

VECTORS IN SPACE

After studying this unit, students will be able to:

- Recognize rectangular coordinates system in space.
- Recognize unit vector, x , y and z components of a vector.
- Find the magnitude of a vector
- Repeat all fundamental mathematical operations for vectors in space which, in the plane, have already been discussed
- Demonstrate and prove the properties of vector addition
 - Commutative law for vector addition
 - Associative law for vector addition
 - 0 as the identity for vector addition
 - $-\vec{r}$ as the inverse for $\vec{r}$
 - Dot or scalar product
- Explain dot or scalar product of two vectors and give the mathematical interpretation.
- Express dot or scalar product of two vectors and give its geometrical interpretation.
- Express dot product in terms of components.
- Find the condition for the orthogonality of two vectors
- Find the projection of a vector along the other vector
- Find the work done by a constant force in moving an object along a given vector
- Solve daily life problems based on vectors. Cross or vector product
- Explain the cross or the vector product of two vectors and give its geometrical interpretation
- Apply a cross product to find the angle between two vectors
- Solve situations in daily based on cross or dot vector product
- Describe scalar triple product of vectors
- Express scalar triple product of vectors in terms of components(determinant form)
- Prove that
 - $\hat{i} \cdot \hat{j} \times \hat{k} = \hat{j} \cdot \hat{k} \times \hat{i} = \hat{k} \cdot \hat{i} \times \hat{j} = 1$
 - $\hat{i} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot \hat{i} \times \hat{k} = \hat{k} \cdot \hat{j} \times \hat{i} = -1$

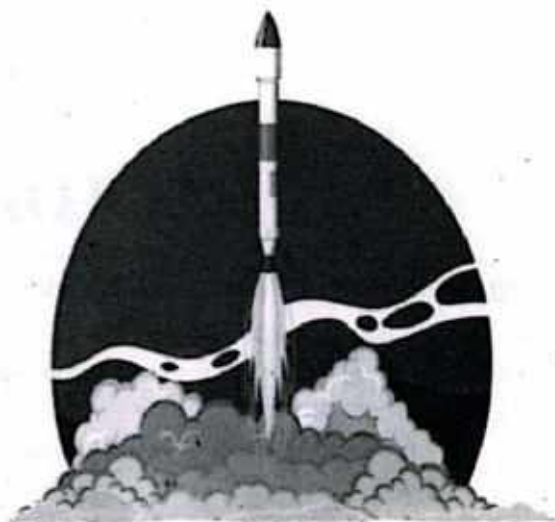
Prove that dot and cross are inter-changeable in scalar triple product

- Find the volume of a parallelepiped and tetrahedron

Determined by three given vectors

- Define co-planar vectors and find the condition for planarity of three vectors
- Apply concept of vector in space to real-world problems such as (design and execute optimal navigation paths in transportation and logistics, graphing complex 3D motion, vector operations in engineering and computer graphics, practical proficiency for work, flux, and circulation)

Vectors are utilised in day-to-day life to assist in the localization of people, places, and things. They are also used to describe things that are acting in response to an external force being applied to them. A quantity that possesses both a magnitude and a direction is known as a vector. The first, second, and third laws of Newton are all relationships between vectors that precisely describe the motion of bodies when they are subjected to the influence of an outside force. Newton's laws cover a wide range of phenomena and can be used to describe anything from a ball in free fall to a rocket on its way to the moon.



3.1 Vectors

i. Scalar

A physical quantity which can be completely specified by its magnitude only is called a scalar. e.g., mass, time, distance and volume etc.

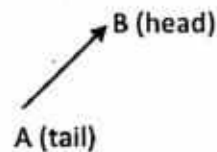
ii. Vector

A physical quantity which is completely specified by its magnitude as well as direction. e.g., weight, displacement, velocity and acceleration etc.

3.1.1 Geometrical Representation of a Vector

Geometrically a vector is represented by a line segment with an arrow head at its one end. The length of the line segment describes the magnitude and the arrow head indicates the direction of the vector.

The end A is called the tail or the initial point of the vector and the end B is called the terminal point. In the figure vector AB is shown. It is denoted by $\overrightarrow{AB}$.

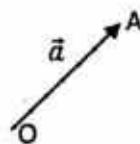


Usually, the vectors are denoted by bold face letters $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ etc.; or $\vec{a}$, $\vec{b}$, $\vec{c}$. There are also other notations to denote a vector like $\underline{a}$, $\underline{b}$ and $\underline{c}$ etc.

3.1.2 Some Fundamental Definitions and Terms Related to Vectors

i. Magnitude of a Vector

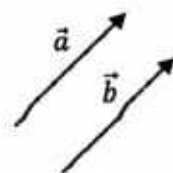
In the figure vector $\overrightarrow{OA}$ is denoted by $\vec{a}$. The magnitude or the length or the norm of the vector $\overrightarrow{OA}$ denoted by $|\overrightarrow{OA}|$ or $|\vec{a}|$.



ii. Equal Vectors

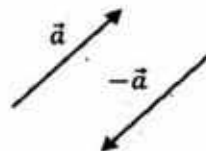
Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if both have the same magnitude and direction.

It is not necessary for the equal vectors to have the same position. If vectors $\vec{a}$ and $\vec{b}$ are equal then we write $\vec{a} = \vec{b}$. Geometrically two vectors are equal if they are translation of one another.



iii. Negative of a Vector

A vector having the same magnitude but opposite in direction of a vector $\vec{a}$ is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$.



iv. Zero or Null Vector

If the initial and terminal points of a vector coincide then the vector has zero length. This vector is called zero vector and is denoted by $\vec{0}$. The zero vector has no direction. It can be assigned as convenient direction according to the situation.

v. Unit Vector

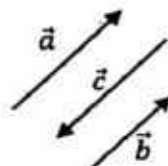
A vector which is in the direction of non-zero vector $\vec{a}$ and has magnitude 1 is called unit vector of $\vec{a}$ and is denoted by $\hat{a}$. If $\vec{a}$ is non-zero vector of arbitrary length $|\vec{a}|$ then

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = |\vec{a}|\hat{a}.$$

This means any vector $\vec{a}$ can be obtained by multiplying the magnitude of the vector to its unit vector. The process of finding the unit vector of a vector $\vec{a}$, is called normalizing vector $\vec{a}$.

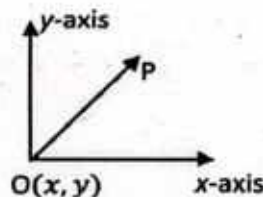
vi. Parallel Vectors

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if $\vec{a} = \lambda\vec{b}$; where λ is a scalar. If value of λ is positive then both vectors have the same direction and if value of λ is negative then both are in the opposite direction. The vectors which are in the opposite direction are known as antiparallel vectors.



vii. Position Vector

The vector used to specify the position of a point P with respect to origin O is called position vector of P. The tail of this vector is at origin and tip at the point P. Thus $\vec{OP}$ is the position vector of point P with respect to O.



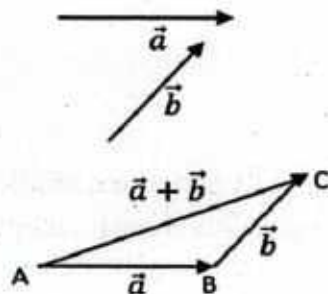
3.1.3 Addition of Vectors

i. Head to Tail Rule or Triangle Law of Addition

To add non-zero vectors $\vec{a}$ and $\vec{b}$, join the tail of the second vector with the head of the first vector. Now the vector obtained by joining the tail of the first vector to head of the second vector is the vector $\vec{a} + \vec{b}$.

$\vec{a} + \vec{b}$ is known as resultant vector of $\vec{a}$ and $\vec{b}$.

This method for the addition of two vectors is called Head to Tail rule of addition. Since $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ are along the sides of a triangle ABC, so the rule of addition is also called triangle law of addition.



ii. Parallelogram Law of Addition

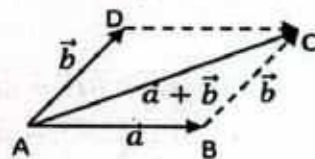
Consider any parallelogram ABCD. Let $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$.

Since the vector $\vec{BC}$ has the same magnitude and direction as that of $\vec{AD}$; so $\vec{BC} = \vec{AD}$. Also $\vec{DC}$ has the same magnitude and direction as that of $\vec{AB}$ so $\vec{DC} = \vec{AB}$.

Using triangle law of addition, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\text{i.e.; } \vec{a} + \vec{b} = \vec{AC}$$



This mean diagonal vector $\overrightarrow{AC}$ of the parallelogram is the sum of the vectors of $\vec{a}$ and $\vec{b}$.

This is known as parallelogram law of addition.

Example 1:

Let us consider two vectors that is $\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} - 5\hat{j} + 2\hat{k}$ in space according to head to tail rule

$$\vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - 5\hat{k}) + (\hat{i} - 5\hat{j} + 2\hat{k})$$

$$\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k} + \hat{i} - 5\hat{j} + 2\hat{k} = 3\hat{i} - 2\hat{j} - 3\hat{k}$$

similarly

$$\vec{b} + \vec{a} = 3\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\text{So, } \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

iii. Polygon Law of Addition of Vectors

The process for the addition of vectors can be extended to any number of vectors. For instance, let we have five vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $\vec{e}$.

The sum of vectors $\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$.

For this draw $\overrightarrow{IA} = \vec{a}, \overrightarrow{AB} = \vec{b}, \overrightarrow{BC} = \vec{c}, \overrightarrow{CD} = \vec{d}, \overrightarrow{DE} = \vec{e}$

$$\begin{aligned} \text{Now } \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} &= \overrightarrow{IA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} \\ &= (\overrightarrow{IA} + \overrightarrow{AB}) + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} \\ &= \overrightarrow{IB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} \quad (\because \overrightarrow{IA} + \overrightarrow{AB} = \overrightarrow{IB}) \\ &= (\overrightarrow{IB} + \overrightarrow{BC}) + \overrightarrow{CD} + \overrightarrow{DE} \\ &= \overrightarrow{IC} + \overrightarrow{CD} + \overrightarrow{DE} \quad (\because \overrightarrow{IB} + \overrightarrow{BC} = \overrightarrow{IC}) \\ &= (\overrightarrow{IC} + \overrightarrow{CD}) + \overrightarrow{DE} \\ &= \overrightarrow{ID} + \overrightarrow{DE} \quad (\because \overrightarrow{IC} + \overrightarrow{CD} = \overrightarrow{ID}) \\ &= \overrightarrow{IE} \quad (\because \overrightarrow{ID} + \overrightarrow{DE} = \overrightarrow{IE}) \end{aligned}$$

Then $\overrightarrow{IE}$ is the sum of all these five vectors.

Same method can be adopted to find the sum of any number of vectors. This is called polygon law of addition of vectors.

iv. Subtraction of Two Vectors

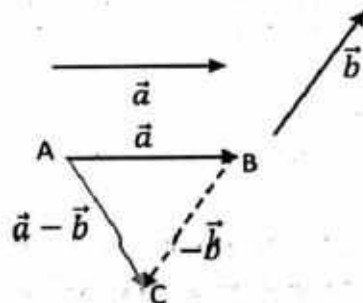
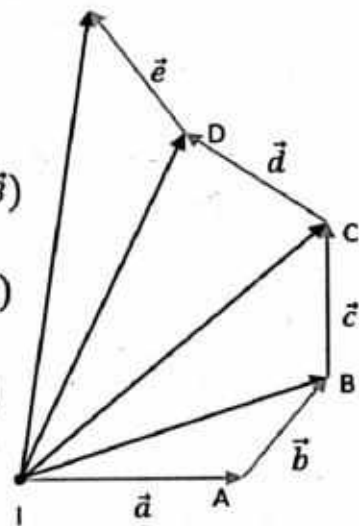
Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ then $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

To find $\vec{a} - \vec{b}$ draw $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = -\vec{b}$; then

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\vec{a} + (-\vec{b}) = \overrightarrow{AC}$$

Thus $\overrightarrow{AC}$ is the vector which represents $\vec{a} - \vec{b}$.



Example 2:

$$\vec{a} = 2\hat{i} + 7\hat{j} + 4\hat{k} \text{ and } \vec{b} = 6\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\vec{a} - \vec{b} = 2\hat{i} + 7\hat{j} + 4\hat{k} - (6\hat{i} - 4\hat{j} - 3\hat{k})$$

$$\vec{a} - \vec{b} = 2\hat{i} + 7\hat{j} + 4\hat{k} - 6\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\vec{a} - \vec{b} = (2 - 6)\hat{i} + (7 + 4)\hat{j} + (4 + 3)\hat{k}$$

$$\vec{a} - \vec{b} = 8\hat{i} - \hat{j} + \hat{k}$$

v. Scalar Multiplication

If λ is a non-zero scalar and $\vec{a}$ is a non-zero vector then the scalar multiple $\lambda\vec{a}$ is a vector whose magnitude is $|\lambda|$ times magnitude of $\vec{a}$. $\lambda\vec{a}$ has the same direction as that of $\vec{a}$ if λ is positive and if λ is negative then direction of $\lambda\vec{a}$ is in the opposite direction of $\vec{a}$.

