-4 + 6i) = 3(-4) + 3(6i) + (7i)(-4) + (7i)(6i) \\ &= -12 + 18i - 28i + 42i^2 = -12 - 10i - 42 \end{aligned}$$

$$= -54 - 10i$$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{3+7i}{-4+6i} = \left(\frac{3+7i}{-4+6i}\right) \left(\frac{-4-6i}{-4-6i}\right) = \frac{-12-18i-28i-42i^2}{(-4)^2-(6i)^2} \\ &= \frac{-12-46i+42}{16-36i^2} = \frac{30-46i}{16+36} = \frac{30-46i}{52} \\ &= \frac{30}{52} - \frac{46}{52}i = \frac{15}{26} - \frac{23}{26}i\end{aligned}$$

Example 7: Write the complex number $\frac{(2+3i)(2+i)}{1-i}$ in the form $x + iy$.

$$\begin{aligned}\text{Solution: } \frac{(2+3i)(2+i)}{1-i} &= \frac{4+2i+6i+3i^2}{1-i} = \frac{4+8i-3}{1-i} \\ &= \frac{1+8i}{1-i} = \frac{1+8i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+8i+8i^2}{1^2-i^2} \\ &= \frac{1+9i-8}{1-(-1)} = \frac{-7+9i}{2} = \frac{-7}{2} + \frac{9i}{2}\end{aligned}$$

Example 8: Find the values of x and y if, $\frac{x}{2+3i} - y(1+2i) = 1+i$

Solution:

$$\begin{aligned}\frac{x}{2+3i} - y(1+2i) &= 1+i \\ \Rightarrow \frac{x(2-3i)}{(2+3i)(2-3i)} - y(1+2i) &= 1+i \\ \Rightarrow \frac{2x-3xi}{(2)^2-(3i)^2} - y(1+2i) &= 1+i \\ \Rightarrow \frac{2x-3xi}{4-9i^2} - y(1+2i) &= 1+i \\ \Rightarrow \frac{2x-3xi}{13} - y(1+2i) &= 1+i \\ \Rightarrow \frac{2x}{13} - \frac{3ix}{13} - y - 2iy &= 1+i \\ \Rightarrow \left(\frac{2x}{13} - y\right) - \left(\frac{3x}{13} + 2y\right)i &= 1+i\end{aligned}$$

Comparing real and imaginary parts

$$\frac{2x}{13} - y = 1 \quad (1)$$

$$\text{and } -\frac{3x}{13} - 2y = 1 \quad (2)$$

Multiplying equation (1) with 2 then adding them, we get:

$$\begin{array}{rcl}\frac{4x}{13} - 2y & = & 2 \\ \frac{3x}{13} + 2y & = & -1 \\ \hline \frac{7x}{13} & = & 1\end{array}$$

$$x = \frac{13}{7}$$

Putting value of x in equation (1).

$$\frac{2}{13} \times \frac{13}{7} - y = 1$$

$$\frac{2}{7} - y = 1 \Rightarrow -y = 1 - \frac{2}{7} = \frac{7-2}{7}$$

$$y = -\frac{5}{7}$$

1.3 Conjugate of a Complex Number

Conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined as $\bar{z} = x - iy$.

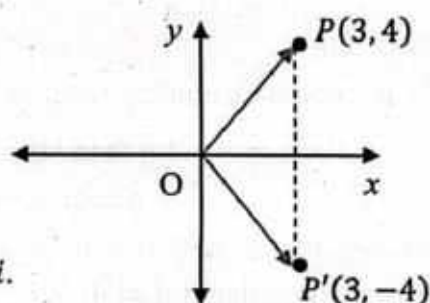
Geometrically, conjugate of a complex number is its mirror image about x -axis. For example, if $z = 3 + 4i$ then $\bar{z} = 3 - 4i$.

Example 9: Find the conjugate of $z = (1 + i)(2 - i)$.

Solution:

$$z = (1 + i)(2 - i) = 2 - i + 2i - i^2 = 2 + i + 1 = 3 + i$$

$$\text{Now } \bar{z} = \overline{3 + i} = 3 - i$$



Key Facts

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

1.4 Magnitude or Modulus of a Complex Number

Consider the complex number $z = x + iy = (x, y)$ its magnitude is denoted by $|z|$ and find it as follows

Draw a complex number $z = x + iy = (x, y)$ on the complex plane.

Draw perpendicular from P on the real axis.

It is clear that PAO is a right-angled triangle.

So, by using Pythagoras theorem; we have

$$|\overline{OP}|^2 = |\overline{OA}|^2 + |\overline{AP}|^2$$

$$\Rightarrow |z|^2 = x^2 + y^2$$

$$\Rightarrow |z| = \sqrt{x^2 + y^2}$$

which is the magnitude of the complex number z .

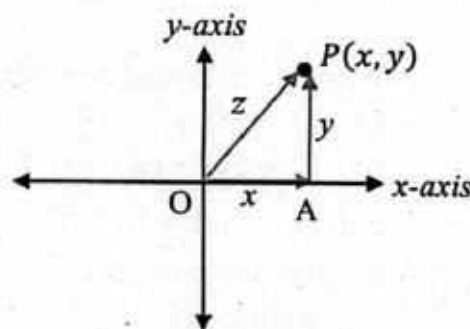
$$\text{Also, } |z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}$$

Obviously $|z|$ is the distance of $z = (x, y)$ from origin.

Example 10: Find the conjugate and magnitude of $z = \frac{(3+2i)(1-2i)}{4+3i}$

Solution:

$$\begin{aligned} \bar{z} &= \overline{\left[\frac{(3+2i)(1-2i)}{4+3i}\right]} = \frac{\overline{(3+2i)(1-2i)}}{\overline{(4+3i)}} = \frac{\overline{(3+2i)}\overline{(1-2i)}}{4-3i} = \frac{(3-2i)(1+2i)}{4-3i} \\ &= \frac{3+6i-2i-4i^2}{4-3i} = \frac{3+4+4i}{4-3i} = \frac{7+4i}{4-3i} \times \frac{4+3i}{4+3i} \\ &= \frac{28+21i+16i+12i^2}{4^2-(3i)^2} = \frac{28-12+37i}{16-9i^2} = \frac{16+37i}{25} = \frac{16}{25} + \frac{37}{25}i \end{aligned}$$



And

$$|z| = \left| \frac{(3+2i)(1-2i)}{4+3i} \right| = \frac{|3+2i||1-2i|}{|4+3i|}$$

$$= \frac{\sqrt{3^2+2^2}\sqrt{1^2+(-2)^2}}{\sqrt{4^2+3^2}} = \frac{\sqrt{13}\sqrt{5}}{\sqrt{25}}$$

$$= \frac{\sqrt{65}}{5}$$

Key Facts

$$|z_1 z_2| = |z_1| |z_2|$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

Theorem:

If z is a complex number then $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.

Proof: Let $z = x + iy$; then $|z| = \sqrt{x^2 + y^2}$, where x and y are real. Since the square of any real number is always non-negative. Thus, value of $x^2 + y^2$ is non-negative, also square-root of the non-negative number is non-negative. Hence $\sqrt{x^2 + y^2}$ is non-negative; i.e.; $|z| \geq 0$.

Now suppose that $|z| = 0$

$$\Rightarrow \sqrt{x^2 + y^2} = 0 \Rightarrow x^2 + y^2 = 0$$

Which is possible only if $x = 0$ and $y = 0$

Thus $z = x + iy = 0 + i0 = 0$

Conversely, suppose that $z = 0 = 0 + i0$

$$\Rightarrow |z| = \sqrt{0^2 + 0^2} = 0$$

Key Facts

In order to calculate conjugate of a complex number we may simplify it first then take conjugate or we take first conjugate than simplify the complex number.

