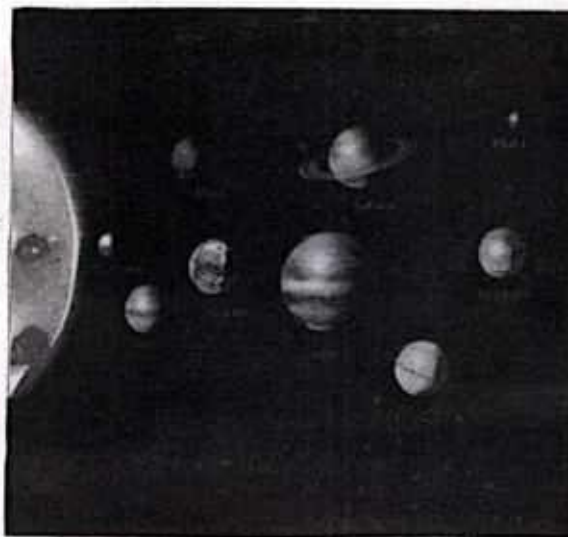


FUNDAMENTAL LAWS OF TRIGONOMETRY

- ❖ Establish fundamental law of trigonometry.
- ❖ Apply fundamental law and its deductions to derive:
 - trigonometric ratios of allied angles.
 - double angle identities.
 - half angle identities.
 - triple angle identities.
- ❖ Express the product of sines and cosines as sums and differences of sines and cosines.
- ❖ Express the sums and differences of sines and cosines as product of sines and cosines.



Trigonometry has a wide range of applications in the sciences, such as, in the measurement of distances between celestial bodies or in satellite navigation systems.

The solar system has fascinated human beings everywhere since the start of civilization. We use trigonometry to find heights of high buildings, trees and mountains etc. and distance of the shore from a point in the sea. Astronomers use trigonometry to calculate how far stars and planets are from Earth. Even though, we know the distances between planets and stars.

8.1 Distance Formula

The formula for the distance between two points whose coordinates are (x_1, y_1) and (x_2, y_2) is:

$$d = AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is called the distance formula.

For example, if $A(5, 4)$ and $B(3, 2)$ are two points in the plane then distance between A and B is:

$$\begin{aligned} AB &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(5 - 3)^2 + (4 - 2)^2} \\ &= \sqrt{(2)^2 + (2)^2} = \sqrt{8} \text{ units} \end{aligned}$$



8.2 Fundamental Law of Trigonometry

This law is stated as:

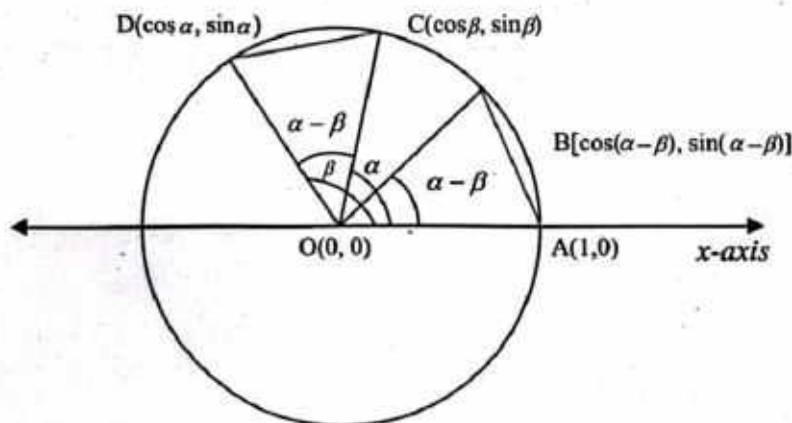
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \text{ where } \alpha > \beta$$

Proof:

Consider a unit circle with centre at O as shown in the figure below. $\overline{OC}$ and $\overline{OD}$ are terminal sides of angles α and β respectively in standard position where $\alpha > \beta$.

The coordinates of points D and C are respectively $(\cos \alpha, \sin \alpha)$ and $(\cos \beta, \sin \beta)$.

Measure of $\angle COD$ is $\alpha - \beta$. An angle $\angle AOB$ equal to measure of $\angle COD = \alpha - \beta$ is constructed in standard position in the same unit circle.



As $\angle COD = \angle AOB = \alpha - \beta$, therefore

$$CD = AB$$

$$\sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

Taking square on both sides, we have:

$$(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2$$

$$\begin{aligned} \Rightarrow \cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \\ = \cos^2(\alpha - \beta) + 1 - 2 \cos(\alpha - \beta) + \sin^2(\alpha - \beta) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \\ &= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta) \quad \dots\dots \text{(Rearranging)} \\ &\Rightarrow 1 + 1 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta = 1 + 1 - 2\cos(\alpha - \beta) \quad \dots\dots (\text{as } \cos^2 \theta + \sin^2 \theta = 1) \\ &\Rightarrow -2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta = -2\cos(\alpha - \beta) \end{aligned}$$

After simplification, we get

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \dots\dots\dots (1)$$

This law is known as fundamental law of trigonometry.

Does the identity (1) make it possible to find

$\cos(90^\circ - 60^\circ) = ?$ Let us check.

$$\cos(90^\circ - 60^\circ) = \cos 90^\circ \cos 60^\circ + \sin 90^\circ \sin 60^\circ$$

$$\cos 30^\circ = 0 \times 0.5 + 1 \times 0.866$$

$$0.866 = 0.866 \text{ which is true.}$$

Now replacing β by $-\beta$ in the above law, we get:

$$\cos(\alpha + \beta) = \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \dots\dots\dots (2)$$

Before proving the other identities, we deduce some important results from the fundamental law.

(i) Letting $\alpha = 0$ in the fundamental law, we get:

$$\cos(0 - \beta) = \cos 0 \cos \beta + \sin 0 \sin \beta$$

$$\Rightarrow \cos(-\beta) = 1 \times \cos \beta + 0 \times \sin \beta$$

$$\cos(-\beta) = \cos \beta$$

(ii) Substituting $\alpha = \frac{\pi}{2}$ in the fundamental law, we have:

$$\cos\left(\frac{\pi}{2} - \beta\right) = \cos \frac{\pi}{2} \cos \beta + \sin \frac{\pi}{2} \sin \beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \beta\right) = 0 \times \cos \beta + 1 \times \sin \beta$$

$$\cos\left(\frac{\pi}{2} - \beta\right) = \sin \beta$$

(iii) Now substituting $\beta = -\frac{\pi}{2}$ in the fundamental law, we have:

$$\cos\left(\alpha - \left(-\frac{\pi}{2}\right)\right) = \cos \alpha \cos\left(-\frac{\pi}{2}\right) + \sin \alpha \sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos\left(\alpha + \frac{\pi}{2}\right) = \cos \alpha \times 0 + \sin \alpha \times (-1)$$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$$

Key Facts

Trigonometric identities are used in simplifying complicated problems in various fields such as calculus, physical and social sciences.