If $\lambda\vec{a} = 0$ then either $\lambda = 0$ or $\vec{a} = 0$.

Example 3:

$$\text{Let } \lambda = 5 \text{ and } \vec{a} = 10\hat{i} + 9\hat{j} + 17\hat{k}$$

$$\lambda\vec{a} = 5(10\hat{i} + 9\hat{j} + 17\hat{k})$$

$$\lambda\vec{a} = 50\hat{i} + 45\hat{j} + 85\hat{k}$$

3.1.4 Position Vector of a Point Dividing the Line Segment in a Given Ratio

Let $\overline{AB}$ be any line segment and P is the point which divides this line segment in the given ratio $m : n$ internally.

The position vectors of the given points A and B are $\vec{a}$ and $\vec{b}$ respectively. Let $\vec{r}$ be the position vector of point P .

Given that

$$|\overline{AP}| : |\overline{PB}| = m : n$$

$$\Rightarrow \frac{|\overline{AP}|}{|\overline{PB}|} = \frac{m}{n}$$

$$\Rightarrow n|\overline{AP}| = m|\overline{PB}|$$

Because $\overline{AP}$ and $\overline{PB}$ have the same direction; so

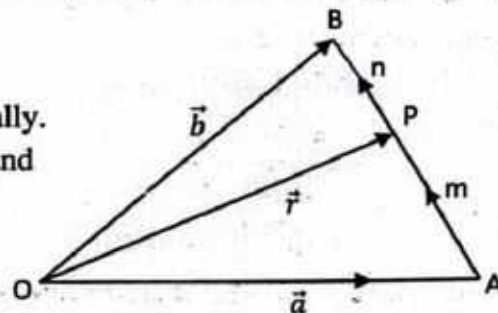
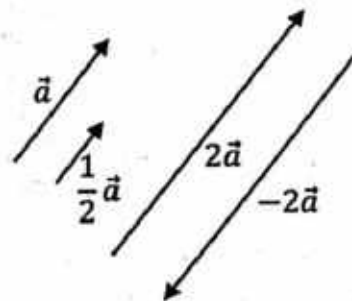
$$n\overline{AP} = m\overline{PB} \quad (1)$$

From figure

$$\overline{OA} + \overline{AP} = \overline{OP}$$

$$\Rightarrow \vec{a} + \overline{AP} = \vec{r}$$

$$\Rightarrow \overline{AP} = \vec{r} - \vec{a}$$



And

$$\begin{aligned}\overrightarrow{OP} + \overrightarrow{PB} &= \overrightarrow{OB} \\ \Rightarrow \vec{r} + \overrightarrow{PB} &= \vec{b} \\ \Rightarrow \overrightarrow{PB} &= \vec{b} - \vec{r}\end{aligned}$$

Substituting values in equation (1)

$$\begin{aligned}n(\vec{r} - \vec{a}) &= m(\vec{b} - \vec{r}) \\ \Rightarrow n\vec{r} - n\vec{a} &= m\vec{b} - m\vec{r} \\ \Rightarrow n\vec{r} + m\vec{r} &= m\vec{b} + n\vec{a} \\ \Rightarrow (n + m)\vec{r} &= m\vec{b} + n\vec{a} \\ \Rightarrow \vec{r} &= \frac{m\vec{b} + n\vec{a}}{m + n}\end{aligned}$$

Case I

If $m : n = 1 : 1$ then $\frac{m}{n} = \frac{1}{1}$ or $m = n$. In this case P will be the midpoint of $\overline{AB}$ and position vector of P in this case equation (2) becomes:

$$\vec{r} = \frac{n\vec{b} + n\vec{a}}{n + n}$$

$$\because (m = n)$$

$$\Rightarrow \vec{r} = \frac{n(\vec{b} + \vec{a})}{2n} = \frac{\vec{a} + \vec{b}}{2}$$

Case II

When the point P divides the line segment $\overline{AB}$ in the ratio $m : n$ externally then

$$|\overrightarrow{AP}| : |\overrightarrow{BP}| = m : n$$

$$\Rightarrow \frac{|\overrightarrow{AP}|}{|\overrightarrow{BP}|} = \frac{m}{n}$$

$$\Rightarrow n|\overrightarrow{AP}| = m|\overrightarrow{BP}|$$

Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ have the same direction thus,

$$\begin{aligned}n\overrightarrow{AP} &= m\overrightarrow{BP} \\ n(\vec{r} - \vec{a}) &= m(\vec{r} - \vec{b}) \\ \Rightarrow n\vec{r} - n\vec{a} &= m\vec{r} - m\vec{b} \\ \Rightarrow n\vec{r} - m\vec{r} &= n\vec{a} - m\vec{b} \\ \Rightarrow (n - m)\vec{r} &= n\vec{a} - m\vec{b} \\ \Rightarrow \vec{r} &= \frac{n\vec{a} - m\vec{b}}{n - m}\end{aligned}$$

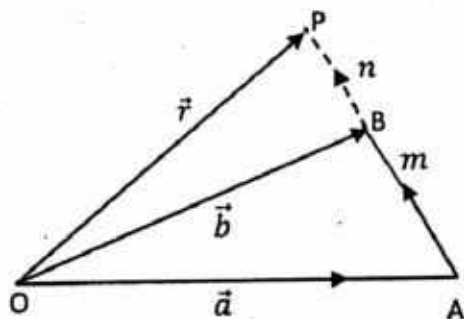
Example 4:

Find the vector which divides $\overrightarrow{OA} = \underline{a} = 3\hat{i} + \hat{j} + 10\hat{k}$ and $\overrightarrow{OB} = \underline{b} = 3\hat{i} + 7\hat{j} + \hat{k}$ in the ratio 3:2 (Internally)

Solution: we know that

$$\vec{r} = \frac{2\underline{b} + 3\underline{a}}{2 + 3}$$

$$\vec{r} = \frac{2(3\hat{i} + 7\hat{j} + \hat{k}) + 3(3\hat{i} + \hat{j} + 10\hat{k})}{2 + 3} = \frac{6\hat{i} + 14\hat{j} + 2\hat{k} + 9\hat{i} + 3\hat{j} + 30\hat{k}}{5}$$



$$= \frac{15\hat{i} + 17\hat{j} + 32\hat{k}}{5}$$

$$\vec{r} = 5\hat{i} + \frac{17}{5}\hat{j} + \frac{32}{5}\hat{k}$$

Example 5:

Find the vector which divides $\overrightarrow{OA} = \underline{a} = 3\hat{i} + \hat{j} + 10\hat{k}$ and $\overrightarrow{OB} = \underline{b} = 3\hat{i} + 7\hat{j} + \hat{k}$ in the ratio 3:2(externally)

Solution:

$$\vec{r} = \frac{2\underline{b} - 3\underline{a}}{2 - 3}$$

$$\vec{r} = \frac{2(3\hat{i} + 7\hat{j} + \hat{k}) - 3(3\hat{i} + \hat{j} + 10\hat{k})}{-1} = -(6\hat{i} + 14\hat{j} + 2\hat{k} - 9\hat{i} - 3\hat{j} - 30\hat{k})$$

$$\vec{r} = -(-3\hat{i} + 11\hat{j} - 28\hat{k}) = 3\hat{i} - 11\hat{j} + 28\hat{k}$$

3.1.5 Application to Geometry

Here we are giving some simple geometrical proofs by using vector methods.

Theorem 1: The diagonals of a parallelogram bisect each other.

Proof:

Consider any parallelogram ABCD. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of the vertices A, B, C and D respectively.

Now the position vector of the midpoint M_1 of the diagonal $\overrightarrow{AC}$ is $\frac{\vec{a} + \vec{c}}{2}$.

$$\text{i.e., p.v of } M_1 = \frac{\vec{a} + \vec{c}}{2} \quad (1)$$

And the position vector of the midpoint M_2 of the diagonal $\overrightarrow{BD}$ is $\frac{\vec{b} + \vec{d}}{2}$ i.e.,

$$\text{p.v of } M_2 = \frac{\vec{b} + \vec{d}}{2} \quad (2)$$

Since ABCD is a parallelogram so

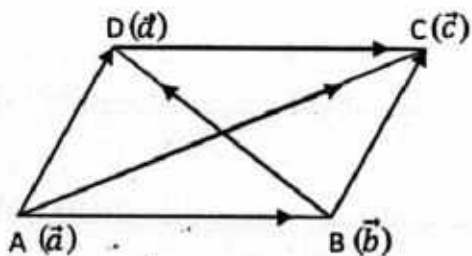
$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{DC} \\ \Rightarrow \vec{b} - \vec{a} &= \vec{c} - \vec{d} \\ \Rightarrow \vec{b} + \vec{d} &= \vec{a} + \vec{c} \end{aligned}$$

Dividing by 2:

$$\text{p.v of } M_2 = \frac{\vec{b} + \vec{d}}{2} = \frac{\vec{a} + \vec{c}}{2} = \text{p.v of } M_1$$

Since the position vectors of the point of intersection of both the diagonals are same. Thus, they bisect each other.

Theorem 2: Line joining the midpoints of any two sides of a triangle is parallel to the third side and half in length of the third side.



Proof:

Consider any triangle ABC. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C respectively.

Let M_1 and M_2 be the midpoints of the sides $\overline{CA}$ and $\overline{BC}$ respectively. Therefore:

$$\text{Position vector of } M_1 = \frac{\vec{a} + \vec{c}}{2}$$

$$\text{Position vector of } M_2 = \frac{\vec{b} + \vec{c}}{2}$$

$$\text{Now } \overrightarrow{M_1 M_2} = \frac{\vec{b} + \vec{c}}{2} - \frac{\vec{a} + \vec{c}}{2}$$

$$= \frac{1}{2}(\vec{b} + \vec{c} - \vec{a} - \vec{c}) = \frac{1}{2}(\vec{b} - \vec{a}) = \frac{1}{2}\overrightarrow{AB} \quad (\because \overrightarrow{AB} = \vec{b} - \vec{a})$$

This shows that $\overrightarrow{M_1 M_2}$ is parallel to $\overrightarrow{AB}$. Also

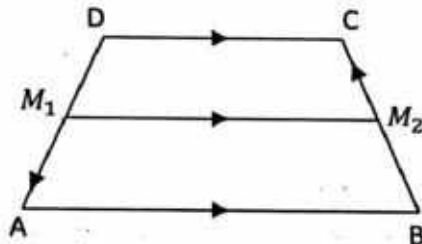
$$|\overrightarrow{M_1 M_2}| = \left| \frac{1}{2}\overrightarrow{AB} \right| = \frac{1}{2}|\overrightarrow{AB}|$$

This shows that length of $\overrightarrow{M_1 M_2}$ is half the length of $\overrightarrow{AB}$.

Theorem 3: The joining of the midpoints of the two non-parallel sides of a trapezium is parallel to its parallel sides and its length is half the sum of the lengths of the parallel sides.

Proof

Consider any trapezium ABCD with two parallel sides $\overline{AB}$ and $\overline{DC}$.



Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the position vectors of the vertices A, B, C and D respectively. Also suppose that M_1 and M_2 be the midpoints of the non-parallel sides $\overline{BC}$ and $\overline{DA}$ respectively.

Therefore,

$$\text{Position vector of } M_1 = \frac{\vec{d} + \vec{a}}{2}$$

$$\text{Position vector of } M_2 = \frac{\vec{b} + \vec{c}}{2}$$

$$\text{Now } \overrightarrow{M_1 M_2} = \frac{\vec{b} + \vec{c}}{2} - \frac{\vec{d} + \vec{a}}{2}$$

$$\begin{aligned} &= \frac{1}{2}(\vec{b} + \vec{c} - \vec{d} - \vec{a}) = \frac{1}{2}[(\vec{b} - \vec{a}) + (\vec{c} - \vec{d})] \\ &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC}) \end{aligned}$$

(1)

Since $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ (given).

Thus $\overrightarrow{AB} = \lambda \overrightarrow{DC}$; where λ is some scalar.

$$\text{Therefore } \overrightarrow{M_1 M_2} = \frac{1}{2}(\lambda \overrightarrow{DC} + \overrightarrow{DC}) = \frac{1}{2}(\lambda + 1)\overrightarrow{DC}$$

$$\overrightarrow{M_1 M_2} = \mu \overrightarrow{DC} \quad \text{where } \mu = \frac{1}{2}(\lambda + 1) \text{ is a scalar.}$$

This shows that $\overline{M_1M_2}$ is parallel to $\overline{DC}$ and $\overline{AB}$.

Thus $\overline{M_1M_2}$ is parallel to the parallel sides. Also, from (1) it is clear that length of $\overline{M_1M_2}$ is half the sum of the lengths of the parallel sides $\overline{AB}$ and $\overline{DC}$.

3.2 Vectors in Space (Three-Dimensional Space)

3.2.1 Rectangular Coordinate System

To represent a vector in space we need a 3-dimensional coordinate system. For this we consider three mutually perpendicular lines intersecting at a common point O known as origin.