Exercise 1.1

1. Evaluate the following:

(i) i^{31} (ii) $(-i)^6$ (iii) $(-1)^{\frac{-13}{2}}$ (iv) $\frac{2}{(-1)^{\frac{3}{2}}}$ (v) $i^{23} + i^{58} + i^{21}$

2. Write the following complex numbers in the form $x + iy$.

(i) $(3+2i) + (2+4i)$ (ii) $(4+3i) - (2+5i)$ (iii) $(4+7i) + (4-7i)$
 (iv) $(2+5i) - (2-5i)$ (v) $(3+2i)(4-3i)$ (vi) $(3,2) \div (3,-1)$
 (vii) $(1+i)(1-i)(2+i)$ (viii) $\frac{1}{2+3i}$

3. Simplify the following:

(i) $\frac{(2+i)(3-2i)}{1+i}$ (ii) $\frac{1+i}{(2+i)^2}$ (iii) $\frac{1}{3+i} - \frac{1}{3-i}$
 (iv) $(1+i)^{-2} + (1-i)^{-2}$ (v) $(2+i)^2 + \frac{7-4i}{2+i}$

4. Find the values of the real numbers x and y in each of the following:

(i) $(2+3i)x + (1+3i)y + 2 = 0$ (ii) $\frac{x}{1+i} + \frac{y}{1-2i} = 1$
 (iii) $\frac{x}{2+i} = \frac{1-5i}{3-2i} + \frac{y}{2-i}$ (iv) $x(1+i)^2 + y(2-i)^2 = 3 + 10i$

5. Find the complex number z if $4z - 3\bar{z} = \frac{1-18i}{2-i}$

6. Find the conjugate of the following complex numbers:

(i) $4 - 3i$ (ii) $3i + 8$ (iii) $2 + \sqrt{\frac{-1}{5}}$ (iv) $\frac{5i}{2} - \frac{7}{8}$

7. Find the magnitude of the following complex numbers:

(i) $11 + 12i$ (ii) $(2 + 3i) - (2 + 6i)$ (iii) $(2 - i)(6 + 3i)$
 (iv) $\frac{3-2i}{2+i}$ (v) $(\sqrt{3} - \sqrt{-8})(\sqrt{3} + \sqrt{-8})$

1.5 Real and Imaginary Parts of the Complex Numbers of the Types

(i) $(x + iy)^n$; $n = \pm 1$

(ii) $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$; $x_2 + iy_2 \neq 0$; $n = \pm 1$

Type-I Consider the complex number of the type $(x + iy)^n$.

When $n = 1$

$$z = x + iy$$

$$\text{Its real part} = x = \text{Re}(z)$$

$$\text{Imaginary part} = y = \text{Im}(z)$$

When $n = -1$

$$\begin{aligned} z^{-1} &= (x + iy)^{-1} = \frac{1}{x + iy} = \left(\frac{1}{x + iy}\right) \left(\frac{x - iy}{x - iy}\right) = \frac{x - iy}{x^2 - i^2 y^2} \\ &= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \end{aligned}$$

$$\text{Thus } \text{Re}(x + iy)^{-1} = \text{Re} z^{-1} = \frac{x}{x^2 + y^2} = \frac{\text{Re}(z)}{|z|^2}$$

$$\text{Im}(x + iy)^{-1} = \text{Im} z^{-1} = \frac{-y}{x^2 + y^2} = -\frac{\text{Im}(z)}{|z|^2}$$

Example-11: Find the real and imaginary parts of the following.

(i) $3 + 4i$ (ii) $(3 + 4i)^{-1}$ (iii) $(3 + 4i)^2$ (iv) $(3 + 4i)^{-2}$

Solution: Let $z = 3 + 4i$

(i) $\text{Re}(3 + 4i) = \text{Re}(z) = 3$

$$\text{Im}(3 + 4i) = \text{Im}(z) = 4$$

(ii) $\text{Re}(3 + 4i)^{-1} = \text{Re}(z)^{-1} = \frac{\text{Re}(z)}{|z|^2} = \frac{3}{(\sqrt{3^2 + 4^2})^2} = \frac{3}{25}$

$$\text{Im}(3 + 4i)^{-1} = \text{Im}(z)^{-1} = \frac{-\text{Im}(z)}{|z|^2} = \frac{-4}{(\sqrt{3^2 + 4^2})^2} = \frac{-4}{25}$$

(iii) $\text{Re}(3 + 4i)^2 = \text{Re}(z)^2 = (\text{Re}(z))^2 - (\text{Im}(z))^2$
 $= 3^2 - 4^2 = 9 - 16 = -7$

$$\begin{aligned} \text{Im}(3 + 4i)^2 &= \text{Im}(z)^2 = 2\text{Re}(z)\text{Im}(z) \\ &= 2(3)(4) = 24 \end{aligned}$$

$$(iv) \quad Re(3 + 4i)^{-2} = Re(z)^{-2} = \frac{(Re(z))^2 - (Im(z))^2}{|z|^4} = \frac{3^2 - 4^2}{(\sqrt{3^2 + 4^2})^4} = \frac{9 - 16}{5^4} = \frac{-7}{625}$$

$$Im(3 + 4i)^{-2} = Im(z)^{-2} = \frac{-2Re(z)Im(z)}{|z|^4} = \frac{-2(3)(4)}{(\sqrt{3^2 + 4^2})^4} = \frac{-24}{5^4} = \frac{-24}{625}$$

Type-II Consider the complex number of the form $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$; where $x_2 + iy_2 \neq 0$.

Let $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$

So $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n = \left(\frac{z_1}{z_2}\right)^n$ where $z_2 \neq 0$.

When $n = 1$

$$\begin{aligned} \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \times \frac{(x_2 - iy_2)}{(x_2 - iy_2)} \\ &= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{(x_2)^2 - (iy_2)^2} = \frac{x_1x_2 + i(x_2y_1 - x_1y_2) + y_1y_2}{x_2^2 - i^2y_2^2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i \frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \end{aligned}$$

$$\therefore Re\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right) = Re\left(\frac{z_1}{z_2}\right) = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2}$$

$$Re\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right) = \frac{(Re(z_1))(Re(z_2)) + (Im(z_1))(Im(z_2))}{|z_2|^2}$$

$$Im\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right) = Im\left(\frac{z_1}{z_2}\right) = \frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$Im\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right) = \frac{(Im(z_1))(Re(z_2)) - (Re(z_1))(Im(z_2))}{|z_2|^2}$$

When $n = -1$

$$\begin{aligned} \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n &= \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-1} = \frac{x_2 + iy_2}{x_1 + iy_1} = \frac{x_2 + iy_2}{x_1 + iy_1} \times \frac{x_1 - iy_1}{x_1 - iy_1} \\ &= \frac{x_1x_2 + ix_1y_2 - ix_2y_1 - i^2y_1y_2}{x_1^2 - (iy_1)^2} = \frac{x_1x_2 + ix_1y_2 - ix_2y_1 + y_1y_2}{x_1^2 + y_1^2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1)}{x_1^2 + y_1^2} = \frac{(x_1x_2 + y_1y_2)}{x_1^2 + y_1^2} + i \frac{(x_1y_2 - x_2y_1)}{x_1^2 + y_1^2} \end{aligned}$$

$$Re\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-1} = \frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2} = \frac{Re(z_1)Re(z_2) + Im(z_1)Im(z_2)}{|z_1|^2}$$

$$Im\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-1} = \frac{x_1y_2 - x_2y_1}{x_1^2 + y_1^2} = \frac{Re(z_1)Im(z_2) - Re(z_2)Im(z_1)}{|z_1|^2}$$

Key Facts



$$Re(z_1)Re(z_2) \neq Re(z_1z_2)$$

$$Im(z_1)Im(z_2) \neq Im(z_1z_2)$$

Example 12: If $x_1 + iy_1 = 12 + 5i$ and $x_2 + iy_2 = 3 + 2i$ then find the real and imaginary parts of the following:

(i) $\frac{x_1 + iy_1}{x_2 + iy_2}$ (ii) $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-1}$ (iii) $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^2$ (iv) $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-2}$

Solution:

(i)

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{12 + 5i}{3 + 2i}$$

Now,

$$\operatorname{Re} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right) = \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} = \frac{(12)(3) + (5)(2)}{3^2 + 2^2} = \frac{36 + 10}{9 + 4} = \frac{46}{13}$$

And

$$\operatorname{Im} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right) = \frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \frac{(3)(5) - (12)(2)}{3^2 + 2^2} = \frac{15 - 24}{9 + 4} = \frac{-9}{13}$$

(ii)

$$\left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^{-1} = \left(\frac{12 + 5i}{3 + 2i} \right)^{-1}$$

$$\operatorname{Re} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^{-1} = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} = \frac{(12)(3) + (5)(2)}{12^2 + 5^2} = \frac{36 + 10}{144 + 25} = \frac{46}{169}$$

And

$$\operatorname{Im} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^{-1} = \frac{x_1 y_2 - x_2 y_1}{x_1^2 + y_1^2} = \frac{(12)(2) - (3)(5)}{12^2 + 5^2} = \frac{24 - 15}{144 + 25} = \frac{9}{169}$$

(iii)

$$\left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^2 = \left(\frac{12 + 5i}{3 + 2i} \right)^2$$

Now,

$$\begin{aligned} \operatorname{Re} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^2 &= \frac{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1 x_2 y_1 y_2}{(x_2^2 + y_2^2)^2} \\ &= \frac{(12^2 - 5^2)(3^2 - 2^2) + 4(12)(3)(5)(2)}{(3^2 + 2^2)^2} \\ &= \frac{(144 - 25)(9 - 4) + 1440}{(9 + 4)^2} = \frac{(119)(5) + 1440}{13^2} = \frac{595 + 1440}{169} \\ &= \frac{2035}{169} \end{aligned}$$

And

$$\begin{aligned} \operatorname{Im} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^2 &= \frac{2[x_1 y_1 (x_2^2 - y_2^2) - x_2 y_2 (x_1^2 - y_1^2)]}{(x_2^2 + y_2^2)^2} \\ &= \frac{2[(12)(5)(3^2 - 2^2) - (3)(2)(12^2 - 5^2)]}{(3^2 + 2^2)^2} = \frac{2[60(9 - 4) - 6(144 - 25)]}{(9 + 4)^2} \end{aligned}$$

$$= \frac{2(300 - 714)}{13^2} = \frac{2(-414)}{169} = \frac{-828}{169}$$

(iv)

$$\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-2} = \left(\frac{12 + 5i}{3 + 2i}\right)^{-2}$$

Now,

$$\begin{aligned} \operatorname{Re}\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-2} &= \frac{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1x_2y_1y_2}{(x_1^2 + y_1^2)^2} \\ &= \frac{(12^2 - 5^2)(3^2 - 2^2) + 4(12)(3)(5)(2)}{(12^2 + 5^2)^2} \\ &= \frac{(144 - 25)(9 - 4) + 1440}{(144 + 25)^2} = \frac{(119)(5) + 1440}{169^2} = \frac{595 + 1440}{28561} \\ &= \frac{2035}{28561} \end{aligned}$$

And

$$\begin{aligned} \operatorname{Im}\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-2} &= \frac{-2[x_1y_1(x_2^2 - y_2^2) - x_2y_2(x_1^2 - y_1^2)]}{(x_1^2 + y_1^2)^2} \\ &= \frac{-2[(12)(5)(3^2 - 2^2) - (3)(2)(12^2 - 5^2)]}{(12^2 + 5^2)^2} \\ &= \frac{-2[60(9 - 4) - 6(144 - 25)]}{(144 + 25)^2} = \frac{-2(300 - 714)}{169^2} \\ &= \frac{2(-414)}{28561} = \frac{828}{169} \end{aligned}$$

When $n = 2$

$$\begin{aligned} \operatorname{Re}\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^2 &= \frac{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1x_2y_1y_2}{(x_1^2 + y_1^2)^2} \\ &= \frac{[(\operatorname{Re}z_1)^2 - (\operatorname{Im}z_1)^2][(\operatorname{Re}z_2)^2 - (\operatorname{Im}z_2)^2] + 4\operatorname{Re}z_1\operatorname{Re}z_2\operatorname{Im}z_1\operatorname{Im}z_2}{|z_2|^4} \end{aligned}$$

And

$$\begin{aligned} \operatorname{Im}\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^2 &= \frac{2[x_1y_1(x_2^2 - y_2^2) - x_2y_2(x_1^2 - y_1^2)]}{(x_1^2 + y_1^2)^2} \\ &= \frac{2[\operatorname{Re}z_1\operatorname{Im}z_1\{(\operatorname{Re}z_2)^2 - (\operatorname{Im}z_2)^2\} - \operatorname{Re}z_2\operatorname{Im}z_2\{(\operatorname{Re}z_1)^2 - (\operatorname{Im}z_1)^2\}]}{|z_2|^4} \end{aligned}$$

When $n = -2$

$$\begin{aligned} \operatorname{Re}\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^{-2} &= \frac{(x_2^2 - y_2^2)(x_1^2 - y_1^2) + 4x_2x_1y_2y_1}{(x_1^2 + y_1^2)^2} \\ &= \frac{(x_1^2 - y_1^2)(x_2^2 - y_2^2) + 4x_1x_2y_1y_2}{(x_1^2 + y_1^2)^2} \\ &= \frac{[(\operatorname{Re}z_1)^2 - (\operatorname{Im}z_1)^2][(\operatorname{Re}z_2)^2 - (\operatorname{Im}z_2)^2] + 4\operatorname{Re}z_1\operatorname{Re}z_2\operatorname{Im}z_1\operatorname{Im}z_2}{|z_1|^4} \end{aligned}$$

$$\begin{aligned}
 \operatorname{Im} \left(\frac{x_1 + iy_1}{x_2 + iy_2} \right)^{-2} &= \frac{2[x_2 y_2 (x_1^2 - y_1^2) - x_1 y_1 (x_2^2 - y_2^2)]}{(x_1^2 + y_1^2)^2} \\
 &= \frac{-2[x_1 y_1 (x_2^2 - y_2^2) - x_2 y_2 (x_1^2 - y_1^2)]}{(x_1^2 + y_1^2)^2} \\
 &= \frac{-2[\operatorname{Re} z_1 \operatorname{Im} z_1 \{(\operatorname{Re} z_2)^2 - (\operatorname{Im} z_2)^2\} - \operatorname{Re} z_2 \operatorname{Im} z_2 \{(\operatorname{Re} z_1)^2 - (\operatorname{Im} z_1)^2\}]}{|z_1|^4}
 \end{aligned}$$

Example 13: Write the equation $|z - 2i| = |\bar{z} + 3|$ in terms of x and y , by taking

$$z = x + iy.$$

Solution:

$$\text{Since, } z = x + iy$$

$$\Rightarrow \bar{z} = x - iy$$

$$\therefore |z - 2i| = |\bar{z} + 3|$$

$$\Rightarrow |x + iy - 2i| = |x - iy + 3|$$

$$\Rightarrow |x + i(y - 2)| = |(x + 3) - iy|$$

$$\Rightarrow \sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 3)^2 + (-y)^2}$$

Squaring both sides.