Key Facts

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \sin \theta$$

$$\tan(-\theta) = -\sin \theta$$



Check Point

1. Are the equations

$$\sin \alpha = \cos(90^\circ - \alpha) \text{ and}$$

$\cos \alpha = \sin(90^\circ - \alpha)$ true for all real numbers or only for values of α in the interval

$$0 < \alpha < 90^\circ?$$

2. Amna said that, without finding the values on a calculator, she knows that $\sin 100^\circ = \cos(-10^\circ)$. Do you agree with her? Explain why or why not.

(iv) Replacing β with $\frac{\pi}{2} + \alpha$, the identity $\cos\left(\frac{\pi}{2} - \beta\right) = \sin \beta$ gives:

$$\cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + \alpha\right)\right) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\Rightarrow \cos(-\alpha) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha$$

(v) Replacing α with $\frac{\pi}{2} + \alpha$, the identity (2) gives:

$$\cos\left(\frac{\pi}{2} + \alpha + \beta\right) = \cos\left(\frac{\pi}{2} + \alpha\right)\cos \beta - \sin\left(\frac{\pi}{2} + \alpha\right)\sin \beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2} + (\alpha + \beta)\right) = \cos\left(\frac{\pi}{2} + \alpha\right)\cos \beta - \sin\left(\frac{\pi}{2} + \alpha\right)\sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad \dots\dots\dots (3)$$

Now replacing β by $-\beta$ in identity (3), we get:

$$\sin(\alpha + (-\beta)) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad \dots\dots\dots (4)$$

Example 1: Prove that:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Solution: $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

$$\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

Dividing numerator and denominator by $\cos \alpha \cos \beta$, we get:

$$\frac{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

$$= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \times \frac{\sin \beta}{\cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Check Point

Prove that:

(i) $\sin(-\alpha) = -\sin \alpha$

(ii) $\tan(-\alpha) = -\tan \alpha$

(iii) $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$



Key Facts

The identities of tangent ratio are true for all values of α and β for which $\cos \alpha \neq 0$ and $\cos \beta \neq 0$, and for which $\tan(\alpha + \beta)$ or $\tan(\alpha - \beta)$ are defined.



8.3 Trigonometric Ratios of Allied Angles

The angles connected with basic angles of measure θ by a right angle or its multiple, are called allied angles.

If θ is a basic angle, then angles of measure $\frac{\pi}{2} \pm \theta, \pi \pm \theta, \frac{3\pi}{2} \pm \theta, 2\pi \pm \theta$ etc. are called allied angles.

The following trigonometric ratios can be derived easily with the help of fundamental theorem of trigonometry and its deductions.

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$	$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta$	$\sin(\pi \mp \theta) = \pm \sin \theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$	$\cos(\pi \pm \theta) = -\cos \theta$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta$	$\tan(\pi \pm \theta) = \pm \tan \theta$
$\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos \theta$	$\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm \sin \theta$	$\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp \cot \theta$
$\sin(2\pi \pm \theta) = \pm \sin \theta$	$\cos(2\pi \pm \theta) = \cos \theta$	$\tan(2\pi \pm \theta) = \pm \tan \theta$

Key Facts



- A trigonometric ratio changes to its co-ratio when allied angle contains an odd multiple of right angle. For example sine ratio changes to cosine ratio and vice versa.
- A trigonometric ratio does not change when allied angle contains an even multiple of right angle.
- The sign of ratio will change according the position of terminal arm of angle in the quadrant.
- The above results are also valid for the reciprocals of ratios of sine, cosine and tangent.

Example 2: Use $(60^\circ - 45^\circ) = 15^\circ$ to find the exact value of:

- (i) $\cos 15^\circ$ (ii) $\sin 15^\circ$ (iii) $\tan 15^\circ$

Solution: Letting $\alpha = 60^\circ$ and $\beta = 45^\circ$, we have:

(i) $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$$\cos 15^\circ = \cos(60^\circ - 45^\circ) = \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$$

$$\cos 15^\circ = 0.5 \times 0.707 + 0.866 \times 0.707$$

$$\cos 15^\circ = 0.3535 + 0.6123 = 0.966$$

(ii) $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$$\sin 15^\circ = \sin(60^\circ - 45^\circ) = \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$$

$$\sin 15^\circ = 0.866 \times 0.707 - 0.5 \times 0.707$$

$$\sin 15^\circ = 0.6123 - 0.3535 = 0.259$$

$$(iii) \tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{0.259}{0.966} = 0.268$$

Example 3: If $\sin A = \frac{1}{5}$, and $\sin B = \frac{1}{3}$, where angles A and B are in quadrant II, then find $\cos(A+B)$. In which quadrant does the terminal arm of angle $(A+B)$ lie?

Solution : We use the identity $\cos^2 \theta = 1 - \sin^2 \theta$ to find $\cos A$ and $\cos B$.

$$\cos^2 A = 1 - \sin^2 A$$

$$\cos^2 A = 1 - \left(\frac{1}{5}\right)^2 = 1 - \frac{1}{25} = \frac{24}{25}$$

$$\cos A = -\frac{\sqrt{24}}{5} \quad (\text{Terminal arm of angle is in quad. II})$$

Similarly, $\cos^2 B = 1 - \sin^2 B$

$$\cos^2 B = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\cos B = -\frac{\sqrt{8}}{3} \quad (\text{Terminal arm of angle is in quad. II})$$

Now

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A+B) = \left(-\frac{\sqrt{24}}{5}\right) \times \left(-\frac{\sqrt{8}}{3}\right) - \left(\frac{1}{5}\right) \times \left(\frac{1}{3}\right)$$

$$\cos(A+B) = \frac{\sqrt{192}}{15} - \frac{1}{15} = \frac{8\sqrt{3}-1}{15}$$

As the value of $(A+B)$ is positive, therefore terminal arm of angle lies in fourth quadrant.