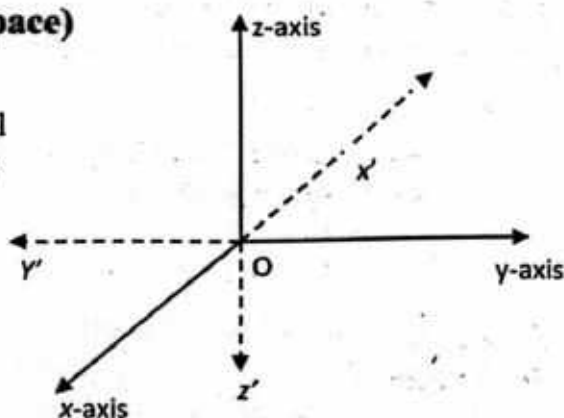
Any point in the space has some specific position w. r. t. origin O i.e., We can locate the point by moving specific distance along these three lines.

These three lines are known as coordinate axes and are named as x-axis, y-axis and z-axis. The distance along x-axis is called x-coordinate, the distance along y-axis is y-coordinate and the distance along z-axis is z-coordinate of the point.

A general point in the space has coordinate (x, y, z) .

This coordinate system to represent or locate a point is known as rectangular coordinate system or Cartesian coordinate system and is denoted by $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$ or $\mathcal{R}^3$. The set of all the points in space is:

$$\mathcal{R}^3 = \{(x, y, z) : x, y, z \in \mathcal{R}\}.$$



3.2.2 Unit Vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

To represent a vector in space we need unit vectors in the direction of coordinate axes. For this we have three fundamental unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ along x-axis, y-axis and z-axis respectively.

3.2.3 Components of a Vector

Let $\overline{OP}$ be the position vector of the point $P(x, y, z)$, then

$$\overline{OA} = x\hat{i}; \quad \overline{OB} = y\hat{j} \text{ and}$$

$$\overline{OC} = z\hat{k}$$

From figure it is clear that

$$\overline{OP} = \overline{OQ} + \overline{QP} \quad (1)$$

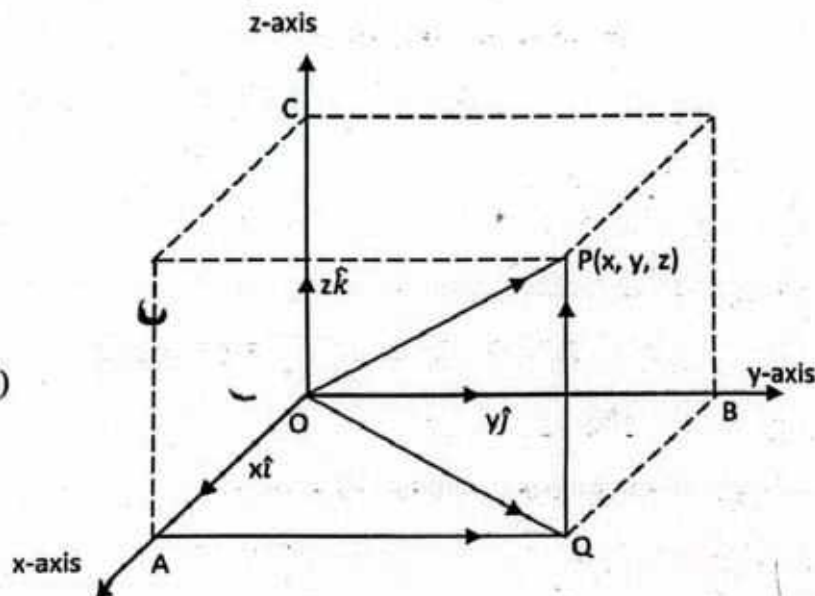
$$\text{Since } \overline{OA} + \overline{AQ} = \overline{OQ}$$

$$\Rightarrow \overline{OA} + \overline{OB} = \overline{OQ} \quad (\because \overline{AQ} = \overline{OB})$$

$$\Rightarrow x\hat{i} + y\hat{j} = \overline{OQ}$$

$$\text{Or } \overline{OQ} = x\hat{i} + y\hat{j}$$

$$\text{Also } \overline{QP} = z\hat{k}$$



Putting values in equation (1), we have:

$$\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

Which is the position vector of the point $P(x, y, z)$.

In the representation of the position vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, x, y and z are known as components of the vector along x -axis, y -axis and z -axis respectively.

3.2.4 Analytical Representation of the Vector

The representation of a vector in space having its component form $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is known as analytic representation of the vector $\vec{r}$.

3.2.5 Magnitude of a Vector

Consider a vector in space $\vec{r} = \overrightarrow{OP}$.

From figure

(1)

$$|\overrightarrow{OP}|^2 = |\overrightarrow{OQ}|^2 + |\overrightarrow{QP}|^2 \quad (\because \overrightarrow{OQ} = x\hat{i} + y\hat{j})$$

$$\Rightarrow |\overrightarrow{OQ}| = \sqrt{x^2 + y^2}$$

$$\text{and } |\overrightarrow{QP}| = z$$

Putting values in equation (1).

$$|\vec{r}|^2 = (\sqrt{x^2 + y^2})^2 + z^2$$

$$\Rightarrow |\vec{r}|^2 = x^2 + y^2 + z^2$$

$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Which is the magnitude of a vector in space.

Example 6:

Find the magnitude of the following vectors

$$\text{i. } \vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k} \quad \text{ii. } \vec{b} = 2\hat{i} - 3\hat{j} + 7\hat{k} \quad \text{iii. } \vec{c} = 7\hat{i} + 4\hat{j} - \hat{k}$$

Solution:

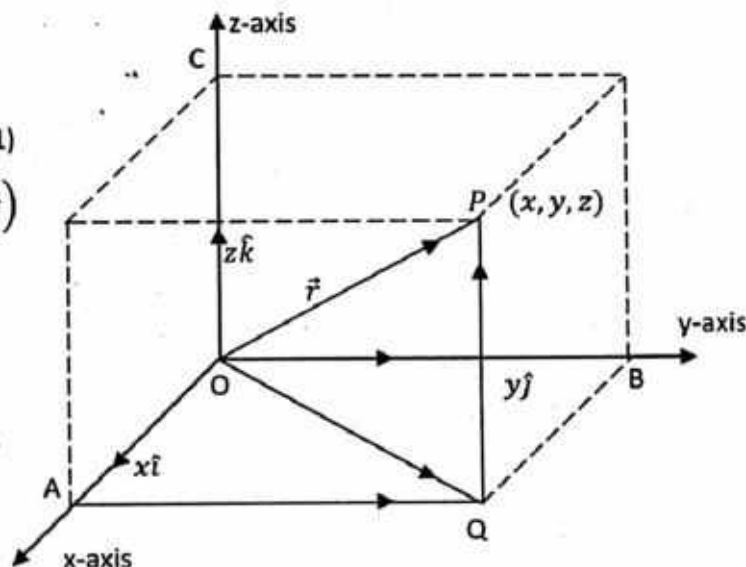
$$\text{i. } \vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}$$

Using formula for the magnitude of the vector

$$|\vec{a}| = \sqrt{(2)^2 + (3)^2 + (-5)^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$$

$$\text{ii. } \vec{b} = 2\hat{i} - 3\hat{j} + 7\hat{k}$$

Using formula for the magnitude of a vector



$$|\vec{b}| = \sqrt{(2)^2 + (-3)^2 + (7)^2} = \sqrt{4 + 9 + 49} = \sqrt{62}$$

$$\text{iii. } \vec{c} = \hat{i} + 4\hat{j} - 9\hat{k}$$

Using formula for vector magnitude

$$|\vec{c}| = \sqrt{(1)^2 + (4)^2 + (-9)^2} = \sqrt{1 + 16 + 81} = \sqrt{98} = 7\sqrt{2}$$

3.2.6 Fundamental Definitions for Vectors in Space

i. Unit Vector

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector in space. A unit vector $\hat{r}$ in the direction of $\vec{r}$ is given by

$$\begin{aligned}\hat{r} &= \frac{\vec{r}}{|\vec{r}|} \\ \Rightarrow \hat{r} &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \\ \Rightarrow \hat{r} &= \frac{x}{\sqrt{x^2 + y^2 + z^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\hat{k}\end{aligned}$$

ii. Equal Vectors

Two vectors in space $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are said to be equal if they have the same corresponding components. i.e.;

$$\begin{aligned}\vec{r}_1 &= \vec{r}_2 \\ \Rightarrow x_1\hat{i} + y_1\hat{j} + z_1\hat{k} &= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \\ \Rightarrow x_1 = x_2; \quad y_1 = y_2; \quad z_1 = z_2\end{aligned}$$

iii. Zero Vector

A vector in space which has all its three components equal to zero is called zero vector. It is usually denoted by $\vec{0}$. i.e.;

$$\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

iv. Negative of a Vector

For a vector in space $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ negative of $\vec{r}$ is denoted by $-\vec{r}$ and is defined as:

$$-\vec{r} = (-x)\hat{i} + (-y)\hat{j} + (-z)\hat{k}$$

v. Scalar Multiplication

The product of a scalar λ with a vector in space $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is denoted by $\lambda\vec{r}$ and is defined as

$$\lambda\vec{r} = (\lambda x)\hat{i} + (\lambda y)\hat{j} + (\lambda z)\hat{k}$$

vi. Parallel Vectors

Two non-zero vectors in space $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are said to be parallel if there exists some non-zero scalar λ such that $\vec{r}_1 = \lambda\vec{r}_2$. i.e.;

$$\begin{aligned}x_1\hat{i} + y_1\hat{j} + z_1\hat{k} &= \lambda(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ \Rightarrow x_1\hat{i} + y_1\hat{j} + z_1\hat{k} &= (\lambda x_2)\hat{i} + (\lambda y_2)\hat{j} + (\lambda z_2)\hat{k}\end{aligned}$$

$$\begin{aligned}\Rightarrow x_1 &= \lambda x_2; & y_1 &= \lambda y_2; & z_1 &= \lambda z_2 \\ \Rightarrow \frac{x_1}{x_2} &= \lambda; & \frac{y_1}{y_2} &= \lambda; & \frac{z_1}{z_2} &= \lambda \\ \Rightarrow \frac{x_1}{x_2} &= \frac{y_1}{y_2} = \frac{z_1}{z_2} = \lambda\end{aligned}$$

Which is the condition for two vectors to be parallel. For positive value of λ vectors will have the same direction and will be in opposite direction if λ is negative.

Addition of Vectors

Consider two vectors $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ in space. Their sum $\vec{r}_1 + \vec{r}_2$ is defined as:

$$\begin{aligned}\vec{r}_1 + \vec{r}_2 &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}.\end{aligned}$$

Subtraction of two vectors

Consider two vectors $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ in space. Their difference $\vec{r}_1 - \vec{r}_2$ is defined as:

$$\begin{aligned}\vec{r}_1 - \vec{r}_2 &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}.\end{aligned}$$

3.3 Properties of Vector Addition

3.3.1 Commutative Law for Vector Addition

Statement: For any two vectors $\vec{r}_1$ and $\vec{r}_2$ in space $\vec{r}_1 + \vec{r}_2 = \vec{r}_2 + \vec{r}_1$.

Proof: Let $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Thus

$$\begin{aligned}\vec{r}_1 + \vec{r}_2 &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}\end{aligned}$$

Since $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and commutative law w. r. t. addition holds in $\mathbb{R}$, so we may write

$$\begin{aligned}\vec{r}_1 + \vec{r}_2 &= (x_2 + x_1)\hat{i} + (y_2 + y_1)\hat{j} + (z_2 + z_1)\hat{k} \\ &= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) + (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= \vec{r}_2 + \vec{r}_1\end{aligned}$$

3.3.2 Associative Law for Vector Addition

Statement: For any three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ in space; $\vec{r}_1 + (\vec{r}_2 + \vec{r}_3) = (\vec{r}_1 + \vec{r}_2) + \vec{r}_3$

Proof: Let $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{r}_3 = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$

$$\begin{aligned}\vec{r}_2 + \vec{r}_3 &= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} + x_3\hat{i} + y_3\hat{j} + z_3\hat{k} \\ &= (x_2 + x_3)\hat{i} + (y_2 + y_3)\hat{j} + (z_2 + z_3)\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{r}_1 + (\vec{r}_2 + \vec{r}_3) &= x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + [(x_2 + x_3)\hat{i} + (y_2 + y_3)\hat{j} + (z_2 + z_3)\hat{k}] \\ &= [x_1 + (x_2 + x_3)]\hat{i} + [y_1 + (y_2 + y_3)]\hat{j} + [z_1 + (z_2 + z_3)]\hat{k}\end{aligned}$$

Since $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in \mathbb{R}$ and associative law holds in $\mathbb{R}$ w. r. t. addition so, we may write:

$$\begin{aligned}\vec{r}_1 + (\vec{r}_2 + \vec{r}_3) &= [(x_1 + x_2) + x_3]\hat{i} + [(y_1 + y_2) + y_3]\hat{j} + [(z_1 + z_2) + z_3]\hat{k} \\ &= [(x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}] + (x_3\hat{i} + y_3\hat{j} + z_3\hat{k})\end{aligned}$$

$$= [(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})] + (x_3\hat{i} + y_3\hat{j} + z_3\hat{k})$$

$$= (\vec{r}_1 + \vec{r}_2) + \vec{r}_3$$

3.4.2 Identity Vector for Addition

For any vector $\vec{r}$ and null vector $\vec{0}$, $\vec{r} + \vec{0} = \vec{0} + \vec{r} = \vec{r}$

Let $\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ be the null vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be any vector in space. Now

$$\vec{0} + \vec{r} = (0\hat{i} + 0\hat{j} + 0\hat{k}) + (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (0 + x)\hat{i} + (0 + y)\hat{j} + (0 + z)\hat{k}$$

Since 0 is the additive identity of real numbers; so,

$$\vec{0} + \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$$

Also

$$\vec{r} + \vec{0} = (x\hat{i} + y\hat{j} + z\hat{k}) + (0\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= (x + 0)\hat{i} + (y + 0)\hat{j} + (z + 0)\hat{k}$$

$$= x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$$

Therefore, $\vec{0} + \vec{r} = \vec{r} + \vec{0} = \vec{r}$

This shows that $\vec{0}$ is the identity for the vector addition.