$$x^2 + (y - 2)^2 = (x + 3)^2 + (y)^2$$

$$\Rightarrow x^2 + y^2 - 4y + 4 = x^2 + 6x + 9 + y^2$$

$$\Rightarrow -4y + 4 = 6x + 9$$

$$\Rightarrow 6x + 4y + 5 = 0$$

Example 14: Write the inequation $\operatorname{Re}(z - 3) \leq 2$ in terms of x and y , by taking

$$z = x + iy.$$

Solution:

$$\operatorname{Re}(z - 3) \leq 2$$

$$\operatorname{Re}(x + iy - 3) \leq 2$$

$$\operatorname{Re}\{(x - 3) + iy\} \leq 2$$

$$x - 3 \leq 2$$

$$\Rightarrow x \leq 5$$

Exercise 1.2

1. Show that for any complex number.

$$\text{i) } \operatorname{Re}(iz) = -\operatorname{Im}(z) \quad \text{ii) } \operatorname{Im}(iz) = \operatorname{Re}(z)$$

2. Use the algebraic properties of complex numbers to prove that:

$$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4) = z_3(z_1 z_2)z_4$$

3. Prove that for $z \in \mathbb{C}$.

$$\text{i) } z \text{ is real iff } z = \bar{z} \quad \text{ii) } \frac{z - \bar{z}}{z + \bar{z}} = i \left(\frac{\operatorname{Im} z}{\operatorname{Re} z} \right)$$

$$\text{iii) } z \text{ is either real or pure imaginary iff } (\bar{z})^2 = z^2$$

4. If $z_1 = 2 - 3i$ and $|z_1 z_2| = 16$ find $|z_2|$.

5. If z_1 and z_2 are any two complex numbers then prove that
 $|z_1 + z_2|^2 - |z_1 - z_2|^2 = 4\operatorname{Re}(z_1)\operatorname{Re}(z_2) + 4\operatorname{Im}(z_1)\operatorname{Im}(z_2)$
6. Find the value of λ ; if $\left|\frac{z_1}{z_2} + \lambda\right| = \sqrt{\lambda + 2}$; where $z_1 = 3 + i$ and $z_2 = 1 + i$.
7. Verify that $\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ Hint: (Start with $(|x| - |y|)^2 \geq 0$)
8. Write the following equations and inequations in terms of x and y by taking $z = x + iy$.
 (i) $|2z - i| = 4$ (ii) $|z - 1| = |\bar{z} + i|$ (iii) $|z - 4i| + |z + 4i| = 10$
 (iv) $\frac{1}{2}\operatorname{Re}(iz) = 4$ (v) $\operatorname{Im}\left(\frac{z-1}{2i}\right) = -5$ (vi) $-2 \leq \operatorname{Im}(z + i) \leq 3$
9. Find real and imaginary parts of the followings:
 (i) $(2 + 4i)^{-1}$ (ii) $(3 - \sqrt{-4})^{-2}$ (iii) $\left(\frac{7+2i}{3-i}\right)^{-1}$
 (iv) $\left(\frac{4+2i}{2+5i}\right)^{-2}$ (v) $\left(\frac{5-4i}{5+4i}\right)^2$ (vi) $\frac{3-7i}{2+5i}$
10. For $z_1 = -3 + 2i$ and $z_2 = 1 - 3i$ verify the followings:
 (i) $|z_1| = |-z_1| = |\bar{z}_1| = |-\bar{z}_1|$ (ii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ (iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
 (iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (v) $|z_1 z_2| = |z_1| |z_2|$ (vi) $|z_1 + z_2| \leq |z_1| + |z_2|$

1.6 Solution of Equations

When we consider more than one equation then it is called system of equations and if we find the values of variables which satisfies all the equations under considerations simultaneously, is called the simultaneous solutions of the equations.

If z and ω are the two complex variables then an equation of the form $az + b\omega = p$ is called equation with complex variables z and ω . Here a and b cannot be zero at the same time. If a and b belong to the set of complex numbers (i.e.; they are itself complex numbers) then the equation is called linear equation in two variables with complex coefficients.

Here we will find the solution of system of two simultaneous equations in two variables with complex coefficients.

1.6.1 Working Rule to Find the Solution by Elimination Method

Consider the two linear equations:

$$a_1 z + b_1 \omega = p_1 \text{ and } a_2 z + b_2 \omega = p_2$$

Step 1: Multiply the equation or both equations by suitable numbers so that the coefficients of one of the variables become same.

Step 2: By adding or subtracting the equations thus obtained in Step 1, eliminate the term involving the variable having same coefficients.

Step 3: The equation obtained in Step 2 will have only one variable. From here find the value of this variable.

- A system of equations is **consistent** if it has at least one solution.
- A system of equations which has no solution is called **inconsistent**.

Step 4: Substitute the value of the variable found in **Step 3** in any one of the given equations and get the value of the other variable.

Step 5: Writing the values of z and ω in the form of ordered pair (z, ω) is the solution of the system of equations.

1.6.2 Working Rule to Find the Solution by Substitution Method

Step 1: Find the value of any one of the variables in terms of the other variable from any one of the equations given above.

Step 2: Substitute the value of the variable obtained in **Step 1** to the equation which is not used yet.

Step 3: Equation obtained in the **Step 2** will involve only one variable. Find its value.

Step 4: Substitute the value of variable obtained in **Step 3** in any one the given equations and get the value of the other variable.

Step 5: Writing the values of the both unknowns z and ω in the ordered pair (z, ω) is the solution of the system.

Example 15: Solve the following system of simultaneous equations:

$$2z - (1 - 3i)\omega = 1 + 2i, \quad (1 + i)z + (2 - i)\omega = 2 + i$$

Solution:

$$2z - (1 - 3i)\omega = 1 + 2i \quad (1)$$

$$(1 + i)z + (2 - i)\omega = 2 + i \quad (2)$$

Multiplying Eq. (1) by $(1 + i)$ and Eq. (2) by 2 then subtracting Eq. (2) from (1).

$$(1) \Rightarrow 2(1 + i)z - (1 + i)(1 - 3i)\omega = (1 + i)(1 + 2i)$$

$$(2) \Rightarrow 2(1 + i)z + 2(2 - i)\omega = 2(2 + i)$$

$$-(1 + i)(1 - 3i)\omega - 2(2 - i)\omega = (1 + i)(1 + 2i) - 2(2 + i)$$

$$\Rightarrow -[(1 + i)(1 - 3i) + 2(2 - i)]\omega = (1 + i)(1 + 2i) - 2(2 + i)$$

$$\Rightarrow -(1 - 3i + i - 3i^2 + 4 - 2i)\omega = 1 + 2i + i + 2i^2 - 4 - 2i$$

$$\Rightarrow -(1 - 3i + i + 3 + 4 - 2i)\omega = 1 + 2i + i - 2 - 4 - 2i$$

$$\Rightarrow -(8 - 4i)\omega = -5 + i$$

$$\Rightarrow (-8 + 4i)\omega = -5 + i$$

$$\begin{aligned} \Rightarrow \omega &= \frac{-5 + i}{-8 + 4i} = \frac{-5 + i}{-8 + 4i} \times \frac{-8 - 4i}{-8 - 4i} = \frac{40 + 20i - 8i - 4i^2}{(-8)^2 - (4i)^2} = \frac{40 + 12i + 4}{64 + 16} \\ &= \frac{44 + 12i}{80} = \frac{44}{80} + i \frac{12}{80} = \frac{11}{20} + i \frac{3}{20} \end{aligned}$$

Substituting value of ω in Eq. (1)

$$(1) \Rightarrow 2z - (1 - 3i)\left(\frac{11}{20} + i \frac{3}{20}\right) = 1 + 2i$$

$$\Rightarrow 2z - (1 - 3i)\left(\frac{11 + 3i}{20}\right) = 1 + 2i$$

Multiplying both sides of the equation with 20.

$$\Rightarrow 40z - (11 + 3i - 33i - 9i^2) = 20(1 + 2i)$$

$$\Rightarrow 40z - (11 - 30i + 9) = 20 + 40i$$

Challenge

Solve the system by Cramer's rule.

$$\begin{aligned}
 &\Rightarrow 40z - (20 - 30i) = 20 + 40i \\
 &\Rightarrow 40z = 20 + 40i + (20 - 30i) \\
 &\Rightarrow 40z = 40 + 10i \\
 &\Rightarrow z = \frac{40}{40} + \frac{10}{40}i = 1 + \frac{1}{4}i
 \end{aligned}$$

$\therefore \left(1 + \frac{1}{4}i, \frac{11}{20} + \frac{3}{20}i\right)$ is the solution of the system of equations.