Example 4: If α, β, γ are interior angles of a triangle, then prove that:

$$\cot \beta \cot \gamma + \cot \alpha \cot \gamma + \cot \alpha \cot \beta = 1$$

Solution : Given that $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + \beta = 180^\circ - \gamma$$

$$\cot(\alpha + \beta) = \cot(180^\circ - \gamma)$$

$$\frac{1}{\tan(\alpha + \beta)} = \frac{1}{\tan(180^\circ - \gamma)} \Rightarrow \tan(\alpha + \beta) = \tan(180^\circ - \gamma)$$

$$\Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = -\tan \gamma \Rightarrow \tan \alpha + \tan \beta = -\tan \gamma (1 - \tan \alpha \tan \beta)$$

$$\Rightarrow \tan \alpha + \tan \beta = -\tan \gamma + \tan \alpha \tan \beta \tan \gamma \Rightarrow \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$$

Check Point

Find the exact value of $\tan y$, when

$$\tan(y - 45^\circ) = \frac{1}{3}$$

Check Point

Find the exact value of

$$(i) \cos 105^\circ \quad (ii) \cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

Dividing both sides by $\tan \alpha \tan \beta \tan \gamma$.

$$\frac{\tan \alpha}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \beta}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \gamma}{\tan \alpha \tan \beta \tan \gamma} = \frac{\tan \alpha \tan \beta \tan \gamma}{\tan \alpha \tan \beta \tan \gamma}$$

$$\cot \beta \cot \gamma + \cot \alpha \cot \gamma + \cot \alpha \cot \beta = 1$$

8.4 Expressing $a \sin \theta + b \cos \theta$ in the form $r \sin(\theta + \phi)$

Let $P(a, b)$ be a point in the coordinate plane and let θ be the angle that $\overrightarrow{OP}$ makes with x -axis as shown in the figure.

If we let $a = r \cos \phi$ and $b = r \sin \phi$, then

$$\begin{aligned} a \sin \theta + b \cos \theta &= r \cos \phi \sin \theta + r \sin \phi \cos \theta \\ &= r (\cos \phi \sin \theta + \sin \phi \cos \theta) \\ &= r \sin(\theta + \phi) \end{aligned}$$

Where $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$.

For more illustration, let us solve following example.

Example 5:

Express $12 \sin \theta + 5 \cos \theta$ in the form of $r \sin(\theta + \phi)$.

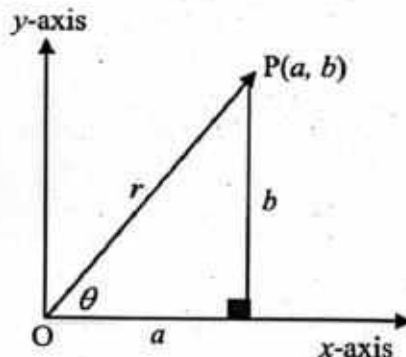
Solution: If we compare $12 \sin \theta + 5 \cos \theta$ with $r \cos \phi \sin \theta + r \sin \phi \cos \theta$, then:

$$a = 12 = r \cos \phi \text{ and } b = 5 = r \sin \phi$$

$$\text{So, } r = \sqrt{a^2 + b^2} = \sqrt{(12)^2 + (5)^2} = \sqrt{169} = 13$$

$$\begin{aligned} \text{Now } 12 \sin \theta + 5 \cos \theta &= 13 \left(\frac{12}{13} \times \sin \theta + \frac{5}{13} \times \cos \theta \right) = 13 \left(\sin \theta \times \frac{12}{13} + \cos \theta \times \frac{5}{13} \right) \\ &= r (\sin \theta \cos \phi + \cos \theta \sin \phi) = r \sin(\theta + \phi) \end{aligned}$$

$$\text{Where } r = 13 \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{5}{12}\right).$$



Check Point



Express $\cos \theta + \sin \theta$ in the form of $r \sin(\theta + \phi)$.

Exercise 8.1

1. Find the values of $\cos(\alpha \pm \beta)$, $\sin(\alpha \pm \beta)$ and $\tan(\alpha \pm \beta)$ for each given pair of angles.

(i) $\alpha = 180^\circ$, $\beta = 60^\circ$ (ii) $\alpha = 90^\circ$, $\beta = 60^\circ$ (iii) $\alpha = 180^\circ$, $\beta = 30^\circ$

(iv) $\alpha = \pi$, $\beta = \frac{\pi}{3}$ (v) $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{6}$ (vi) $\alpha = \frac{3\pi}{2}$, $\beta = \frac{\pi}{4}$

2. a. Find the exact value of $\cos 15^\circ$ by using $\cos(45^\circ - 30^\circ)$.

b. Use the value of $\cos 15^\circ$ found in a to find $\cos 165^\circ$ by using $\cos(180^\circ - 15^\circ)$.

c. Use the value of $\cos 15^\circ$ found in a to find $\cos 345^\circ$ by using $\cos(360^\circ - 15^\circ)$.

d. Find the exact value of $\sin 75^\circ$.

3. a. Find the exact value of $\cos 120^\circ$ by using $\cos(180^\circ - 60^\circ)$ and $\cos(90^\circ + 30^\circ)$.

b. Find the exact value of $\sin 120^\circ$ and then $\tan 120^\circ$.

c. Find the exact value of $\cos 75^\circ$ by using $\cos(120^\circ - 45^\circ)$.

d. Use the value of $\cos 75^\circ$ found in c to find $\cos 105^\circ$ by using $\cos(180^\circ - 75^\circ)$.

e. Use the value of $\cos 75^\circ$ found in c to find $\cos 285^\circ$ by using $\cos(360^\circ - 75^\circ)$.

f. Find the exact value of $\sin 15^\circ$.

4. Rewrite as a single expression.

(i) $\cos 6\theta \cos 3\theta - \sin 6\theta \sin 3\theta$

(ii) $\cos 7\theta \cos 2\theta + \sin 7\theta \sin 2\theta$

(iii) $\sin\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{6}\right) + \cos\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{6}\right)$

(iv) $\sin 138^\circ \cos 46^\circ - \cos 138^\circ \sin 46^\circ$

(v) $\frac{\tan 75^\circ - \tan 45^\circ}{1 + \tan 75^\circ \tan 45^\circ}$

(vi) $\frac{\tan \frac{4\pi}{3} + \tan \frac{2\pi}{3}}{1 - \tan \frac{4\pi}{3} \tan \frac{2\pi}{3}}$

5. For $\sin \alpha = \frac{4}{5}$, $\tan \beta = -\frac{5}{12}$ with terminal sides of angles α and β lie quad. II, find $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