3.3.4 Additive Inverse in Vectors

Let $\vec{r}$ be any vector then $-\vec{r}$ is called additive inverse of $\vec{r}$ if $\vec{r} + (-\vec{r}) = (-\vec{r}) + \vec{r} = \vec{0}$

Consider any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in space, then $-\vec{r} = (-x)\hat{i} + (-y)\hat{j} + (-z)\hat{k}$. Now

$$\vec{r} + (-\vec{r}) = (x\hat{i} + y\hat{j} + z\hat{k}) + \{(-x)\hat{i} + (-y)\hat{j} + (-z)\hat{k}\}$$

$$= (x + (-x))\hat{i} + (y + (-y))\hat{j} + (z + (-z))\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

and

$$(-\vec{r}) + \vec{r} = \{(-x)\hat{i} + (-y)\hat{j} + (-z)\hat{k}\} + (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= ((-x) + x)\hat{i} + ((-y) + y)\hat{j} + ((-z) + z)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Therefore, $\vec{r} + (-\vec{r}) = (-\vec{r}) + \vec{r} = \vec{0}$

This shows that $-\vec{r}$ is the additive inverse of $\vec{r}$.

3.4 Properties of Scalar Multiplication

3.4.1 Commutative Law of Scalar Multiplication

Statement: For a scalar λ and a vector $\vec{r}$ in space $\lambda\vec{r} = \vec{r}\lambda$.

Proof: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector in space then:

$$\lambda\vec{r} = \lambda(x\hat{i} + y\hat{j} + z\hat{k}) = (\lambda x)\hat{i} + (\lambda y)\hat{j} + (\lambda z)\hat{k}$$

$$= (x\lambda)\hat{i} + (y\lambda)\hat{j} + (z\lambda)\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k})\lambda$$

$$= \vec{r}\lambda$$

Distance Between the Two Points in $\mathbb{R}^3$ (Distance Formula)

Consider any two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in $\mathbb{R}^3$.

The distance between P and Q is the magnitude of the vector $\overrightarrow{PQ}$.

The position vectors of P and Q are $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively; where

$$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\text{and } \overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

It is clear that:

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

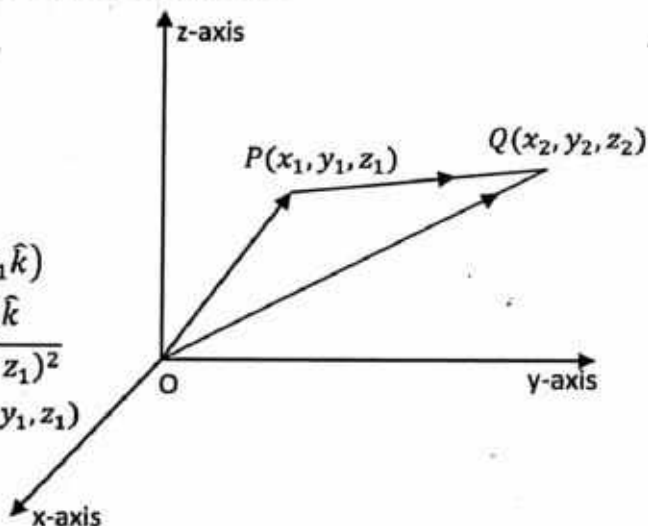
$$\Rightarrow \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\Rightarrow |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Which is the distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.



Exercise 3.1

1. In the following find the required vector in its component form. Given that $P = (3, -1)$; $Q = (-4, -6)$; $R = (1, 4)$ and $S = (2, 5)$

(i) $\overrightarrow{PQ}$

(ii) $3\overrightarrow{PQ} - \overrightarrow{RS}$

(iii) $2\overrightarrow{PR} + 3\overrightarrow{PS}$

(iv) $\frac{1}{2}\overrightarrow{PQ} + \frac{5}{2}\overrightarrow{PR} - \frac{3}{2}\overrightarrow{QS}$

(v) $3^2\overrightarrow{PS} - 4^2\overrightarrow{SP} + \overrightarrow{QP}$

2. Show that:

(i) the points $A(1, 0)$; $B(6, 0)$ and $C(0, 0)$ are collinear.

(ii) if $\vec{a}$ and $\vec{b}$ are the position vectors of points $(2, -7)$ and $(\frac{m}{2}, 11)$, find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

3. If $\vec{u} = \langle -1, 1 \rangle$; $\vec{v} = \langle 0, 1 \rangle$ and $\vec{w} = \langle 3, 4 \rangle$ then

(i) Find $\vec{x}$ that satisfies $\vec{u} - 2\vec{x} = \vec{x} - \vec{w} + 3\vec{v}$ ($\langle x, y \rangle$ means $x\hat{i} + y\hat{j}$)

(ii) Find $\vec{u}$ and $\vec{v}$ if $\vec{u} + \vec{v} = \langle 2, -3 \rangle$; $3\vec{u} + 2\vec{v} = \langle -1, 2 \rangle$

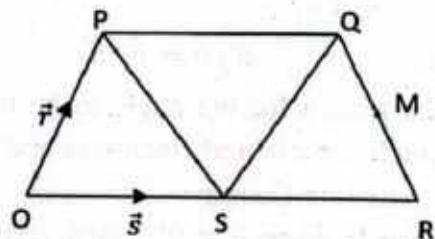
(iii) Find initial point of $\vec{v} = \langle -3, 1, 2 \rangle$ if its terminal point is $(5, 0, 1)$.

4. (i) Find the value of m for which the vector $\vec{a} = 3\hat{i} + 4\hat{j} - 9\hat{k}$ is parallel to $\vec{b} = \hat{i} + m\hat{j} - 3\hat{k}$.

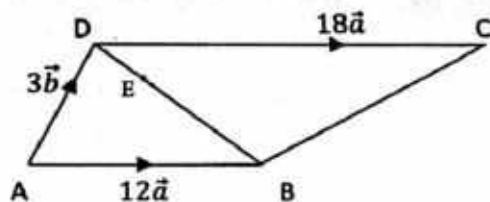
(ii) Find the value of λ for which the points P, Q and R are collinear.

Given that $\hat{i} + 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 2\hat{j} + 6\hat{k}$ and $\lambda\hat{i} - \lambda\hat{k}$ are the position vectors of points P, Q and R respectively.

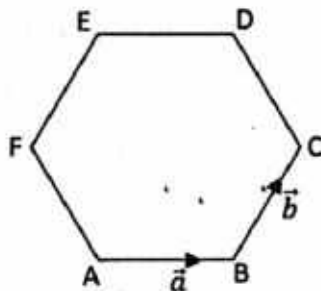
5. (i) If $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$; $\vec{b} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} + \hat{k}$ then find a unit vector in the direction of $2\vec{a} - 3\vec{b} + \vec{c}$.
- (ii) Use vectors to find the length of diagonals of a parallelogram having adjacent sides $\hat{i} + \hat{j}$ and $\hat{i} - 2\hat{j}$.
6. (i) Show that the points with position vectors $\hat{i} - \hat{j}$, $4\hat{i} - 3\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + 5\hat{k}$ are the vertices of a right-angled triangle.
- (ii) Show that the triangle with $2\hat{i} + 3\hat{j} + \sqrt{3}\hat{k}$, $\sqrt{10}\hat{i} - \hat{j} + \sqrt{5}\hat{k}$ and $-3\hat{i} + \sqrt{3}\hat{j} + 2\hat{k}$ are the vertices of an equilateral triangle.
7. (i) Find the value of λ for which $|\vec{a}| = |\vec{b}|$ where $\vec{a} = \hat{i} - 3\hat{j} + \lambda\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$.
- (ii) If $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = -3\hat{i} + 2\hat{j}$. Check whether $|\vec{a}| = |\vec{b}|$ or $\vec{a} = \vec{b}$.
8. If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ then find:
- (i) A vector of magnitude 5 in the direction $\vec{a} - 2\vec{b}$.
- (ii) A vector of magnitude $\frac{3}{7}$ opposite in direction $3\vec{a} + \vec{b}$.
9. The position vectors of points A and B are $\hat{i} - 2\hat{j} + \hat{k}$ and $2\hat{i} + 3\hat{j} - \hat{k}$ respectively.
- (i) Find the position vector of point P dividing the line segment joining A and B in the ratio 2 : 3 internally.
- (ii) Find the position vector of point Q dividing the line segment $\overline{AB}$ in the ratio 3 : 2 externally.
10. (i) The three vertices of a parallelogram ABCD taken in order are $A(3, -4)$; $B(-1, -3)$ and $C(-6, 2)$. Find the fourth vertex D.
- (ii) Find the values of x and y if $A(1, 2)$; $B(4, y)$; $C(x, 6)$ and $D(3, 5)$ taken in order are the vertices of a parallelogram.
11. Show that the line segments joining the mid points of the sides of a quadrilateral consecutively form a parallelogram.
12. Show that line segments joining the mid points of the diagonals and the mid points of any two opposite sides of a quadrilateral consecutively form a parallelogram.
13. Prove that line segments joining the midpoint of the diagonals of a trapezium is parallel to the parallel sides and its length is half the difference of the lengths of the parallel sides.
14. OPQR is a trapezium made from three equilateral triangles with $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OS} = \vec{s}$ and M is the midpoint of $\overline{QR}$.
- (i) Write $\overrightarrow{PS}$ in terms of $\vec{r}$ and $\vec{s}$.
- (ii) Show that $\overrightarrow{OQ}$ is parallel to $\overrightarrow{SM}$.



15. ABCD is a trapezium with $\overline{AB}$ parallel to $\overline{DC}$. E is the point on the diagonal $\overline{DB}$ such that $DE = \frac{1}{3}DB$. Show that $\overline{BC}$ is parallel to $\overline{AE}$.



16. ABCDEF is a regular hexagon as shown in the figure. If $\overline{AB} = \vec{a}$ and $\overline{BC} = \vec{b}$, then express $\overline{AC}$, $\overline{CD}$, $\overline{EF}$, $\overline{DA}$, $\overline{EB}$, $\overline{FA}$ and $\overline{FC}$ in terms of $\vec{a}$ and $\vec{b}$.



3.5 Dot or Scalar Product

3.5.1 Dot or Scalar Product of Two Vectors and its Geometrical Interpretation

If θ is the angle between the two non-zero vectors $\vec{a}$ and $\vec{b}$ then their dot product is denoted by $\vec{a} \cdot \vec{b}$ and is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

where, θ is measured from $\vec{a}$ to $\vec{b}$ and $0^\circ \leq \theta \leq 180^\circ$.

θ is positive if measured in anticlockwise and is taken as negative if measured clockwise.

The value of dot product is a scalar quantity that's why it is known as scalar product.

Observe that:

$$\begin{aligned} \vec{b} \cdot \vec{a} &= |\vec{b}||\vec{a}| \cos(-\theta) \\ &= |\vec{b}||\vec{a}| \cos \theta \quad (\because \cos(-\theta) = \cos \theta) \\ &= |\vec{a}||\vec{b}| \cos \theta \\ &= \vec{a} \cdot \vec{b} \end{aligned}$$

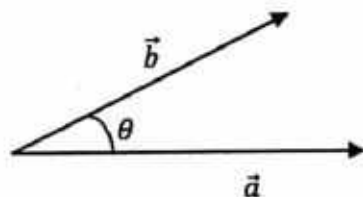
i.e., $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

The reason for the angle to be taken $-\theta$ is that for $\vec{b} \cdot \vec{a}$ angle will be measured from $\vec{b}$ to $\vec{a}$ which is measured clockwise and therefore will be taken negative.

Particular Cases:

Case I: when $\theta = 90^\circ$ then vectors will be perpendicular or orthogonal to each other. In this case

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^\circ = |\vec{a}||\vec{b}|(0) = 0$$



Key Facts



The angle between the vectors is the angle where the tail or head of both vectors meet.