1.7 Complex Polynomial

If z is a complex variable, then the expression $a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ is called complex polynomial of degree n if $a_n \neq 0$ and n is a non-negative integer. Here $a_0, a_1, a_2, \dots, a_n$ are constants and may be real or complex. Let us denote this polynomial by $P(z)$ i.e.;

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

When $n = 1$; then the polynomial is $a_0 + a_1z$ and is called linear polynomial. We are interested to factorize the polynomial of the two types as a product of linear factors.

- (i) $P(z) = z^2 + a^2$; where a is a real number.
- (ii) $P(z) = az^3 + bz^2 + cz + d$; where a, b, c, d are all real numbers.

1.7.1 Factorization of Polynomial of the Type $z^2 + a^2$ as a Polynomial of Linear Factor

The factorization of this type of polynomials is simple. Consider

$$z^2 + a^2 = z^2 - i^2a^2 = z^2 - (ia)^2 = (z + ia)(z - ia)$$

$(z + ia)(z - ia)$ are required linear factors of $z^2 + a^2$.

1.7.2 Factorization of Polynomial of the Type $az^3 + bz^2 + cz + d = 0$ where $a \neq 0$

To factorize this type of polynomial first we find one of its factors with the help of factor theorem and then do the synthetic division to find the depressed equation.

Recall that $z - a$ is a factor of the polynomial $P(z)$ iff $P(a) = 0$. We may say that a is a root or zero of the polynomial.

e.g.; $z - 2$ is a factor (root or zero) of the polynomial $P(z) = 2z^3 + 3z^2 + 6z - 40$; since $P(2) = 2(2)^3 + 3(2)^2 + 6(2) - 40 = 0$.

Example 16: Factorize the polynomial $P(z) = z^3 + 2z^2 - 5z - 6$.

Solution: Product of coefficient of z^3 and the last term is $(1)(-6) = -6$.

The possible roots of the equation are the factors of -6 which are $\pm 1, \pm 2, \pm 3, \pm 6$.

Since $P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = 0$.

So $z - (-1) = z + 1$ is a factor of the polynomial. To factorize it completely use the method of synthetic division.

	1	2	-5	-6
-1	0	-1	-1	6
	1	1	-6	0

$$\begin{aligned}
 \therefore z^3 + 2z^2 - 5z - 6 &= (z+1)(z^2 + z - 6) \\
 &= (z+1)(z^2 + 3z - 2z - 6) \\
 &= (z+1)[z(z+3) - 2(z+3)] \\
 &= (z+1)(z+3)(z-2)
 \end{aligned}$$

1.7.3 Solution by Completing Square Method:

Example 17: Solve the equation $2z^2 - 6z - 9 = 0$ by completing square method.

Solution:

The given equation is

$$2z^2 - 6z - 9 = 0$$

Dividing both sides by 2 (coefficient of z^2)

$$z^2 - 3z - \frac{9}{2} = 0$$

$$z^2 - 3z = \frac{9}{2}$$

Adding $\left(-\frac{3}{2}\right)^2$ on both sides

$$z^2 - 3z + \left(-\frac{3}{2}\right)^2 = \frac{9}{2} + \left(-\frac{3}{2}\right)^2$$

$$\left(z - \frac{3}{2}\right)^2 = \frac{9}{2} + \frac{9}{4} = \frac{27}{4}$$

Taking square root on both sides

$$z - \frac{3}{2} = \pm \frac{3\sqrt{3}}{2}$$

$$\Rightarrow z = \frac{3}{2} \pm \frac{3\sqrt{3}}{2} = \frac{3 \pm 3\sqrt{3}}{2}$$

$$S.S = \left\{ \frac{3 \pm 3\sqrt{3}}{2} \right\}$$

Exercise 1.3

1. Factorize the following polynomials into linear functions:

- (i) $z^2 + 169$ (ii) $2z^2 + 18$ (iii) $3z^2 + 363$ (iv) $z^2 + \frac{3}{25}$
 (v) $2z^3 + 3z^2 - 10z - 15$ (vi) $z^3 - 7z + 6$ (vii) $z^3 + 2z^2 - 23z - 60$
 (viii) $2z^3 + 9z^2 - 11z - 30$ (ix) $z^2 - 7z - 8$ (x) $4z^2 - 7z - 11$

2. Solve the following equations by completing square method

- (i) $z^2 - 6z + 2 = 0$ (ii) $-\frac{1}{2}z^2 - 5z + 2 = 0$
 (iii) $4z^2 + 5z = 14$ (iv) $z^2 = 5z - 3$

3. Solve the following quadratic equations:

- (i) $\frac{1}{3}z^2 + 2z - 16 = 0$ (ii) $z^2 - \frac{1}{2}z + 17 = 0$
 (iii) $z^2 - 6z + 25 = 0$ (iv) $z^2 - 9z + 11 = 0$

4. Solve the simultaneous system of linear equations with complex coefficients:

(i) $(1 - i)z + (1 + i)\omega = 3; \quad 2z - (2 + 5i)\omega = 2 + 3i$

(ii) $2iz + (3 - 2i)\omega = 1 + i; \quad (1 - 2i)z + (3 + 2i)\omega = 5 + 6i$

(iii) $\frac{3}{i}z - (6 + 2i)\omega = 5; \quad \frac{i}{2}z + \left(\frac{3}{4} - \frac{1}{2}i\right)\omega = \left(\frac{1}{2} + 2i\right)$

(iv) $\frac{1}{1-i}z + (1 + i)\omega = 3; \quad \frac{2}{i}z - (2 - 3i)\omega = 2 + 6i$

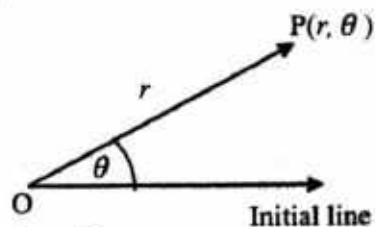
1.8 Polar Coordinate System

Another way to locate a point in the plane is polar coordinate system consists of a fixed-point O called the pole and the horizontal line emerging from the pole is called initial line (polar axis).

For a point P in the plane if r is its distance from the pole and θ is the angle which is measured anticlockwise from the initial line to the line $\overrightarrow{OP}$ then the ordered pair (r, θ) are the polar coordinates of the point P.

The angle θ is also called the $\arg(z)$.

- For $z = 0$ the $\arg(z)$ is undefined so it is understood that $z \neq 0$ whenever we use polar coordinate system.
- If a complex number $z = x + iy$ has polar coordinates (r, θ) then its conjugate is $\bar{z} = x - iy$ has polar coordinates $(r, -\theta)$.



1.9 Complex Numbers in Polar Form

1.9.1 Polar Representation of a Complex Number

Consider a complex number $z = x + iy$ in cartesian form. Draw it on the complex plane as shown in the figure.

Let $r = |z|$, and θ be the angle in positive direction

which $\overrightarrow{OP}$ makes with the initial line(x-axis).