6. For $\cos \alpha = -\frac{7}{25}$ and $\cot \beta = \frac{15}{8}$ with terminal sides of α in quad. II and β in quad. III

find: (i) $\sin(\alpha - \beta)$ (ii) $\cos(\alpha - \beta)$ (iii) $\tan(\alpha - \beta)$

7. Given α and β are acute angles with $\sin \alpha = \frac{12}{13}$ and $\tan \beta = \frac{4}{3}$ find:

(i) $\sin(\alpha + \beta)$ (ii) $\cos(\alpha + \beta)$ (iii) $\tan(\alpha + \beta)$

8. If $\sin \alpha = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$ and $\cos \beta = \frac{12}{13}$, where $\frac{3\pi}{2} < \beta < 2\pi$ find:

(i) $\csc(\alpha + \beta)$ (ii) $\sec(\alpha + \beta)$ (iii) $\cot(\alpha + \beta)$

9. Given α and β are obtuse angles with $\sin \alpha = \frac{1}{\sqrt{2}}$ and $\cos \beta = -\frac{3}{5}$ find:

(i) $\sin(\alpha \pm \beta)$ (ii) $\cos(\alpha \pm \beta)$ (iii) $\tan(\alpha \pm \beta)$

10. Verify:

$$(i) \sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$$

$$(ii) \cos(\pi - \alpha) = -\cos \alpha$$

$$(iii) \cos\left(\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos \alpha - \sin \alpha) \quad (iv) \sin\left(\beta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos \beta + \sin \beta)$$

$$(v) \tan\left(\gamma - \frac{\pi}{4}\right) = \frac{\tan \gamma - 1}{\tan \gamma + 1} \quad (vi) \tan\left(\gamma + \frac{\pi}{4}\right) = \frac{1 + \tan \gamma}{1 - \tan \gamma} = \frac{\cos \gamma + \sin \gamma}{\cos \gamma - \sin \gamma}$$

$$(vii) \cos(x+y) + \cos(x-y) = 2\cos x \cos y \quad (viii) \sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

11. Show that:

$$(i) \frac{\sin(180^\circ + \lambda)\cos(270^\circ + \lambda)}{\sin(180^\circ - \lambda)\cos(270^\circ - \lambda)} = 1 \quad (ii) \frac{\sin(90^\circ + \alpha) - \cos(360^\circ - \alpha) + \cos \alpha}{\sin(180^\circ - \alpha) + \sin(270^\circ - \alpha) + \cos(90^\circ + \alpha)} = -1$$

$$(iii) \tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$$

$$(iv) \sin(\alpha + \beta)\sin(\alpha - \beta) = \cos^2 \beta - \cos^2 \alpha = \sin^2 \alpha - \sin^2 \beta$$

$$(v) \frac{\tan(x+y)}{\cot(x-y)} = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y} \quad (vi) \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan \alpha \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$(vii) \cot(\alpha - \beta) = \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha} \quad (viii) \frac{\cos 4\theta}{\csc \theta} + \frac{\sin 4\theta}{\sec \theta} = \sin 5\theta$$

12. If $\alpha + \beta + \gamma = 180^\circ$, prove that:

$$(i) \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma \quad (ii) \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

$$(iii) \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} - 1 = 0$$

13. Express the following in the form of $r \sin(\theta + \phi)$.

$$(i) 12\sin \theta - 5\cos \theta \quad (ii) 3\sin \theta + 4\cos \theta \quad (iii) \sin \theta - \cos \theta$$

14. A telephone pole is braced by two wires that are both fastened to the ground at a point 3m from the base of the pole. The shorter wire is fastened to the pole 3m above the ground and the longer wire 7m above the ground.

- What is the measure, in degrees, of the angle that the shorter wire makes with the ground?
- Let θ be the measure of the angle that the longer wire makes with the ground. Find $\sin \theta$ and $\cos \theta$.
- Find the cosine of the angle between the wires where they meet at the ground.
- Find, to the nearest degree, the measure of the angle between the wires.

8.5 Double, Half and Triple Angle Identities

(i) Double Angle Identities

The double-angle identities for sine, cosine, and tangent can be derived by putting $\alpha = \beta$ in the following identities.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (1)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (2)$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (3)$$

Putting $\beta = \alpha$ in identity (1), we get:

$$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha \quad (4)$$

Now putting $\beta = \alpha$ in identity (2), we get:

$$\cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \quad (5)$$

Using relation $\cos^2 \alpha + \sin^2 \alpha = 1$, the identity (5) becomes:

$$\cos(2\alpha) = 2 \cos^2 \alpha - 1 \quad (6)$$

$$\cos(2\alpha) = 1 - 2 \sin^2 \alpha \quad (7)$$

Relations (6) and (7) also imply:

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \Rightarrow \cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}} \quad (8)$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \Rightarrow \sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}} \quad (9)$$

Dividing identity (9) by (8), we have:

$$\tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} \quad (10)$$

Again putting $\beta = \alpha$ in identity (3), we get:

$$\tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

$$\tan(2\alpha) = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad (11)$$

Check Point



Does $\cos 2\theta = \sin 2(90^\circ - \theta)$?

Justify your answer.

Check Point



Use the double angle formula to find the exact value of $\sin 120^\circ$.

(ii) Half Angle Identities

Substituting $\alpha = \frac{\theta}{2}$ in above identities, we get the following relations.

From identity (4), we get:

$$\sin\left(2 \times \frac{\theta}{2}\right) = 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \Rightarrow \sin(\theta) = 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \quad (12)$$

Similarly, identities (5) to (7) imply:

$$\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2 \cos^2\left(\frac{\theta}{2}\right) - 1 = 1 - 2 \sin^2\left(\frac{\theta}{2}\right) \quad (13)$$

And from identity (11):

$$\tan\left(2 \times \frac{\theta}{2}\right) = \frac{2 \tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)} \Rightarrow \tan(\theta) = \frac{2 \tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)} \quad (14)$$

From identities (8) and (9), we have:

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad (15)$$

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad (16)$$

Now dividing identity (16) by (15), we have:

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \quad (17)$$

These identities are useful in simplifying complex trigonometric expressions.