Case II: When $\theta = 0^\circ$ then both the vectors have the same direction. i.e., Both are parallel to each other, in this case

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 0^\circ = |\vec{a}||\vec{b}|(1) = |\vec{a}||\vec{b}|$$

Case III: When $\vec{a} = \vec{b}$ then in that case:

$$\begin{aligned}\vec{a} \cdot \vec{a} &= |\vec{a}||\vec{a}| \cos 0^\circ \\ &= |\vec{a}||\vec{a}|(1)\end{aligned}$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$$\Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

3.5.2 Dot Product of Fundamental Unit Vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

The fundamental unit vectors in $\mathbb{R}^3$ are $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$\hat{i}$ is along x-axis; $\hat{j}$ is along y-axis and $\hat{k}$ is along z-axis. So $|\hat{i}| = 1$; $|\hat{j}| = 1$ and $|\hat{k}| = 1$. Now

$$\hat{i} \cdot \hat{i} = |\hat{i}||\hat{i}| \cos 0^\circ = 1.1.1 = 1$$

$$\hat{j} \cdot \hat{j} = |\hat{j}||\hat{j}| \cos 0^\circ = 1.1.1 = 1$$

$$\hat{k} \cdot \hat{k} = |\hat{k}||\hat{k}| \cos 0^\circ = 1.1.1 = 1$$

$$\hat{i} \cdot \hat{j} = |\hat{i}||\hat{j}| \cos 90^\circ = 1.1.0 = 0$$

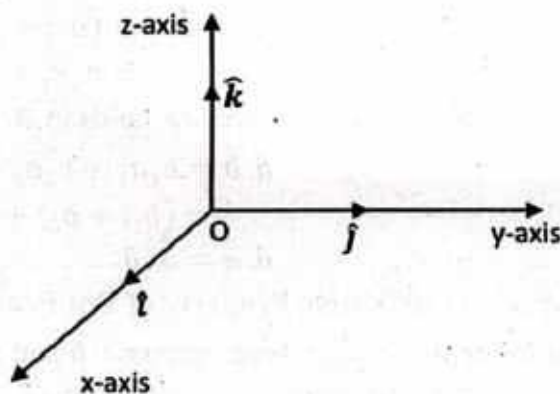
$$\hat{j} \cdot \hat{k} = |\hat{j}||\hat{k}| \cos 90^\circ = 1.1.0 = 0$$

$$\hat{k} \cdot \hat{i} = |\hat{k}||\hat{i}| \cos 90^\circ = 1.1.0 = 0$$

$$\text{Also } \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{j} = 0$$

$$\hat{k} \cdot \hat{j} = \hat{j} \cdot \hat{k} = 0$$

$$\hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$



3.5.3 Dot Product in Terms of Components

Consider any two non-zero vectors $\vec{a}$ and $\vec{b}$ in space.

$$\text{Let } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\text{and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\begin{aligned}\text{Then } \vec{a} \cdot \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= (a_1\hat{i}) \cdot (b_1\hat{i}) + (a_1\hat{i}) \cdot (b_2\hat{j}) + (a_1\hat{i}) \cdot (b_3\hat{k}) + (a_2\hat{j}) \cdot (b_1\hat{i}) + (a_2\hat{j}) \cdot (b_2\hat{j}) \\ &\quad + (a_2\hat{j}) \cdot (b_3\hat{k}) + (a_3\hat{k}) \cdot (b_1\hat{i}) + (a_3\hat{k}) \cdot (b_2\hat{j}) + (a_3\hat{k}) \cdot (b_3\hat{k})\end{aligned}$$

Since the dot product is defined between the vectors, so,

$$\begin{aligned}\vec{a} \cdot \vec{b} &= a_1b_1(\hat{i} \cdot \hat{i}) + a_1b_2(\hat{i} \cdot \hat{j}) + a_1b_3(\hat{i} \cdot \hat{k}) + a_2b_1(\hat{j} \cdot \hat{i}) + a_2b_2(\hat{j} \cdot \hat{j}) + a_2b_3(\hat{j} \cdot \hat{k}) \\ &\quad + a_3b_1(\hat{k} \cdot \hat{i}) + a_3b_2(\hat{k} \cdot \hat{j}) + a_3b_3(\hat{k} \cdot \hat{k})\end{aligned}$$

$$\begin{aligned}\vec{a} \cdot \vec{b} &= a_1b_1(1) + a_1b_2(0) + a_1b_3(0) + a_2b_1(0) + a_2b_2(1) + a_2b_3(0) + \\ &\quad a_3b_1(0) + a_3b_2(0) + a_3b_3(1)\end{aligned}$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

This is known as analytical expression for dot product.

3.5.4 Condition for Orthogonality of Two Vectors

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ in space. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a}$ and $\vec{b}$ will be orthogonal (perpendicular) to each other if and only if $\vec{a} \cdot \vec{b} = 0$

$$\Rightarrow (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$$

$$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$$

Which is the condition for the two vectors $\vec{a}$ and $\vec{b}$ to be orthogonal to each other.

3.5.5 Commutative Property of Dot Product

Statement: For any two vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Proof: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be two vectors

$$\begin{aligned}\therefore \vec{a} \cdot \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

Since commutative law holds in $\mathbb{R}$, so

$$\begin{aligned}\vec{a} \cdot \vec{b} &= b_1a_1 + b_2a_2 + b_3a_3 \\ &= (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \cdot (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a}\end{aligned}$$

3.5.6 Distributive Property of Dot Product

Statement: For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Proof: Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then

$$\begin{aligned}\vec{b} + \vec{c} &= (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\ &= (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k} \\ \Rightarrow \vec{a} \cdot (\vec{b} + \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot [(b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}] \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= (a_1b_1 + a_1c_1) + (a_2b_2 + a_2c_2) + (a_3b_3 + a_3c_3) \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}\end{aligned}$$

3.5.7 Direction Angles

The angles which a non-zero vector $\vec{r}$ makes with the coordinate axes in the positive direction are known as direction angles of $\vec{r}$. Let these angles be α, β and γ ; then

$$0 \leq \alpha \leq \pi; \quad 0 \leq \beta \leq \pi; \quad 0 \leq \gamma \leq \pi$$

3.5.8 Direction Cosines

If α, β and γ be the direction angles of a non-zero vector $\vec{r}$ with x-axis; y-axis and z-axis respectively, then $\cos \alpha, \cos \beta$ and $\cos \gamma$ are called the direction cosines of $\vec{r}$.

Here $\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

From right-angled triangle AOP:

$$\frac{|\overrightarrow{OA}|}{|\overrightarrow{OP}|} = \cos \alpha \text{ or } \Rightarrow$$

$$\cos \alpha = \frac{x}{|\vec{r}|}$$

From right-angled triangle BOP:

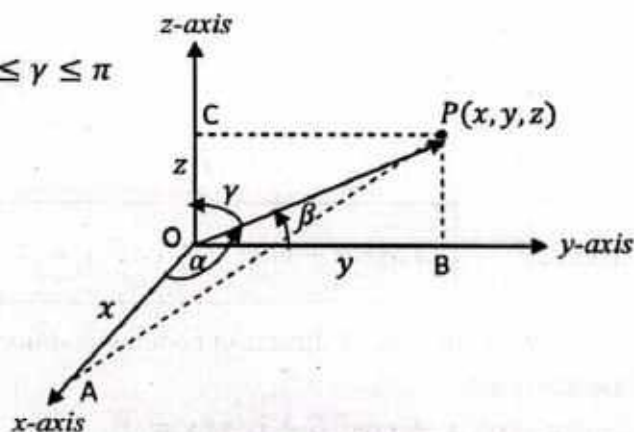
$$\frac{|\overrightarrow{OB}|}{|\overrightarrow{OP}|} = \cos \beta \text{ or } \Rightarrow$$

$$\cos \beta = \frac{y}{|\vec{r}|}$$

From right-angled triangle COP:

$$\frac{|\overrightarrow{OC}|}{|\overrightarrow{OP}|} = \cos \gamma \text{ or } \Rightarrow$$

$$\cos \gamma = \frac{z}{|\vec{r}|}$$



In general $\cos \alpha, \cos \beta$ and $\cos \gamma$ are denoted by l, m and n respectively. i.e.;

$$l = \cos \alpha = \frac{x}{|\vec{r}|}$$

$$m = \cos \beta = \frac{y}{|\vec{r}|}$$

$$n = \cos \gamma = \frac{z}{|\vec{r}|}$$

3.5.9 Sum of Squares of Direction Cosines is Unity

Let $\vec{r}$ be a non-zero vector and α, β and γ be its direction angles. If

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Then

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\vec{r}|^2 = x^2 + y^2 + z^2$$

The direction cosines of $\vec{r}$ are $\cos \alpha = \frac{x}{|\vec{r}|}, \cos \beta = \frac{y}{|\vec{r}|}$ and $\cos \gamma = \frac{z}{|\vec{r}|}$.

$$\text{Now } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \left(\frac{x}{|\vec{r}|}\right)^2 + \left(\frac{y}{|\vec{r}|}\right)^2 + \left(\frac{z}{|\vec{r}|}\right)^2$$

$$= \frac{x^2}{|\vec{r}|^2} + \frac{y^2}{|\vec{r}|^2} + \frac{z^2}{|\vec{r}|^2} = \frac{x^2 + y^2 + z^2}{|\vec{r}|^2}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^2} = 1$$

Hence

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$\therefore$ Sum of squares of direction cosines is unity.

Deduction:

$$\text{Since } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1^2$$

$$\Rightarrow (1 - \sin^2 \alpha) + (1 - \sin^2 \beta) + (1 - \sin^2 \gamma) = 1$$

$$\Rightarrow 3 - \sin^2 \alpha - \sin^2 \beta - \sin^2 \gamma = 1$$

$$\Rightarrow \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$$

3.5.10 Direction Ratios

The numbers which are proportional to the direction cosines of a non-zero vector $\vec{r}$ are known as direction ratios.

Let a, b and c be the numbers which are proportional to $\cos \alpha, \cos \beta$ and $\cos \gamma$. i.e.;

$$a \propto \cos \alpha; \quad b \propto \cos \beta; \quad c \propto \cos \gamma$$

$$\Rightarrow a = k \cos \alpha; \quad b = k \cos \beta; \quad c = k \cos \gamma$$

where k is constant of proportionality and $k \neq 0$

$$\Rightarrow a^2 + b^2 + c^2 = k^2 \cos^2 \alpha + k^2 \cos^2 \beta + k^2 \cos^2 \gamma$$

$$= k^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = k^2$$

$$\Rightarrow k = \pm \sqrt{a^2 + b^2 + c^2}$$

$$\therefore a = k \cos \alpha \Rightarrow \cos \alpha = \frac{a}{k}$$

$$\text{or } \cos \alpha = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore b = k \cos \beta \Rightarrow \cos \beta = \frac{b}{k}$$

$$\text{or } \cos \beta = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore c = k \cos \gamma \Rightarrow \cos \gamma = \frac{c}{k}$$

$$\text{or } \cos \gamma = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

These relations are used to find direction cosines when its direction ratios are given.

Key Facts



- Direction ratios are also known as direction numbers or direction components.
- Sum of squares of direction ratios need not to be unity.



Let $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point $P(x, y, z)$.

If $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are its direction cosines then

$$\cos \alpha = \frac{x}{|\vec{r}|} \Rightarrow x = |\vec{r}| \cos \alpha$$

$$\cos \beta = \frac{y}{|\vec{r}|} \Rightarrow y = |\vec{r}| \cos \beta$$

$$\cos \gamma = \frac{z}{|\vec{r}|} \Rightarrow z = |\vec{r}| \cos \gamma$$

The coordinates of point P in the form of direction cosines can be written as:

$$(x, y, z) = (|\vec{r}| \cos \alpha, |\vec{r}| \cos \beta, |\vec{r}| \cos \gamma)$$

Example 9: Find the coordinates of point P , if $\vec{OP}$ is a vector of magnitude 2 and is parallel to the vector $2\hat{i} - 3\hat{j} + 4\hat{k}$.

Solution:

$$\text{Let } \vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\Rightarrow |\vec{a}| = \sqrt{(2)^2 + (-3)^2 + (4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$\text{Thus } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{29}}$$

$$\hat{a} = \frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$$

$$\text{As } \vec{OP} = 2\hat{a} = 2\left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right)$$

$$\Rightarrow \vec{OP} = \frac{4}{\sqrt{29}}\hat{i} - \frac{6}{\sqrt{29}}\hat{j} + \frac{8}{\sqrt{29}}\hat{k}$$

$$\therefore \left(\frac{4}{\sqrt{29}}, -\frac{6}{\sqrt{29}}, \frac{8}{\sqrt{29}}\right) \text{ are the coordinates of point } P.$$

Example 10: Two direction angles of vector $\vec{r}$ are 30° and 60° . Find the third direction angle. Also find unit vector $\hat{r}$.