Draw a perpendicular from P on the initial line then by Pythagoras theorem, we have

$$|OL|^2 + |LP|^2 = |OP|^2$$

$$\Rightarrow x^2 + y^2 = |z|^2$$

$$\Rightarrow x^2 + y^2 = r^2$$

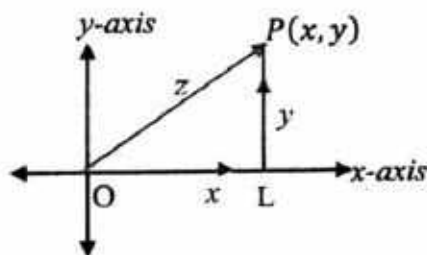
Or $r = \sqrt{x^2 + y^2}$

Also $\frac{x}{|z|} = \cos \theta \Rightarrow \frac{x}{r} = \cos \theta$

$$\Rightarrow x = r \cos \theta$$

And $\frac{y}{|z|} = \sin \theta \Rightarrow \frac{y}{r} = \sin \theta$

$$\Rightarrow y = r \sin \theta$$



Key Facts
 θ is called argument of z and is written as
 $\theta = \arg(z)$

Key Facts
 $\cos \theta + i \sin \theta$ can also be written as $\text{cis } \theta$
 and $\cos \theta + i \sin \theta = e^{i\theta}$ is known as
 Euler's formula.

By substituting the values of x and y in $z = x + iy$

We have

$$z = r \cos \theta + ir \sin \theta$$

$$z = r(\cos \theta + i \sin \theta)$$

This form of the complex number is called polar form of a complex number.

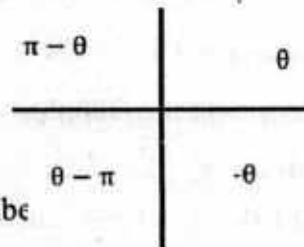
1.9.2 Principal Argument

The symbol $\arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta \leq \pi$ is called the **principal value** of $\arg(z)$ or the **principal argument** of z . The principal argument of z is unique and is represented by the symbol $\text{Arg}(z)$, that is, $-\pi < \text{Arg}(z) \leq \pi$.

If θ is the principal argument then principal values can be calculated

As in first quadrant value is θ , in second quadrant value is $\pi - \theta$,

In fourth quadrant value is $-\theta$ then in third quadrant is $\theta - \pi$.



Example 18:

Find the modulus and principal argument of the following complex number

- (i) $\sqrt{3} + i$ (ii) $-\sqrt{3} + i$ (iii) $-\sqrt{3} - i$ (iv) $\sqrt{3} - i$

Solution: (i) $\sqrt{3} + i$

Since the complex number $\sqrt{3} + i$ lying in the first quadrant, has the principal value $\theta = \alpha = \pi/6$.

$$\text{Modulus} = r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2$$

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{6}$$

Therefore, the modulus and principal argument of

$\sqrt{3} + i$ are 2 and $\pi/6$ respectively.

- (ii) $-\sqrt{3} + i$

$$\text{Modulus} = r = 2 \text{ and } \alpha = \frac{\pi}{6}$$

Since the complex number $-\sqrt{3} + i$ lying in the second quadrant has the principal value

Therefore, the modulus and principal argument of $-\sqrt{3} + i$ are 2 and $\frac{5\pi}{6}$ respectively.

$$\theta = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

- (iii) $-\sqrt{3} - i$

$$r = 2 \text{ and } \alpha = \frac{\pi}{6}$$

Since the complex number $-\sqrt{3} - i$ lying in the third quadrant, has the principal value,

$$\theta = -\pi + \alpha = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

Therefore, the modulus and principal argument of $-\sqrt{3} - i$ are 2 and $-5\pi/6$ respectively.

$$(iv) \sqrt{3} - i$$

$$r = 2 \text{ and } \alpha = \pi/6$$

Since the complex number lying in the fourth quadrant, has the principal value,

$$\theta = -\alpha = -\frac{\pi}{6}$$

Therefore, the modulus and principal argument of $\sqrt{3} - i$ are 2 and $-\pi/6$.

In all the four cases, modulus are equal, but the arguments are depending on the quadrant in which the complex number lies.

Example 19:

Represent the complex number (i) $-1 - i$ (ii) $1 + i\sqrt{3}$ in polar form.

Solution:

$$(i) \text{ Let } -1 - i = r(\cos \theta + i \sin \theta)$$

$$\text{We have } r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} 1 = \frac{\pi}{4}$$

Since the complex number $-1 - i$ lies in the third quadrant, it has the principal value,

$$\theta = \alpha - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

$$\therefore -1 - i = \sqrt{2} \left[\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right] = \sqrt{2} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)$$

$$-1 - i = \sqrt{2} \left[\cos \left(\frac{3\pi}{4} + 2k\pi \right) - i \sin \left(\frac{3\pi}{4} + 2k\pi \right) \right]$$

$$(ii) 1 + i\sqrt{3}$$

$$r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1} (\sqrt{3}) = \frac{\pi}{3}$$

Hence

$$\arg(z) = \frac{\pi}{3}$$

Therefore, the polar form of $1 + i\sqrt{3}$ can be written as

$$1 + i\sqrt{3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

$$= 2 \left[\cos \left(\frac{\pi}{3} + 2k\pi \right) + i \sin \left(\frac{\pi}{3} + 2k\pi \right) \right]$$

Key Facts



The value of
 $\sin(\theta + 2k\pi) = \sin \theta$ and
 $\cos(\theta + 2k\pi) = \cos \theta$



Key Facts

$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$
 $\text{Arg} \left(\frac{z_1}{z_2} \right) = \text{Arg}(z_1) - \text{Arg}(z_2)$

Example 20: Find the principal arg z , when $z = \frac{-2}{1+i\sqrt{3}}$.

Solution:

$$\begin{aligned} \arg z &= \arg \frac{-2}{1+i\sqrt{3}} = \arg(-2) - \arg(1+i\sqrt{3}) \quad \left(\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \right) \\ &= \left[\pi - \tan^{-1}\left(\frac{0}{2}\right) \right] - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

This implies that one of the values of $\arg z$ is $\frac{2\pi}{3}$.

Since $\frac{2\pi}{3}$ lies between $-\pi$ and π , the principal argument $\text{Arg } z$ is $\frac{2\pi}{3}$.

1.9.3 Properties of Complex Numbers in Polar Form

Property 1:

If $z = r(\cos\theta + i\sin\theta)$, then $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$.

Proof:

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} \\ &= \frac{1}{r(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)} \\ &= \frac{(\cos\theta - i\sin\theta)}{r(\cos^2\theta + \sin^2\theta)} \\ z^{-1} &= \frac{1}{r}(\cos\theta - i\sin\theta) \end{aligned}$$

Property 2:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers in polar form then their sum is given by $z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$.

Proof:

$$\begin{aligned} z_1 + z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1\cos\theta_1 + i r_1\sin\theta_1 + r_2\cos\theta_2 + i r_2\sin\theta_2 \\ &= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2) \end{aligned}$$

Property 3:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers in polar form then their difference is given by

$$z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$

Proof:

$$\begin{aligned} z_1 - z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1\cos\theta_1 + i r_1\sin\theta_1 - r_2\cos\theta_2 - i r_2\sin\theta_2 \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2) \end{aligned}$$

Property 4:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers in polar form then their product is given as

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Proof:

$$\begin{aligned}z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) r_2(\cos \theta_2 + i \sin \theta_2) \\&= r_1 r_2 [(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)] \\&= r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2] \\z_1 z_2 &= r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] \\&= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\&= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]\end{aligned}$$

Or $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Property 5:

If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ are two complex numbers in polar form then their division is given as $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$

Proof:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\&= \frac{r_1}{r_2} \left[\frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \right] \\&= \frac{r_1}{r_2} \left[\frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - i^2 \sin \theta_1 \sin \theta_2}{(\cos \theta_2)^2 - (i \sin \theta_2)^2} \right] \\&= \frac{r_1}{r_2} \left[\frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{(\cos \theta_2)^2 - i^2 (\sin \theta_2)^2} \right] \\&= \frac{r_1}{r_2} \left[\frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \right]\end{aligned}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Or $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Example 21: Find the product $\frac{2}{3} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \times 6 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$ in rectangular form.