(iii) Triple Angle Identities

$\sin 3\alpha$, $\cos 3\alpha$ and $\tan 3\alpha$ etc. are called triple angle identities. Let's prove these identities.

$$\begin{aligned} (a) \sin(3\alpha) &= \sin(2\alpha + \alpha) = \sin(2\alpha)\cos\alpha + \cos(2\alpha)\sin\alpha \\ &= (2\sin\alpha\cos\alpha)\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha \quad \dots \text{by (4) and (7)} \\ &= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha \\ &= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \quad (\because \sin^2\alpha + \cos^2\alpha = 1) \\ &= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha \\ \sin(3\alpha) &= 3\sin\alpha - 4\sin^3\alpha \quad (18) \end{aligned}$$

$$\begin{aligned}
 (b) \cos(3\alpha) &= \cos(2\alpha + \alpha) = \cos(2\alpha)\cos\alpha - \sin(2\alpha)\sin\alpha \\
 &= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha \quad \dots \text{by (4) and (6)} \\
 &= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha \\
 &= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha \quad \dots (\because \sin^2\alpha + \cos^2\alpha = 1) \\
 &= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha
 \end{aligned}$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha \quad (19)$$

$$(c) \tan(3\alpha) = \tan(2\alpha + \alpha)$$

$$\begin{aligned}
 &= \frac{\tan(2\alpha) + \tan\alpha}{1 - \tan(2\alpha)\tan\alpha} = \frac{\frac{2\tan\alpha}{1 - \tan^2\alpha} + \tan\alpha}{1 - \left(\frac{2\tan\alpha}{1 - \tan^2\alpha}\right)\tan\alpha} \quad \dots \text{by (11)} \\
 &= \frac{2\tan\alpha + \tan\alpha(1 - \tan^2\alpha)}{1 - \tan^2\alpha - 2\tan^2\alpha} = \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}
 \end{aligned}$$

$$\tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \quad (20)$$

Example 6: Given $\sin\theta = \frac{3}{5}$, where θ lies in quad. I, find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

Solution: First we find the value of $\cos\theta$.

$$\cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos^2\theta = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$$

$$\cos\theta = \frac{4}{5}$$

Now (i) $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin 2\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

(ii) $\cos 2\theta = 1 - 2\sin^2\theta$

$$\cos 2\theta = 1 - 2 \times \left(\frac{3}{5}\right)^2 = 1 - \frac{18}{25} = \frac{7}{25}$$

Check Point

Find $\sin\left(\frac{\theta}{2}\right)$ when

$\cos\theta = -\frac{3}{5}$ where $\frac{\pi}{2} < \theta < \pi$.

$$(iii) \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24/25}{7/25} = \frac{24}{7}$$

Example 7: Use the half-angle identities to find exact values for: (a) $\sin 15^\circ$ (b) $\tan 15^\circ$

Solution:

$$(a) \sin 15^\circ = \sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - 0.866}{2}} = \sqrt{\frac{0.134}{2}} = \sqrt{0.067} = 0.259$$

$$(b) \tan 15^\circ = \tan\left(\frac{30^\circ}{2}\right) = \frac{1 - \cos 30^\circ}{\sin 30^\circ} = \sqrt{\frac{1 - 0.866}{0.5}} = \frac{0.134}{0.5} = 0.268$$

Since, $\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{\sin \theta}$

Example 8: For $\cos \alpha = \frac{-7}{25}$ and α in quad. III, find values of $\sin\left(\frac{\alpha}{2}\right)$ and $\cos\left(\frac{\alpha}{2}\right)$.

Solution: When $\pi < \alpha < \frac{3\pi}{2}$ then $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$. Thus, $\sin\left(\frac{\alpha}{2}\right) > 0$ and $\cos\left(\frac{\alpha}{2}\right) < 0$.

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - \left(\frac{-7}{25}\right)}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{1 + \left(\frac{-7}{25}\right)}{2}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

Example 9: Express $4\sin^4 x$ in terms of an expression containing only cosines to the power 1.

$$\begin{aligned} \text{Solution: } 4\sin^4 x &= 4(\sin^2 x)^2 = 4\left(\frac{1 - \cos 2x}{2}\right)^2 \\ &= 4\left(\frac{1 - 2\cos 2x + \cos^2(2x)}{4}\right) = 1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \\ &= \frac{2 - 4\cos 2x + 1 + \cos 4x}{2} = \frac{3 - 4\cos 2x + \cos 4x}{2} \end{aligned}$$

Example 10: Find the exact value of $\sin 22.5^\circ \cdot \cos 22.5^\circ$.

Solution: As $\sin(2\alpha) = 2\sin \alpha \cos \alpha$

$$\therefore \sin \alpha \cos \alpha = \frac{\sin(2\alpha)}{2}$$

Check Point

Verify

$(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$

Substituting the value of angle, we get:

$$\begin{aligned}\sin 22.5^\circ \cdot \cos 22.5^\circ &= \frac{\sin(2 \cdot 22.5^\circ)}{2} \\ &= \frac{\sin(2 \cdot 22.5^\circ)}{2} = \frac{\sin 45^\circ}{2} = 0.345\end{aligned}$$

Example 11: Prove that: $\tan\left(\frac{\theta}{2}\right) = \pm \left(\frac{1 - \cos \theta}{\sin \theta}\right)$

Solution: We know that:

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \pm \sqrt{\frac{(1 - \cos \theta)(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}} \\ &= \pm \sqrt{\frac{(1 - \cos \theta)^2}{1 - \cos^2 \theta}} = \pm \sqrt{\frac{(1 - \cos \theta)^2}{\sin^2 \theta}} = \pm \left(\frac{1 - \cos \theta}{\sin \theta}\right)\end{aligned}$$

Example 12: Prove that: $\frac{\sin \theta \cos \theta}{0.5 \cos 2\theta} = \tan 2\theta$

$$\begin{aligned}\text{Solution: L.H.S} &= \frac{\sin \theta \cos \theta}{0.5 \cos 2\theta} = \frac{2 \cdot \sin \theta \cos \theta}{2 \cdot 0.5 \cos 2\theta} \\ &= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \text{R.H.S}\end{aligned}$$

Exercise 8.2

- Suppose P (-3, 4) lies on the terminal side of θ when θ is plotted in standard position. Find $\cos 2\theta$ and $\sin 2\theta$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.
- If $\sin \alpha = y$ and α lies in quad. II. Find expressions for $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ in terms of y .
- Use a half angle formula to find the exact value of $\cos 15^\circ$.
- Find (a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$ (d) $\sin \frac{\theta}{2}$ (e) $\cos \frac{\theta}{2}$ (f) $\tan \frac{\theta}{2}$ when:
 - $\cos \theta = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$
 - $\tan \theta = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
 - $\sin \theta = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$
 - $\sec \theta =$ where $\frac{3\pi}{2} < \theta < 2\pi$
 - $\csc \theta = \sqrt{2}$ where $\frac{\pi}{2} < \theta < \pi$
 - $\cos \theta = -\frac{1}{2}$ where $\frac{\pi}{2} < \theta < \pi$