Solution:

$$\text{Let } \alpha = 30^\circ \text{ and } \beta = 60^\circ$$

$$\text{Since } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 30^\circ + \cos^2 60^\circ + \cos^2 \gamma = 1$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2 \gamma = 1$$

$$\Rightarrow \frac{3}{4} + \frac{1}{4} + \cos^2 \gamma = 1 \Rightarrow 1 + \cos^2 \gamma = 1 \Rightarrow \cos^2 \gamma = 0$$

$$\Rightarrow \cos \gamma = 0$$

$$\Rightarrow \gamma = 90^\circ \text{ or } 270^\circ$$

Since $0 \leq \gamma \leq 180^\circ$

So $\gamma = 90^\circ$

Unit vector of $\vec{r}$ is:

$$\begin{aligned}\hat{r} &= \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \\ \Rightarrow \hat{r} &= \cos 30^\circ \hat{i} + \cos 60^\circ \hat{j} + \cos 90^\circ \hat{k} \\ \Rightarrow \hat{r} &= \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} + 0 \hat{k}\end{aligned}$$

Which is the required unit vector.

3.5.11 Angle Between Two Non-Zero Vectors

Let θ be the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

$$\Rightarrow \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\text{or } \theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

$$\theta = \cos^{-1} \left(\frac{\vec{a}}{|\vec{a}|} \cdot \frac{\vec{b}}{|\vec{b}|} \right)$$

$$\Rightarrow \theta = \cos^{-1}(\hat{a} \cdot \hat{b})$$

In component form we can write it as

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

Then $\vec{a} \cdot \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Substituting values in the equation $\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$, we have:

$$\theta = \cos^{-1} \left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \right)$$

Example 11: Find the angle between the vectors $\hat{i} - 2\hat{j} + \hat{k}$ and $2\hat{i} - 3\hat{j} + \hat{k}$.

Solution:

Let $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$, then

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = (1)(2) + (-2)(-3) + (1)(1) \\ &= 2 + 6 + 1 = 9\end{aligned}$$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

If θ is the angle between $\vec{a}$ and $\vec{b}$ then

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

$$\theta = \cos^{-1} \left(\frac{9}{\sqrt{6}\sqrt{14}} \right) = \cos^{-1} \left(\frac{9}{\sqrt{84}} \right)$$

$$\Rightarrow \theta = 10.89^\circ$$

3.5.12 Projection of a Vector Along Another Vector

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ and let θ be the angle between them.

$|\vec{OL}|$ is the projection of $\vec{b}$ upon $\vec{a}$.

From right-angled triangle BOL;

$$\frac{|\vec{OL}|}{|\vec{OB}|} = \cos \theta$$

$$\begin{aligned} |\vec{OL}| &= |\vec{OB}| \cos \theta \\ &= |\vec{b}| \cos \theta \\ &= \frac{|\vec{a}| |\vec{b}| \cos \theta}{|\vec{a}|} \end{aligned}$$

$$|\vec{OL}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\therefore \text{Projection of } \vec{b} \text{ along } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$= \frac{\vec{a}}{|\vec{a}|} \cdot \vec{b}$$

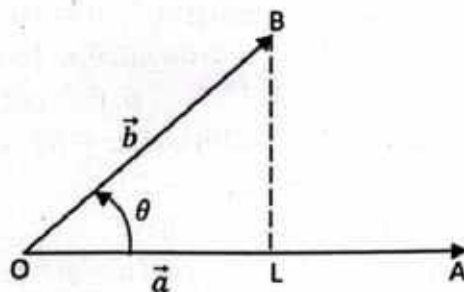
$$= \hat{a} \cdot \vec{b} = \vec{b} \cdot \hat{a}$$

$$\text{Projection of } \vec{b} \text{ along } \vec{a} = \vec{b} \cdot \hat{a}$$

Similarly, we can prove that:

$$\text{Projection of } \vec{a} \text{ along } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$\text{Projection of } \vec{a} \text{ along } \vec{b} = \vec{a} \cdot \hat{b}$$



Example 12: If $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$, find projection of $\vec{a}$ along $\vec{b}$ and projection of $\vec{b}$ along $\vec{a}$.

Solution:

$$\vec{a} \cdot \vec{b} = (\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + \hat{j} + 3\hat{k})$$

$$= (1)(-1) + (-1)(1) + (1)(3) = -1 - 1 + 3 = 1$$

$$|\vec{a}| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(-1)^2 + (1)^2 + (3)^2} = \sqrt{11}$$

$$\text{Now projection of } \vec{a} \text{ along } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{11}}$$

$$\text{Projection of } \vec{b} \text{ along } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{1}{\sqrt{3}}$$

Example 13: Prove that $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

Solution:

Consider two-unit vectors $\hat{u} = \overrightarrow{OP}$

and $\hat{v} = \overrightarrow{OQ}$ making angles α and β with x-axis as shown in the figure. Thus

$\alpha - \beta$ is the angle between $\hat{u}$ and $\hat{v}$. Since $\hat{u} = \overrightarrow{OP}$ is the position vector of point P so, $\hat{u} = \cos \alpha \hat{i} + \sin \alpha \hat{j}$. Similarly, $\hat{v} = \cos \beta \hat{i} + \sin \beta \hat{j}$

$$\text{Now, } \hat{u} \cdot \hat{v} = (\cos \alpha \hat{i} + \sin \alpha \hat{j}) \cdot (\cos \beta \hat{i} + \sin \beta \hat{j})$$

$$\hat{u} \cdot \hat{v} = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (1)$$

$$\text{Also } \hat{u} \cdot \hat{v} = |\hat{u}| |\hat{v}| \cos(\alpha - \beta) = (1)(1) \cos(\alpha - \beta)$$

$$\hat{u} \cdot \hat{v} = \cos(\alpha - \beta) \quad (2)$$

From equation (1) and (2) we have:

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Example 14: For any triangle ABC, with usual notations prove that $|\vec{a}| = |\vec{b}| \cos \gamma + |\vec{c}| \cos \beta$

Solution

Consider a triangle as shown in figure.

It is clear that

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$\Rightarrow \vec{a} = -\vec{b} - \vec{c}$$

Taking dot product with $\vec{a}$ on both sides

$$\Rightarrow \vec{a} \cdot \vec{a} = \vec{a} \cdot (-\vec{b} - \vec{c})$$

$$\Rightarrow |\vec{a}|^2 = -\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$$

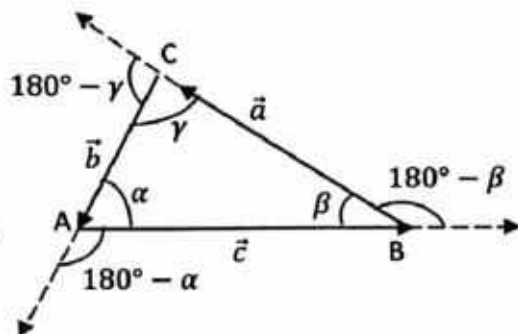
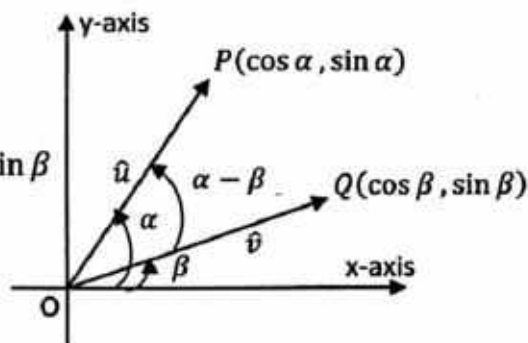
$$= -|\vec{a}| |\vec{b}| \cos(180^\circ - \gamma) - |\vec{a}| |\vec{c}| \cos(180^\circ - \beta)$$

$$= -|\vec{a}| |\vec{b}| (-\cos \gamma) - |\vec{a}| |\vec{c}| (-\cos \beta) \quad \because \cos(180^\circ - \theta) = -\cos \theta$$

$$= |\vec{a}| |\vec{b}| \cos \gamma + |\vec{a}| |\vec{c}| \cos \beta$$

$$\Rightarrow |\vec{a}|^2 = |\vec{a}| (|\vec{b}| \cos \gamma + |\vec{c}| \cos \beta)$$

$$\Rightarrow |\vec{a}| = |\vec{b}| \cos \gamma + |\vec{c}| \cos \beta$$



3.5.13 Work Done by a Constant Force

Let a constant force is applied on an object and it is displaced from A to B.

The force $\vec{F}$ makes an angle θ with the displacement vector $\vec{S}$.

The component of $\vec{F}$ along $\vec{S}$ is $|\vec{F}| \cos \theta$.

Thus work done by the force $\vec{F}$ to move the object from A to B is:

$$\text{work done} = |\vec{F}| \cos \theta (|\vec{S}|) = |\vec{F}| |\vec{S}| \cos \theta$$

$$\text{work done} = \vec{F} \cdot \vec{S}$$

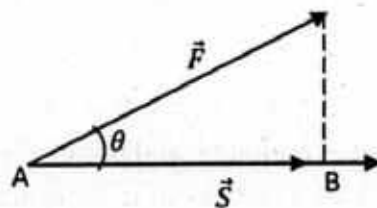
Example 15: Find the work done by a constant force $\vec{F} = 2\hat{i} + \hat{j} - \hat{k}$ in moving an object from $A(0,1,3)$ to $B(-1,2,4)$.

Solution:

$$\vec{F} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\begin{aligned}\vec{S} &= \overrightarrow{AB} = (1-0)\hat{i} + (2-1)\hat{j} + (4-3)\hat{k} \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

$$\begin{aligned}\text{Work done} &= \vec{F} \cdot \vec{S} \\ &= (2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) \\ &= (2)(1) + (1)(1) + (-1)(1) \\ &= 2 + 1 - 1 = 2 \text{ units}\end{aligned}$$



Exercise 3.2

- If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$; $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{c} = -\hat{i} - 2\hat{j} + 5\hat{k}$, then evaluate the followings.
 - $\vec{a} \cdot \vec{b}$
 - $2\vec{a} \cdot 3\vec{b}$
 - $(\vec{a} - \vec{b}) \cdot \vec{c}$
 - $(2\vec{a} + 3\vec{b} - \vec{c}) \cdot (\vec{a} + \vec{b})$
 - $\hat{i} \cdot \vec{a} + \hat{j} \cdot \vec{b} + \hat{k} \cdot \vec{c}$
- If $\vec{a} = \hat{j} - \hat{k}$; $\vec{b} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - 4\hat{k}$, then find the angles between the vectors:
 - $\vec{a}$ and $3\vec{b}$
 - $(2\vec{a} - 3\vec{b})$ and $2\vec{c}$
 - $(-\vec{a} + \vec{c})$ and $(\vec{b} - 2\vec{c})$
 - $(\vec{a} + \vec{b} + \vec{c})$ and $(\vec{a} - \vec{b} - \vec{c})$
 - $(\vec{a} - 2\vec{b} + \vec{c})$ and $\vec{a}$
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 2$; $|\vec{b}| = 3$ and $|\vec{c}| = 4$ then find angle between $\vec{a}$ and $\vec{b}$.
 - If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then find angle between $\vec{a}$ and $\vec{b}$.

4. (i) If $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$; $\vec{b} = 7\hat{i} - 9\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j} + 5\hat{k}$, find the value of λ so that $\vec{a} - \lambda\vec{b}$ is perpendicular to $\vec{c}$.
 (ii) Show that the angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.
5. (i) If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, then find the direction cosine of $\vec{a}$.
 (ii) If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{j} + 8\hat{k}$, then find the projection of $\vec{a} - \vec{b}$ along $\vec{c}$ and projection of $\vec{b}$ along $\vec{c} - \vec{a}$. Also find their vector projections.
6. (i) Three vertices of triangle are $A(0, -1, -2)$; $B(3, 1, 4)$ and $C(5, 7, 1)$. Show that ABC is a right-angled triangle and find the other two angles.
 (ii) A vector $\vec{r}$ is equally inclined with the coordinate axes and $|\vec{r}| = 5$. Find the vector $\vec{r}$.
7. (i) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 2$; $|\vec{b}| = 5$; $|\vec{c}| = 4$ and $\vec{a} + \vec{b} + \vec{c} = 0$ then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$
 (ii) For any vector $\vec{r}$ prove that $\vec{r} = (\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$
8. The dot products of $\vec{r}$ with the vectors $\hat{i} + \hat{j} - 3\hat{k}$; $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ are 0, 5 and 8 respectively. Find the vector $\vec{r}$.
9. Prove that for any non-zero vectors $\vec{a}$ and $\vec{b}$;
 (i) $\vec{a} \cdot \vec{b} = \frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$
 (ii) $|\vec{a}|^2 + |\vec{b}|^2 = \frac{1}{2}|\vec{a} + \vec{b}|^2 + \frac{1}{2}|\vec{a} - \vec{b}|^2$
10. (i) If the sum of two unit vectors is a unit vector, show that magnitude of their difference is $\sqrt{3}$.
 (ii) Show that angle in a semi-circle is a right angle.
11. (i) Prove that altitudes of a triangle are concurrent.
 (ii) Prove that angle bisectors of triangle are concurrent.
12. (i) Prove that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.
 (ii) With usual notations for a triangle ABC; prove that $c^2 = a^2 + b^2 - 2ab \cos \gamma$ and $b = a \cos \gamma + c \cos \alpha$
13. (i) The resultant of two vectors $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{a}$ and $|\vec{a}| = \frac{1}{\sqrt{7}}|\vec{b}|$. Show that resultant of $7\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{b}$.
 (ii) Prove that $\vec{a} + \vec{b}$ is equally inclined with $\vec{a}$ and $\vec{b}$.
14. A force of $\vec{F} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ newtons is applied on a body and moves it at a distance of 14 meters in the direction of the vector $\hat{i} - 3\hat{j} + \hat{k}$. How much work is done?
15. Find the work done by the forces $2\hat{i} + 3\hat{j} - \hat{k}$ and $3\hat{i} + 7\hat{j} + 4\hat{k}$ acting on a particle in moving from point P to Q with position vectors $\hat{i} - 3\hat{j} + \frac{3}{2}\hat{k}$ and $2\hat{i} - \hat{j} + \frac{5}{2}\hat{k}$.
16. A box is dragged on the surface of a floor by a string that is applying a force of 30N at an angle of 30° with the floor. Find the work done by the force when the box is displaced up-to a distance of 10 meters.