Solution:

$$\begin{aligned}\frac{2}{3} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \times 6 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) &= \frac{2}{3} \times 6 \left[\cos \left(\frac{\pi}{3} + \frac{5\pi}{6} \right) + i \sin \left(\frac{\pi}{3} + \frac{5\pi}{6} \right) \right] \\&= 4 \left[\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right] = 4 \left[\cos \left(\pi + \frac{\pi}{6} \right) + i \sin \left(\pi + \frac{\pi}{6} \right) \right] \\&= -4 \cos \frac{\pi}{6} - 4i \sin \frac{\pi}{6} = -4 \left(\frac{\sqrt{3}}{2} \right) - 4i \left(\frac{1}{2} \right) \\&= -2\sqrt{3} - 2i\end{aligned}$$

Which is rectangular form.

Example 22: Find the quotient $\frac{2 \left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right)}{4 \left[\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right]}$ in rectangular form.

Solution:

$$\frac{2 \left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right)}{4 \left[\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right]} = \frac{2}{4} \left[\cos \left(\frac{9\pi}{4} - \left(-\frac{3\pi}{4} \right) \right) + i \sin \left(\frac{9\pi}{4} - \left(-\frac{3\pi}{4} \right) \right) \right]$$

$$= \frac{1}{2} \left[\cos \left(\frac{9\pi}{4} + \frac{3\pi}{4} \right) + i \sin \frac{12\pi}{4} \right] = \frac{1}{2} (\cos 3\pi + i \sin 3\pi) = -\frac{1}{2}$$

Which is in rectangular form.

Example 23: If $z = x + iy$ and $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{2}$, show that $x^2 + y^2 = 1$.

Solution:

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2-1)+y^2}{(x+1)^2+y^2}$$

$$\text{Since, } \arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{2} \Rightarrow \tan^{-1} \frac{0}{\frac{(x^2-1)+y^2}{(x+1)^2+y^2}} = \frac{\pi}{2}$$

$$\frac{0}{x^2-1+y^2} = \tan \frac{\pi}{2} = \frac{1}{0} \Rightarrow x^2 + y^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

Example 24: Find the equation in Cartesian form, if $z = x + iy$ and $\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$.

Solution:

$$\text{Given that } \arg(z-2) - \arg(z+2) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x+iy-2) - \arg(x+iy+2) = \frac{\pi}{4}$$

$$\Rightarrow \arg((x-2)+iy) - \arg((x+2)+iy) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{y}{x-2} - \tan^{-1} \frac{y}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow \tan \left(\tan^{-1} \frac{y}{x-2} - \tan^{-1} \frac{y}{x+2} \right) = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{y}{x-2} - \frac{y}{x+2}}{1 + \left(\frac{y}{x-2} \right) \left(\frac{y}{x+2} \right)} = 1$$

$$\Rightarrow \frac{y(x+2) - y(x-2)}{(x-2)(x+2) + y^2} = 1$$

$$\Rightarrow \frac{xy + 2y - xy + 2y}{x^2 - 4 + y^2} = 1 \quad \Rightarrow 4y = x^2 - 4 + y^2$$

Or

$$\Rightarrow x^2 + y^2 - 4y - 4 = 0$$

1.10 Application of Complex Numbers in Real World

Complex numbers are used in many real-life situations such as cryptography, wave phenomena, pressure and velocity of the fluid and for the calculation of voltage and current in the circuits. These applications are of higher level so will be discussed in higher classes. Here we are going to use the complex numbers by giving an easy example of the simple harmonic motion. In simple harmonic motion we have to determine the position of the microscopic particle from its mean position. The equation which gives the position of the particle from mean position is

$$x = x_{\max} e^{i\theta} \quad (1)$$

Where x is the displacement of the particle from mean position, $x_{\max}$ is the amplitude and $e^{i\theta} = \cos \theta + i \sin \theta$ is the complex number. It is also called Euler's equation.

Example 25: A micro particle is performing to and fro motion. Find its position at an angle of $\frac{\pi}{2}$, when its amplitude is 0.05mm.

Solution:

We are given $x_{\max} = 0.05$

$$\text{and } \theta = \frac{\pi}{2}$$

Using the formula

$$x = x_{\max} e^{i\theta}$$

$$x = 0.05 e^{i\frac{\pi}{2}} = 0.05 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$x = 0.05(0 + i) = 0.05i$$

It means particle is at the position where we cannot see it but just think about it.

The above formula can also be written as $x = x_{\max} e^{i\omega t}$ where ω is the angular velocity and t is the time. Also $\omega = 2\pi f$ where f is the frequency of the particle.

Electrical Engineering:

The flow of electricity I , in a circuit, the resistance to flow Z , called impedance and the electromotive force E , called voltage is given by the formula $E = IZ$. Electrical engineers use j to represent the imaginary units. But for understanding we are representing the imaginary part with i .

Example 26: An electrical engineer is designing a circuit that is to have a current of $(6 - 8i)$ amps. If impedance is $(14 + 8i)$, find the voltage.

Solution:

Here we have $I = (6 - 8i)$

and impedance $Z = (14 + 8i)$,

Using the formula $E = I \times Z$

$$\begin{aligned} E &= I \times Z = (6 - 8i)(14 + 8i) \\ &= 148 - 64i. \end{aligned}$$

Use of complex numbers in the wave phenomenon

Example 27: Compute the potential difference across two AC power supplies with respect to time. When $V_A = 3\cos(t + 30^\circ)$ and $V_B = 4\sin(t + 120^\circ)$.

Solution:

To find the total potential difference, simply adding V_A and V_B is not correct, so we have to follow the method given below

$$\text{Let } Z_A = 3\{\cos(t + 30^\circ) + i\sin(t + 30^\circ)\}$$

$$\text{and } Z_B = 4\{\cos(t + 120^\circ) + i\sin(t + 120^\circ)\}$$

$$\text{therefore, } V_A = \text{Re}(Z_A) \text{ and } V_B = \text{Re}(Z_B)$$

$$\text{Note that } Z_A = 3e^{i(t+30^\circ)} \text{ and } Z_B = 4e^{i(t+120^\circ)}$$

$$Z_{\text{total}} = Z_A + Z_B = 3e^{i(t+30^\circ)} + 4e^{i(t+120^\circ)} = e^{it}(3e^{30^\circ i} + 4e^{120^\circ i})$$

$$Z_{\text{total}} = e^{it}\{3(\cos 30^\circ + i\sin 30^\circ) + 4(\cos 120^\circ + i\sin 120^\circ)\}$$

$$Z_{\text{total}} = e^{it}\left\{3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) + 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right\} = e^{it}\left\{\frac{3\sqrt{3}}{2} + i\frac{3}{2} - 2 + i2\sqrt{3}\right\}$$

$$Z_{\text{total}} = e^{it}\{0.5980762 - 1.964102i\}$$

$$r = \sqrt{(0.5980762)^2 + (1.964102)^2} = \sqrt{25} = 5$$

$$\tan^{-1}\left(\frac{-1.964102}{0.5980762}\right) = 106.936^\circ$$

$$V_{\text{total}} = \text{Re}(Z_{\text{total}}) = 0.5980762e^{it}$$

$$V_{\text{total}} = 5\cos(t + 106.9357^\circ)$$

Use of complex numbers in Resistor in AC circuit

Let us consider a simple AC circuit with a resistor as shown below. Let $v(t)$ be an AC voltage source given by $v(t) = V_o \cos(\omega t) = \text{Re}(V_o e^{i\omega t})$