5. Find exact values for $\sin \theta$, $\cos \theta$ and $\tan \theta$ using the information given.

(i) $\sin 2\theta = \frac{24}{25}$, 2θ in quad. II (ii) $\cos 2\theta = -\frac{7}{25}$, 2θ in quad. III

(iii) $\sin 2\theta = -\frac{240}{289}$, 2θ in quad. III (iv) $\cos 2\theta = \frac{120}{169}$, 2θ in quad. IV

6. Use a double-angle identity to find exact values for the following expressions.

(i) $\sin 15^\circ \cos 15^\circ$ (ii) $\cos^2 15^\circ - \sin^2 15^\circ$ (iii) $1 - 2\sin^2\left(\frac{\pi}{8}\right)$

(iv) $2\cos^2\left(\frac{\pi}{12}\right) - 1$ (v) $\frac{2\tan\left(\frac{\pi}{12}\right)}{1 - \tan^2\left(\frac{\pi}{12}\right)}$

7. Rewrite in terms of an expression containing only cosines to the power 1.

(i) $\sin^2 \alpha \cos^2 \alpha$ (ii) $\sin^4 \alpha \cos^2 \alpha$ (iii) $\sin^4 \alpha \cos^4 \alpha$

8. Verify the following identities.

(i) $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$ (ii) $\tan 2x = \frac{1}{1 - \tan x} - \frac{1}{1 + \tan x}$

(iii) $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$ (iv) $\csc 2\alpha = \frac{\tan \alpha + \cot \alpha}{2}$

(v) $8\sin^4 \theta = 3 + \cos 4\theta - 4\cos 2\theta$ (vi) $\sin 4\theta = 4\sin \theta \cos^3 \theta - 4\sin^3 \theta \cos \theta$

(vii) $\sin 2\theta = 2\cot \theta \sin^2 \theta$ (viii) $\cos^2 2x + 4\sin^2 x \cos^2 x = 1$

(ix) $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$ (x) $\sec 2x = \frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\cos x - \sin x}$

(xi) $\cos^4 x - \sin^4 x = \cos 2x$ (xii) $\tan \frac{\beta}{2} + \cot \frac{\beta}{2} = 2\csc \beta$

(xiii) $\csc 2\alpha - \cot 2\alpha = \tan \alpha$ (xiv) $\frac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = \frac{2 + \sin 2x}{2}$

(xv) $\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} = 2$ (xvi) $\frac{1 - \cos^2 \beta}{2 - 2\cos \beta} = \cos^2 \frac{\beta}{2}$

(xvii) $\frac{\sin \theta}{1 + \cos \theta} = \tan \frac{\theta}{2}$ (xviii) $\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \cos x$

(xix) $\frac{\sin 2\alpha}{\sin \alpha} - \frac{\cos 2\alpha}{\cos \alpha} = \sec \alpha$ (xx) $2\sin^2 \frac{\beta}{2} + \cos \beta = 1$

(xxi) $2\cos y \sec 2y = \frac{1}{\cos y - \sin y} + \frac{1}{\cos y + \sin y}$

(xxii) $2\sin y \sec 2y = \frac{1}{\cos y - \sin y} - \frac{1}{\cos y + \sin y}$

8.6 Sum, Difference and Product of Sines and Cosines

8.6.1 Expressing the Product of Sines and Cosines as Sums or Differences

We have proved the following identities:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (21)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (22)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (23)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (24)$$

Addition of identities (21) and (22), and then (23) and (24) gives:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta \quad (25)$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta \quad (26)$$

Subtracting (22) from (21), we get:

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta \quad (27)$$

Now, subtracting (24) from (23), we get:

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin \alpha \sin \beta \quad (28)$$

Identities (25) to (28) can be re-written as:

$$\begin{aligned} 2 \sin \alpha \cos \beta &= \sin(\alpha + \beta) + \sin(\alpha - \beta) \\ 2 \cos \alpha \sin \beta &= \sin(\alpha + \beta) - \sin(\alpha - \beta) \\ 2 \cos \alpha \cos \beta &= \cos(\alpha + \beta) + \cos(\alpha - \beta) \\ -2 \sin \alpha \sin \beta &= \cos(\alpha + \beta) - \cos(\alpha - \beta) \end{aligned}$$

Example 13: Express the product $2 \cos 6\theta \sin 3\theta$ as a sum or difference of sine and cosine.

Solution: Using the identity (6), we can write:

$$\begin{aligned} 2 \cos 6\theta \sin 3\theta &= \sin(6\theta + 3\theta) - \sin(6\theta - 3\theta) \\ &= \sin(9\theta) - \sin(3\theta) \end{aligned}$$

Check Point

Express $\sin 60^\circ \cos 30^\circ$ as a sum or difference of sine and cosine and simplify.

Example 14: Simplify $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$ after converting into sum or difference of sine and cosine.

Solution:

$$\begin{aligned} \sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ &= \frac{1}{2}(2 \sin 40^\circ \cos 20^\circ) + \frac{1}{2}(2 \cos 40^\circ \sin 20^\circ) \\ &= \frac{1}{2}\{\sin(40^\circ + 20^\circ) + \sin(40^\circ - 20^\circ)\} + \frac{1}{2}\{\sin(40^\circ + 20^\circ) - \sin(40^\circ - 20^\circ)\} \\ &= \frac{1}{2}\{\sin 60^\circ + \sin 20^\circ\} + \frac{1}{2}\{\sin 60^\circ - \sin 20^\circ\} = \frac{1}{2}\{\sin 60^\circ + \sin 20^\circ + \sin 60^\circ - \sin 20^\circ\} \\ &= \frac{1}{2}\{2 \sin 60^\circ\} = \frac{\sqrt{3}}{2} \end{aligned}$$

8.6.2 Expressing the Sums or Differences of Sines and Cosines as Product

Let $\alpha = \frac{p+q}{2}$ and $\beta = \frac{p-q}{2}$, then

$$\alpha + \beta = \frac{p+q}{2} + \frac{p-q}{2} = \frac{2p}{2} = p \text{ and } \alpha - \beta = \frac{p+q}{2} - \frac{p-q}{2} = \frac{2q}{2} = q$$

Substituting values of α and β into identities (5) to (8) of section we get:

$$\sin p + \sin q = 2 \sin \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\sin p - \sin q = 2 \cos \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

$$\cos p + \cos q = 2 \cos \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\cos p - \cos q = -2 \sin \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

Above identities are known as sum to product formulae.