3.6 Cross or Vector Product of Two Vectors

3.6.1 Cross or Vector Product of Two Vectors and its Geometrical Interpretation

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors and θ is the angle between them, then their cross product is also a vector denoted by $\vec{a} \times \vec{b}$ and is defined as:

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{n}$$

where $\hat{n}$ is a unit vector normal to the plane containing both the vectors $\vec{a}$ and $\vec{b}$. θ is positive when measured anticlockwise and is negative when measured clockwise.

While computing $\vec{b} \times \vec{a}$ angle is measured from $\vec{b}$ to $\vec{a}$ which is the clockwise direction so;

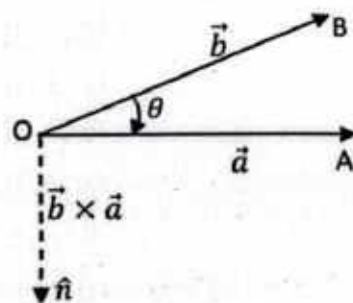
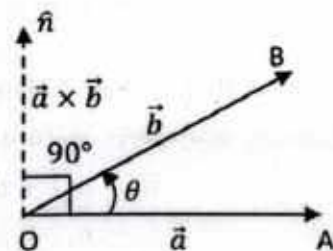
$$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}| \sin(-\theta) \hat{n}$$

$$\begin{aligned} \vec{b} \times \vec{a} &= -|\vec{b}||\vec{a}| \sin \theta \hat{n} (\because \sin(-\theta) = -\sin \theta) \\ &= -\vec{a} \times \vec{b} \end{aligned}$$

This shows that $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ are opposite in direction.

Thus, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

It means cross product is anti-commutative.



3.6.2 Cross Product of Fundamental Unit Vectors

We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{n}$, so

$$\begin{aligned} \hat{i} \times \hat{i} &= |\hat{i}||\hat{i}| \sin 0^\circ \hat{n} \\ &= (1)(1)(0)\hat{n} = \vec{0} \end{aligned}$$

$$\begin{aligned} \hat{j} \times \hat{j} &= |\hat{j}||\hat{j}| \sin 0^\circ \hat{n} \\ &= (1)(1)(0)\hat{n} = \vec{0} \end{aligned}$$

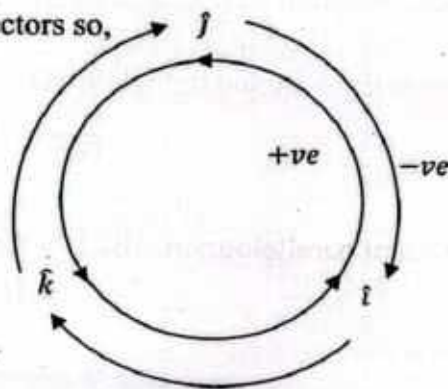
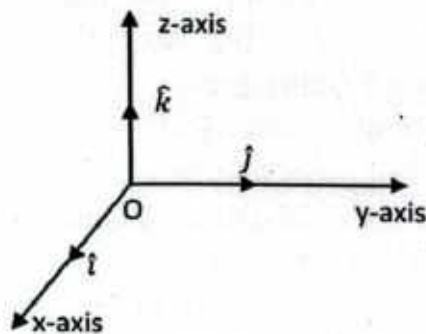
$$\begin{aligned} \hat{k} \times \hat{k} &= |\hat{k}||\hat{k}| \sin 0^\circ \hat{n} \\ &= (1)(1)(0)\hat{n} = \vec{0} \end{aligned}$$

We know that $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the mutually perpendicular unit vectors so,

$$\begin{aligned} \hat{i} \times \hat{j} &= |\hat{i}||\hat{j}| \sin 90^\circ \hat{k} \\ &= (1)(1)(1)\hat{k} = \hat{k} \end{aligned}$$

$$\begin{aligned} \hat{j} \times \hat{k} &= |\hat{j}||\hat{k}| \sin 90^\circ \hat{i} \\ &= (1)(1)(1)\hat{i} = \hat{i} \end{aligned}$$

$$\hat{k} \times \hat{i} = |\hat{k}||\hat{i}| \sin 90^\circ \hat{j}$$



$$= (1)(1)(1)\hat{j} = \hat{j}$$

Also $\hat{j} \times \hat{i} = -\hat{i} \times \hat{j} = -\hat{k}$
 $\hat{k} \times \hat{j} = -\hat{j} \times \hat{k} = -\hat{i}$
 and $\hat{i} \times \hat{k} = -\hat{k} \times \hat{i} = -\hat{j}$

Key Facts



The cross product is defined only for the vector in 3-space; whereas dot product is defined for vector both in 2-space and 3-space.

3.6.3 Cross Product in Terms of its Components

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$, where,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

and

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Now

$$\begin{aligned}\vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= (a_1b_1)(\hat{i} \times \hat{i}) + (a_1b_2)(\hat{i} \times \hat{j}) + (a_1b_3)(\hat{i} \times \hat{k}) + (a_2b_1)(\hat{j} \times \hat{i}) + \\ &\quad (a_2b_2)(\hat{j} \times \hat{j}) + (a_2b_3)(\hat{j} \times \hat{k}) + (a_3b_1)(\hat{k} \times \hat{i}) + (a_3b_2)(\hat{k} \times \hat{j}) + \\ &\quad (a_3b_3)(\hat{k} \times \hat{k}) \\ &= (a_1b_1)(\vec{0}) + (a_1b_2)(\hat{k}) + (a_1b_3)(-\hat{j}) + (a_2b_1)(-\hat{k}) + (a_2b_2)(\vec{0}) \\ &\quad + (a_2b_3)(\hat{i}) + (a_3b_1)(\hat{j}) + (a_3b_2)(-\hat{i}) + (a_3b_3)(\vec{0}) \\ \vec{a} \times \vec{b} &= (a_1b_2)\hat{k} - (a_1b_3)\hat{j} - (a_2b_1)\hat{k} + (a_2b_3)\hat{i} + (a_3b_1)\hat{j} - (a_3b_2)\hat{i} \\ \vec{a} \times \vec{b} &= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}\end{aligned}$$

Which is cross product in component form. Also cross product can be written as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3.6.4 Area of Parallelogram

Consider a parallelogram ABCD.

Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ be the two adjacent sides of the parallelogram and θ is the angle between them.

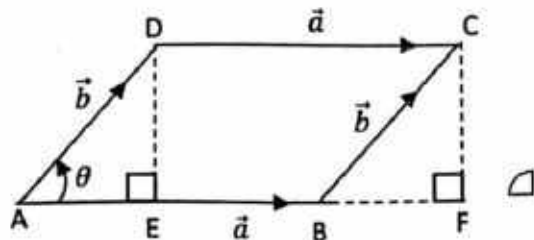
From figure it is clear that area of the parallelogram ABCD is same as that of the area of rectangle EFCD.

From right-angled triangle EAD,

$$\begin{aligned}\frac{|ED|}{|AD|} &= \sin \theta \Rightarrow \frac{|ED|}{|\vec{b}|} = \sin \theta \\ \Rightarrow |ED| &= |\vec{b}| \sin \theta\end{aligned}$$

Area of parallelogram ABCD = Area of rectangle EFCD

$$\begin{aligned}&= |\overrightarrow{EF}| |ED| = |\overrightarrow{DC}| |ED| \\ &= |\vec{a}| |\vec{b}| \sin \theta\end{aligned} \quad (1)$$

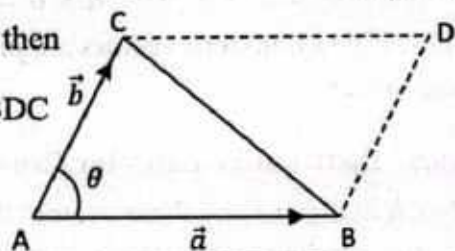


Also

$$\begin{aligned}\vec{a} \times \vec{b} &= |\vec{a}||\vec{b}| \sin \theta \hat{n} \\ |\vec{a} \times \vec{b}| &= |\vec{a}||\vec{b}| \sin \theta |\hat{n}| \\ &= |\vec{a}||\vec{b}| \sin \theta\end{aligned}\quad (2)$$

From equations (1) and (2):

$$|\vec{a} \times \vec{b}| = \text{area of parallelogram ABCD}$$

If $\vec{a}$ and $\vec{b}$ are two adjacent sides of a triangle ABC thenArea of triangle ABC = $\frac{1}{2}$ Area of parallelogram ABDCArea of triangle ABC = $\frac{1}{2} |\vec{a} \times \vec{b}|$ 

Key Facts

Example 16: If $\vec{a} = 2\hat{i} + 5\hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$ are two adjacent sides of a parallelogram, then find its area.

Solution:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -2 \\ 1 & -1 & 3 \end{vmatrix} \\ &= (15 - 2)\hat{i} - (6 + 2)\hat{j} + (-2 - 5)\hat{k}\end{aligned}$$

$$\vec{a} \times \vec{b} = 13\hat{i} - 8\hat{j} - 7\hat{k}$$

$$\begin{aligned}|\vec{a} \times \vec{b}| &= \sqrt{13^2 + (-8)^2 + (-7)^2} \\ &= \sqrt{169 + 64 + 49} = \sqrt{282}\end{aligned}$$

$$\begin{aligned}\text{Area of parallelogram} &= |\vec{a} \times \vec{b}| \\ &= \sqrt{282} \text{ sq. units}\end{aligned}$$

Example 17: Find the area of a triangle with vertices $(0, 0)$, $(2, 9)$, $(3, 5)$.

Solution:Let $A = (0, 0)$; $B = (2, 9)$ and $C = (3, 5)$ Then $\vec{a} = \overrightarrow{AC}$; $\vec{b} = \overrightarrow{AB}$ So $\vec{a} = (3 - 0)\hat{i} + (5 - 0)\hat{j} = 3\hat{i} + 5\hat{j} + 0\hat{k}$ $\vec{b} = (2 - 0)\hat{i} + (9 - 0)\hat{j} = 2\hat{i} + 9\hat{j} + 0\hat{k}$

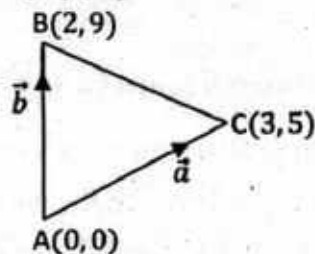
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 0 \\ 2 & 9 & 0 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = (0 - 0)\hat{i} + (0 - 0)\hat{j} + (27 - 10)\hat{k}$$

$$\vec{a} \times \vec{b} = 0\hat{i} + 0\hat{j} + 17\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 17^2} = 17$$

$$\text{Area of the triangle} = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2}(17) = \frac{17}{2} \text{ sq units}$$



3.6.5 Condition for the Two Non-Zero Vectors to be Parallel

Let $\vec{a}$ and $\vec{b}$ are two non-zero vectors. If $\vec{a}$ and $\vec{b}$ are parallel then angle between the vectors is $\theta = 0^\circ$. So

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin 0^\circ \hat{n} = |\vec{a}||\vec{b}|(0)\hat{n} = \vec{0}$$

Also, if $\vec{a}$ and $\vec{b}$ are anti-parallel then angle between the vectors is $\theta = 180^\circ$. So

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin 180^\circ \hat{n} = |\vec{a}||\vec{b}|(0)\hat{n} = \vec{0}$$

Thus, if two non-zero vectors are parallel or anti-parallel then the value of their cross product is zero vector.

3.6.6 Distributive Laws for Cross Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors then:

$$(i) \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(ii) (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

Proof: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\text{Then } \vec{b} + \vec{c} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

$$\begin{aligned} \text{LHS} = \vec{a} \times (\vec{b} + \vec{c}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\ &= \text{RHS} \end{aligned}$$

Similarly, we can prove $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

3.6.7 Angle Between Two Vectors

If θ is the angle between the non-zero vectors $\vec{a}$ and $\vec{b}$ then:

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| |\sin \theta| |\hat{n}|$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$$

$$(\because 0 \leq \theta \leq \pi \text{ so } |\sin \theta| = \sin \theta)$$

$$\Rightarrow \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \theta = \sin^{-1} \left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \right)$$

Example 18: If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$; $\vec{b} = -\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 3\hat{k}$ then find the angle between the vectors $\vec{a} + \vec{b}$ and $\vec{a} + \vec{c}$.