Where V_o and ω are real quantities

The relation between the current i through and voltage $v(t)$ across the resistor R is given by $v(t)_R = Ri$

Using the single loop shown above we have $v(t) = v(t)_R$

$v(t)$ is given by $v(t) = V_o \cos(\omega t)$

$$\text{Hence } v(t)_R = V_o \cos(\omega t) = \text{Re}(V_o e^{i\omega t})$$

$$\text{Combining the above we write } Ri = \text{Re}(V_o e^{i\omega t}) \quad \dots(i)$$

$$\text{Let } V_R = V_o e^{i\omega t} \text{ and rewrite (i) above as } Ri = \text{Re} V_R$$

$$\text{Since } R \text{ is a real quantity, the above may be written as } i = \text{Re}\left(\frac{V_R}{R}\right)$$

$$\text{Let } Z_R \text{ be defined as the impedance of a resistor such that } Z_R = R$$

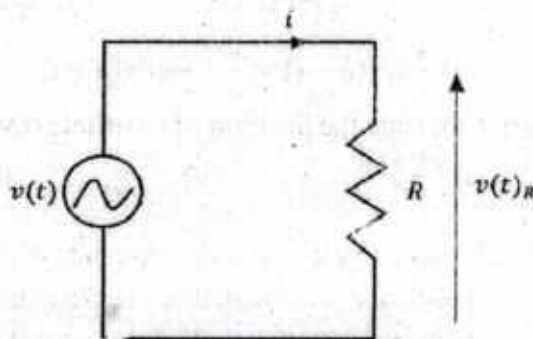
Since R is real, the impedance Z_R of a resistor is a real number

$$\text{Current } i \text{ then may be written as } i = \text{Re}\left(\frac{V_R}{Z_R}\right)$$

$$\text{Let } I = \frac{V_R}{Z_R}$$

The above gives a relationship similar to Ohm's law in DC (Direct Current) circuits. The above relationship between complex quantities I , V_R and R makes calculations much easier.

This simplifies calculations in the sense that we do calculations using complex impedance, voltage and current and then take the real part as the final answer.



Exercise 1.4

1. Write following complex numbers in polar form.

(i) $2 + i2\sqrt{3}$ (ii) $3 - i\sqrt{3}$ (iii) $-2 - i2$

2. Write the complex numbers in rectangular form

(i) $\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ (ii) $\frac{\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}}{2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})}$

3. If $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3) \dots (x_n + iy_n) = a + ib$, show that:

(i) $(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2) \dots (x_n^2 + y_n^2) = a^2 + b^2$

(ii) $\sum_{r=1}^n \tan^{-1} \left(\frac{y_r}{x_r} \right) = \tan^{-1} \left(\frac{b}{a} \right) + 2k\pi, k \in \mathbb{Z}$

4. Write a given complex number in the algebraic form:

(i) $\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$ (ii) $5(\cos 210^\circ + i \sin 210^\circ)$ (iii) $2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$
 (iv) $4 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$ (v) $2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$ (vi) $\cos 135^\circ + i \sin 135^\circ$
 (vii) $10(\cos 50^\circ + i \sin 50^\circ)$ (viii) $\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

5. Convert the following equations and inequations in Cartesian form:

(i) $\arg(z - 1) = -\frac{\pi}{4}$ (ii) $z\bar{z} = 4|e^{i\theta}|$ (iii) $-\frac{\pi}{3} \leq \arg(z - 4) \leq \frac{\pi}{3}$
 (iv) $0 \leq \arg \left(\frac{z-4}{1+i} \right) \leq \frac{\pi}{6}$ (v) $\arg \left(\frac{1-iz}{1-z} \right) = \frac{\pi}{4}; z \neq i$
 (vi) $\frac{1}{2} \arg(z - i) = \frac{\pi}{3} - \frac{1}{2} \arg(z + i)$

6. Calculate the position of a particle from mean position when amplitude is 0.004mm and angle is:

(i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{6}$

7. When particle is at a position of $x = 2 + 3i$ from its mean position and $x_{\max} = 1 + 2\sqrt{3}i$ is the position at maximum distance from mean position as it can be seen under microscope at this point.

- (i) Calculate the angle at time $t = 0$ and find the position of the particle.
 (ii) If $x = 2 + 3i$ and $x_{\max} = 1 + 2\sqrt{3}i$. Calculate the frequency when $t = 2$.

8. Find the impedance Z for the following values:

(i) $E = (-50 + 100i) \text{ volts}, I = (-6 - 2i) \text{ amps}$
 (ii) $E = (100 + 10i) \text{ volts}, I = (-8 + 3i) \text{ amps}$

9. Find the impedance in an AC circuit if $v(t) = 3\cos(t)$ and $i = 10$ amperes.

10. Determine the current flowing through an AC circuit having an impedance of 5 ohm where voltage source is given as $v(t) = 4\cos(3t)$.

11. Compute the potential difference across two AC power supplies with respect to time. Where $V_1 = 5\cos(t + 45^\circ)$ and $V_2 = 3\sin(t + 30^\circ)$.
12. Given that $V_A = 4\cos(t + 120^\circ)$ and $V_B = 5\sin(t + 60^\circ)$. What will be the total potential difference of these AC power supplies?

Review Exercise

1. Choose the correct option:

- (i) Every real number is also a _____ number.
(a) natural (b) integer (c) complex (d) rational
- (ii) Every complex number has _____ part(s).
(a) one (b) two (c) three (d) no
- (iii) Magnitude of a complex number z is the distance of z from _____.
(a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)
- (iv) If z is a complex number then its mirror image is _____.
(a) $|z|$ (b) $1/z$ (c) $-z$ (d) $\bar{z}$
- (v) In complex plane imaginary part is drawn along _____.
(a) x -axis (b) y -axis (c) origin (d) xy -plane
- (vi) If $z_1 = 3 + 2i$ and $z_2 = 5 + 6i$ then
(a) $z_1 > z_2$ (b) $z_1 < z_2$ (c) $|z_1| > |z_2|$ (d) $|z_1| < |z_2|$
- (vii) Diagram representing a complex number is called _____ diagram.
(a) vector (b) Venn (c) argand (d) ordered pair
- (viii) If $z = 3 + 4i$ then z^{-1} is
(a) $(\frac{1}{3}, \frac{1}{4})$ (b) $(-\frac{1}{3}, -\frac{1}{4})$ (c) $(\frac{3}{25}, \frac{-4}{25})$ (d) $(\frac{3}{25}, \frac{-4}{25})$
- (ix) The value of $(\sqrt{-25})(\sqrt{-4})$ is
(a) 10 (b) -10 (c) $10i$ (d) $-10i$
- (x) If $(\frac{1+i}{1-i})^n = 1$ then least positive value of n is
(a) 1 (b) 2 (c) 3 (d) 4

2. Find the values of the following:

- (i) $i^2 + i^4 + i^6 + \dots + i^{100}$
- (ii) $\left| \frac{(3-2i)(1+i)}{2-3i} \right|$
- (iii) $|(3-2i)(4-i)|$
- (iv) $\left(\frac{3+5i}{2-3i} \right)^{-1}$

3. Factorize the following:

- (i) $3x^2 + 108$
- (ii) $4x^2 + 40$

4. Locate the complex number $z = x + iy$ on the complex plane if $\left| \frac{z+2i}{z-2i} \right| = 1$.

5. Find z when $(z - 3i)(2 + 5i) = 3 - 4i$.

6. Evaluate $\left[\frac{1}{i^{10}} + (2 - i)^2 + \sqrt{-25} \right]^3$.

7. Solve by completing square method $2z^2 - 11z + 16 = 0$.

8. When particle is at a position of $\sqrt{2} + i\sqrt{2}$ nm from its mean position calculate its amplitude when $\theta = 45^\circ$.