Example 15: Express $\cos 45^\circ - \cos 15^\circ$ as product.

Solution:
$$\begin{aligned} \cos 45^\circ - \cos 15^\circ &= -2 \sin \left(\frac{45^\circ + 15^\circ}{2} \right) \sin \left(\frac{45^\circ - 15^\circ}{2} \right) \\ &= -2 \sin 30^\circ \sin 15^\circ \end{aligned}$$

Example 16: Show that $\frac{\sin x + \sin y}{\cos x + \sin y} = \tan \frac{x+y}{2}$

Solution:
$$\begin{aligned} \frac{\sin x + \sin y}{\cos x + \sin y} &= \frac{2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}} \\ &= \tan \frac{x+y}{2} \end{aligned}$$

Check Point

Provide two different methods of calculating $\cos 195^\circ \cos 105^\circ$, one of which uses the product to sum. Which method is easier?

Example 17: Show that: $\cos 6\alpha + \cos 5\alpha + \cos 3\alpha + \cos 2\alpha = 4 \cos(4\alpha) \cos(1.5\alpha) \cos(0.5\alpha)$

Solution:
$$\begin{aligned} \cos 6\alpha + \cos 5\alpha + \cos 3\alpha + \cos 2\alpha &= (\cos 6\alpha + \cos 2\alpha) + (\cos 5\alpha + \cos 3\alpha) \\ &= 2 \cos \frac{6\alpha + 2\alpha}{2} \cos \frac{6\alpha - 2\alpha}{2} + 2 \cos \frac{5\alpha + 3\alpha}{2} \cos \frac{5\alpha - 3\alpha}{2} \\ &= 2 \cos 4\alpha \cos 2\alpha + 2 \cos 4\alpha \cos \alpha = 2 \cos 4\alpha (\cos 2\alpha + \cos \alpha) \\ &= 2 \cos 4\alpha \times 2 \cos \frac{2\alpha + \alpha}{2} \cos \frac{2\alpha - \alpha}{2} = 4 \cos(4\alpha) \cos(1.5\alpha) \cos(0.5\alpha) \end{aligned}$$

Example 18: Show that: $\sin 70^\circ \sin 30^\circ \cos 20^\circ \cos 10^\circ = \frac{1}{4} \cos 10^\circ + \frac{\sqrt{3}}{16} + \frac{1}{8} \sin 40^\circ$

Solution:
$$\begin{aligned} \sin 70^\circ \sin 30^\circ \cos 20^\circ \cos 10^\circ \\ &= \frac{1}{2} (\sin 70^\circ \cos 20^\circ \cos 10^\circ) \dots \left(\sin 30^\circ = \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}(2\sin 70^\circ \cos 20^\circ) \cos 10^\circ \\
&= \frac{1}{4}[\sin(70^\circ + 20^\circ) + \sin(70^\circ - 20^\circ)] \cos 10^\circ \\
&= \frac{1}{4}[\sin 90^\circ + \sin 50^\circ] \cos 10^\circ = \frac{1}{4}[1 + \sin 50^\circ] \cos 10^\circ \\
&= \frac{1}{4} \cos 10^\circ + \frac{1}{4} \sin 50^\circ \cos 10^\circ = \frac{1}{4} \cos 10^\circ + \frac{1}{8} \times 2 \sin 50^\circ \cos 10^\circ \\
&= \frac{1}{4} \cos 10^\circ + \frac{1}{8}[\sin(50^\circ + 10^\circ) + \sin(50^\circ - 10^\circ)] \\
&= \frac{1}{4} \cos 10^\circ + \frac{1}{8}[\sin 60^\circ + \sin 40^\circ] = \frac{1}{4} \cos 10^\circ + \frac{1}{8} \left(\frac{\sqrt{3}}{2} + \sin 40^\circ \right) \\
&= \frac{1}{4} \cos 10^\circ + \frac{\sqrt{3}}{16} + \frac{1}{8} \sin 40^\circ
\end{aligned}$$

Exercise 8.3

- Use the product-to-sum formula to change the following to sum or difference.
 - $4\sin 16x \cos 10x$
 - $10\cos 10y \cos 6y$
 - $2\cos 5t \sin 3t$
 - $6\cos 5x \sin 10x$
 - $\sin(-u) \sin 5u$
 - $-2\sin(-100^\circ) \sin(-20^\circ)$
 - $\cos 23^\circ \sin 17^\circ$
 - $2\cos 56^\circ \sin 48^\circ$
 - $2\sin 75^\circ \sin 15^\circ$
 - $4\sin \frac{u+v}{2} \cos \frac{u-v}{2}$
 - $2\cos \frac{2u+2v}{2} \sin \frac{2u-2v}{2}$
- Rewrite the sum or difference as a product of two functions.
 - $\sin 70^\circ + \sin 30^\circ$
 - $\sin 76^\circ - \sin 14^\circ$
 - $\cos 58^\circ - \cos 12^\circ$
 - $\cos \frac{p-q}{2} + \cos \frac{p+q}{2}$
 - $\sin(-10^\circ) + \sin(-20^\circ)$
- Prove the following identities.
 - $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan \alpha \tan \beta}{1 + \tan \alpha \tan \beta}$
 - $\frac{6\cos 8u \sin 2u}{\sin(-6u)} = \frac{-3\sin 10u}{\sin 6u} + 3$
 - $4\cos 4v \sin 3v = 2(\sin 7v - \sin v)$
 - $\sin 3\theta + \sin \theta = 4\cos^2 \theta \sin \theta$
 - $\cos 3x + \cos x = 2\cos x(\cos 2x)$
 - $2\tan y \cos 3y = \sec y(\sin 4y - \sin 2y)$
 - $\frac{\sin 6\beta + \sin 4\beta}{\sin 6\beta - \sin 4\beta} = \tan 5\beta \cot \beta$
 - $\frac{\cos 3\theta + \cos \theta}{\cos 3\theta - \cos \theta} = -\cot 2\theta \cot \theta$

- (ix) $\frac{\cos 6x + \cos 8x}{\sin 6x - \sin 4x} = \cot x \cos 7x \sec 5x$ (x) $\frac{\cos 2\alpha - \cos 4\alpha}{\sin 2\alpha + \sin 4\alpha} = \tan \alpha$
 (xi) $2\cos 2u \cos u + \sin 2u \sin u = 2\cos^3 u$ (xii) $2\sin 2y \sin 3y = \cos y - \cos 5y$
 (xiii) $\frac{\cos 10x + \cos 6x}{\cos 6x - \cos 10x} = \cot 2x \cot 8x$

4. Prove that.

- (i) $\cos 80^\circ \cos 60^\circ \cos 40^\circ \cos 20^\circ = \frac{1}{16}$ (ii) $\sin 70^\circ \sin 50^\circ \sin 30^\circ \sin 10^\circ = \frac{1}{16}$
 (iii) $\sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{3\pi}{9} \sin \frac{4\pi}{9} = \frac{3}{16}$

Review Exercise

1. Select the correct option in the following.

- (i) $\sin(45^\circ - 30^\circ) = \dots$
 (a) $\frac{\sqrt{6} - \sqrt{2}}{4}$ (b) $\frac{\sqrt{6} + \sqrt{2}}{4}$ (c) $\frac{\sqrt{6} - \sqrt{2}}{2}$ (d) $\frac{\sqrt{3} - \sqrt{2}}{2}$
- (ii) $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \dots$
 (a) $\frac{\sqrt{3} - 1}{\sqrt{3} + 1}$ (b) $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ (c) $\frac{\sqrt{3} + 1}{-\sqrt{3} - 1}$ (d) $\frac{\sqrt{3} + 1}{-\sqrt{3} + 1}$
- (iii) $\sin 22.5^\circ \cos 22.5^\circ + \cos 22.5^\circ \sin 22.5^\circ = \dots$
 (a) $\frac{-1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{-1}{\sqrt{2}}$
- (iv) $\cos(\pi - \theta) = \dots$
 (a) $\sec \theta$ (b) $\pm \cos \theta$ (c) $\cos \theta$ (d) $-\cos \theta$
- (v) $\tan\left(\frac{\pi}{2} + \theta\right) = \dots$
 (a) $\cot \theta$ (b) $-\cot \theta$ (c) $\tan \theta$ (d) $-\tan \theta$
- (vi) $2\sin \alpha \cos \alpha = \dots$
 (a) $\sin(\pi - 2\alpha)$ (b) $\sin(\pi + 2\alpha)$ (c) $\sin(-2\alpha)$ (d) $\sin 2(\pi - \alpha)$
- (vii) $\frac{\sin 2\alpha \cos \alpha}{\cos^3 \alpha - \cos \alpha \sin^2 \alpha} = \dots$
 (a) $\csc 2\alpha$ (b) $-\sec 2\alpha$ (c) $\tan 2\alpha$ (d) $-\tan 2\alpha$

(viii) If $\sin \beta = \frac{3}{5}$, then $\cos 2\beta = \dots$

(a) $\frac{-7}{5}$

(b) $\frac{7}{5}$

(c) $\frac{-7}{25}$

(d) $\frac{7}{25}$

(ix) $\cos^2 3x - \sin^2 3x = \dots$

(a) $\sin 6x$

(b) $\cos 6x$

(c) $-\sin 6x$

(d) $-\cos 6x$

(x) $(\sin x - \cos x)^2 = \dots$

(a) $1 + \sin 2x$

(b) $1 - \cos 2x$

(c) $1 - \sin 2x$

(d) $1 + \cos \sin 2x$

(xi) $\cos(60^\circ - 30^\circ) \neq \dots$

(a) $\cos 30^\circ$

(b) $\sin 60^\circ$

(c) $\sqrt{1 - \sin^2 30^\circ}$

(d) $\cos 60^\circ - \cos 30^\circ$

(xii) $\frac{1 - \cos x}{\sin x} = \dots$

(a) $\tan\left(\frac{x}{2}\right)$

(b) $\cot\left(\frac{x}{2}\right)$

(c) $-\tan\left(\frac{x}{2}\right)$

(d) $-\cot\left(\frac{x}{2}\right)$

2. Given that $\sin \theta = \frac{3}{5}$, $\sin \phi = \frac{5}{13}$ where θ is obtuse and ϕ is acute. Find the values of:

(i) $\sin(\theta - \phi)$

(ii) $\tan(\theta - \phi)$

(iii) $\tan(\theta + \phi)$

3. Express the following as single trigonometric ratios.

(i) $\frac{1}{\sqrt{2}}(\sin \beta + \cos \beta)$

(ii) $\frac{1}{\sqrt{2}} \sin 75^\circ + \frac{1}{\sqrt{2}} \cos 75^\circ$

4. Find the values of:

(i) $\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ}$

(ii) $\cos 70^\circ \cos 20^\circ - \sin 70^\circ \sin 20^\circ$

5. Find the values of $\tan \theta$ when $\tan(\theta - 45^\circ) = \frac{1}{3}$.

6. (i) If $\sin(\alpha + \theta) = 2\cos(\alpha - \theta)$ prove that $\tan \alpha = \frac{2 - \tan \theta}{1 - 2 \tan \theta}$.

(ii) If $\sin(\alpha - \theta) = \cos(\alpha + \theta)$ prove that $\tan \alpha = 1$.

7. Show that: (i) $\frac{4\sin^2 \theta \cos \theta}{\cos 3\theta + \cos \theta} = \tan 2\theta \tan \theta$ (ii) $\frac{\sin 10\theta - \sin 4\theta}{\sin 4\theta + \sin 2\theta} = \cos 7\theta \sec \theta$

8. Prove that: (i) $\sqrt{\frac{\cos(90^\circ + x) \sec(-x) \tan(180^\circ - x)}{\sec(360^\circ - x) \sin(180^\circ + x) \cot(90^\circ - x)}} = i$

(ii) $\frac{\tan^2\left(\frac{3\pi}{2} - x\right) \sin^2(\pi + x) \sin(2\pi - x)}{\cos^2(\pi - x) \cot\left(\frac{3\pi}{2} + x\right)} = \cos x$

9. Simplify $\sqrt{\frac{(1 - \tan^2 x \cos(-x) \cos(360^\circ - x)) \tan 45^\circ}{\{\sin 90^\circ - \sin(180^\circ + x)\} \{\sin 90^\circ - \cos(90^\circ - x)\}}}$

10. Prove that: (i) $\sin(16x) = 16 \sin(x) \cos(x) \cos(2x) \cos(4x) \cos(8x)$

(ii) $\frac{1 + \cos 2\theta}{\sin 2\theta - \cos \theta} = \frac{2 \cos \theta}{2 \sin \theta - 1}$

(iii) $\frac{2 \tan^2 \theta}{\tan^2 \theta - 1} = \frac{-2 \tan^2 \theta}{\sec^2 \theta - 2 \tan^2 \theta}$