Solution:

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) + (-\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + \hat{j} + 0\hat{k}$$

and

$$\vec{a} + \vec{c} = (2\hat{i} - \hat{j} + \hat{k}) + (\hat{i} + \hat{j} - 3\hat{k}) = 3\hat{i} + 0\hat{j} - 2\hat{k}$$

now

$$\begin{aligned} (\vec{a} + \vec{b}) \times (\vec{a} + \vec{c}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 3 & 0 & -2 \end{vmatrix} \\ &= (-2 - 0)\hat{i} - (-2 - 0)\hat{j} + (0 - 3)\hat{k} \\ &= -2\hat{i} + 2\hat{j} - 3\hat{k} \end{aligned}$$

$$\begin{aligned} |(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})| &= \sqrt{(-2)^2 + (2)^2 + (-3)^2} \\ &= \sqrt{4 + 4 + 9} = \sqrt{17} \end{aligned}$$

$$|\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$|\vec{a} + \vec{c}| = \sqrt{3^2 + 0^2 + (-2)^2} = \sqrt{13}$$

Let θ be the angle between $\vec{a} + \vec{b}$ and $\vec{a} + \vec{c}$; then

$$\sin \theta = \frac{|(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})|}{|\vec{a} + \vec{b}| |\vec{a} + \vec{c}|}$$

$$\sin \theta = \frac{\sqrt{17}}{\sqrt{2}\sqrt{13}} = \sqrt{\frac{17}{26}}$$

$$\Rightarrow \theta = \sin^{-1} \left(\sqrt{\frac{17}{26}} \right) = 53.96^\circ$$

Example 19: Show that $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

Solution:

Consider two unit vectors $\hat{u} = \overrightarrow{OA}$ and $\hat{v} = \overrightarrow{OB}$ making angles α and β with x -axis respectively. Then $\alpha - \beta$ is the angle between $\hat{u}$ and $\hat{v}$.

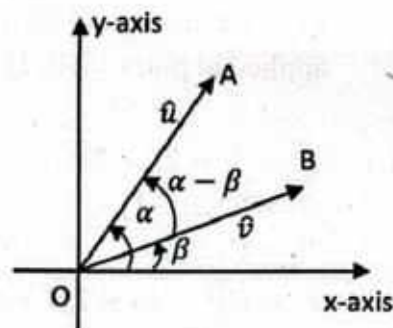
$$\therefore \hat{u} = \overrightarrow{OA} = \cos \alpha \hat{i} + \sin \alpha \hat{j}$$

$$\hat{v} = \overrightarrow{OB} = \cos \beta \hat{i} + \sin \beta \hat{j}$$

$$\begin{aligned} \text{Now } \hat{v} \times \hat{u} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} \\ &= (0 - 0)\hat{i} - (0 - 0)\hat{j} + (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\hat{k} \\ &= 0\hat{i} - 0\hat{j} + (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\hat{k} \end{aligned}$$

$$\begin{aligned} \Rightarrow |\hat{v} \times \hat{u}| &= \sqrt{0^2 + 0^2 + (\sin \alpha \cos \beta - \cos \alpha \sin \beta)^2} \\ &= (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \end{aligned}$$

$$\text{Also } \hat{v} \times \hat{u} = |\hat{v}| |\hat{u}| \sin(\alpha - \beta) \hat{n}$$



(1)

$$\Rightarrow \hat{v} \times \hat{u} = (1)(1)|\sin(\alpha - \beta)||\hat{n}|$$

$$\Rightarrow \hat{v} \times \hat{u} = \sin(\alpha - \beta)$$

(2)

From equation (1) and (2) we find that

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

3.6.8 Moment or Torque of a Given Force About a Given Point

The moment of a force is the turning effect of the force about a point, and is the product of the force and d ; where d is the perpendicular distance of the point from the line of action of the force.

From figure, moment of the force $\vec{F}$ acting at point P about point O is

$$\text{Moment} = |OA||\vec{F}|$$

From the right-triangle OAP;

$$\frac{|OA|}{|OP|} = \sin \theta; \text{ where } \theta \text{ is the angle between } \vec{r} \text{ and } \vec{F}.$$

$$\Rightarrow \frac{|OA|}{|\vec{r}|} = \sin \theta$$

$$\Rightarrow |OA| = |\vec{r}| \sin \theta$$

$$\text{Thus, moment} = (|\vec{r}| \sin \theta)|\vec{F}|$$

$$= |\vec{r}||\vec{F}| \sin \theta$$

$$\text{Moment} = \vec{r} \times \vec{F}$$

The vector $\vec{M} = \vec{r} \times \vec{F}$, is called vector moment of the force $\vec{F}$.

Example 20: Find the moment of the force $\vec{F} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ about the point $(2, 1, -1)$ when it is applied at point $(3, 0, 2)$.

Solution:

$$\text{Here } \vec{F} = 3\hat{i} - 2\hat{j} + 5\hat{k}$$

$$O = (2, 1, -1)$$

$$P = (3, 0, 2)$$

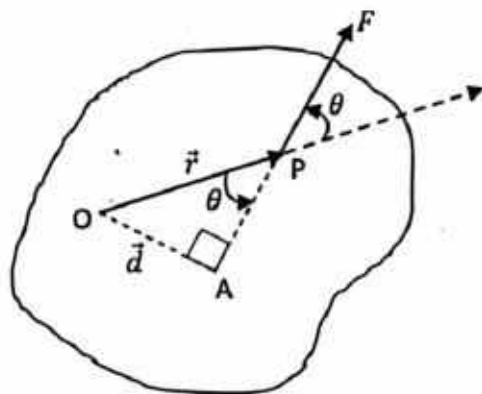
$$\begin{aligned} \vec{r} &= \vec{OP} = (3 - 2)\hat{i} + (0 - 1)\hat{j} + (2 + 1)\hat{k} \\ &= \hat{i} - \hat{j} + 3\hat{k} \end{aligned}$$

$$\text{Vector moment} = \vec{r} \times \vec{F}$$

$$\vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 3 & -2 & 5 \end{vmatrix}$$

$$= (-5 + 6)\hat{i} - (5 - 9)\hat{j} + (-2 + 3)\hat{k}$$

$$\vec{M} = \hat{i} + 4\hat{j} + \hat{k}$$



Example 21: Find the moment of the force $\vec{F} = 7\hat{i} + 4\hat{j} + 2\hat{k}$ when it is applied at the handle of a door at the point $(2, 1, 4)$ about the hinge at point $(0, 0, 1)$.

Solution:

Here $\vec{F} = 7\hat{i} + 4\hat{j} + 2\hat{k}$

$$O = (0, 0, 1)$$

$$H = (2, 1, 4)$$

$$\vec{r} = \vec{OH} = (2 - 0)\hat{i} + (1 - 0)\hat{j} + (4 - 1)\hat{k}$$

$$= 2\hat{i} + \hat{j} + 3\hat{k}$$

Vector moment $= \vec{r} \times \vec{F}$

$$\vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 7 & 4 & 2 \end{vmatrix}$$

$$= (2 - 12)\hat{i} - (4 - 21)\hat{j} + (8 - 7)\hat{k}$$

$$\vec{M} = -10\hat{i} + 17\hat{j} + \hat{k}$$

Is the required moment which is produced in the door.

Exercise 3.3

- For the following vectors, find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ and prove that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
 - $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$; $\vec{b} = \hat{i} - 3\hat{j} + 7\hat{k}$
 - $\vec{a} = 7\hat{i} + 3\hat{j} + 9\hat{k}$; $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$
 - $\vec{a} = \hat{i} - 2\hat{k}$; $\vec{b} = 3\hat{i} + 2\hat{j}$
- For the following vectors, find $\vec{a} \times \vec{b}$ and prove that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
 - $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$; $\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$
 - $\vec{a} = 4\hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{b} = \hat{i} + \hat{j} - 3\hat{k}$
- For the following vectors, find the value of the sine of the angle between them.
 - $\vec{a} = 2\hat{i} - 4\hat{j} + 3\hat{k}$; $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$
 - $\vec{a} = 4\hat{i} - 3\hat{j} + 2\hat{k}$; $\vec{b} = 3\hat{i} - 7\hat{j} + 5\hat{k}$
- Find a vector of magnitude 5 and perpendicular to both the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{b} = 8\hat{i} - 2\hat{j} + \hat{k}$.
 - Express the vector $5\hat{i} + 2\hat{j} - 3\hat{k}$ as a sum of two vectors one of which is parallel and other is perpendicular to the vector $2\hat{i} - \hat{j} + 3\hat{k}$.
- Prove the Lagrange identity $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$
 - For the vectors $\vec{a} = \hat{i} - 2\hat{j}$, $\vec{b} = 2\hat{i} + \hat{k}$ and $\vec{c} = 3\hat{j} + 2\hat{k}$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$. It is given that $\vec{c} \cdot \vec{d} = 1$.

6. (i) Find the vector $\vec{b}$ such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$; where $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$.
- (ii) If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$; where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.
7. (i) For a non-zero vector $\vec{a}$ if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then show that $\vec{b} = \vec{c}$.
- (ii) For three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ if $\vec{a} + \vec{b} + \vec{c} = 0$, then prove that:
- $$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$
8. (i) If $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then prove that $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.
- (ii) Prove that $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$
9. (i) If $|\vec{a}| = 3$; $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = \frac{15}{4}$ then find $|\vec{a} \times \vec{b}|$.
- (ii) If $|\vec{a}| = 2$; $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$ then find $\vec{a} \cdot \vec{b}$.
10. (i) Find the area of a parallelogram if $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 7\hat{k}$ are its two adjacent sides.
- (ii) Find the area of triangle with vertices $(1, -1, 1)$; $(2, 1, 2)$ and $(3, 0, -1)$. Also find its interior angles.
11. (i) Find the area of the parallelogram having diagonals $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.
- (ii) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of A, B and C respectively, then show that area of triangle ABC is $\frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.
12. If $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$, then what conclusion can be drawn about $\vec{a}$ and $\vec{b}$.
13. Show that the three points with position vectors $\vec{a} - \vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear.
14. (i) Find the moment of force $2\hat{i} + 3\hat{j} + 7\hat{k}$ about the point $(1, 2, 3)$ when applied at the point $(-1, 2, 0)$.
- (ii) Two forces $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 4\hat{j} - 2\hat{k}$ are applied at the same point $(1, -2, 4)$. Find the moment of these concurrent forces about $(0, 0, 0)$.
15. (i) How much force is required to produce a moment of magnitude $\sqrt{57} \text{ N.m}$ along the direction $6\hat{i} - 21\hat{j} - 6\hat{k}$ when applied at $(2, 1, -3)$ about $(-1, -1, 1)$.
- (ii) At what point the force $2\hat{i} + 2\hat{j} - 3\hat{k}$ should be applied to produce a vector moment $\vec{M} = 3\hat{i} - 2\hat{j} + \hat{k}$ about $(-1, 2, -3)$.
16. A toy car is located at a point $(2, 3, 5)$ relative to origin, such that when a force of $3\hat{i} + 2\hat{j} + 7\hat{k}$ is applied on car it starts rotating about the origin. Find the moment produced by the force in the car.

17. A seesaw is fixed from its middle point which is at $(0, 2, 3)$. Two forces $F_1[3, 4, 5]$ and $F_2[9, 2, 7]$ are applied at points $(4, 5, 3)$ and $(-4, -1, -3)$ respectively. Calculate the moment produced by each force about the fixed point in seesaw separately. Also find net moment.

3.7 Scalar Triple Product

3.7.1 Scalar Triple Product of Vectors

The scalar product of two vectors in which one vector is already a cross product of two vectors is called scalar triple product. If one vector is $\vec{a}$ and other is $\vec{b} \times \vec{c}$, then their scalar triple product is $\vec{a} \cdot (\vec{b} \times \vec{c})$. It is also denoted by $[\vec{a} \vec{b} \vec{c}]$.

3.7.2 Scalar triple product in form of components

Consider the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ such that:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{b} \times \vec{c} = (b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

Therefore,

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot [(b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}] \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \end{aligned}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If any two vectors in the scalar product are same or parallel then the two rows of the determinant will be same, then the value of the determinant is zero. i.e., $\vec{a} \cdot \vec{b} \times \vec{c} = 0$

If we interchange any two vectors in the scalar triple product then the corresponding rows of the determinant will be interchanged producing the value of new scalar triple product as a negative multiple of the original product.

$$\vec{a} \cdot \vec{b} \times \vec{c} = -\vec{b} \cdot \vec{a} \times \vec{c} = -\vec{a} \cdot \vec{c} \times \vec{b} \text{ etc.}$$

3.7.3 Scalar Triple Product of $\hat{i}, \hat{j}$ and $\hat{k}$ Vectors

$$\hat{i} \cdot \hat{j} \times \hat{k} = \hat{i} \cdot \hat{i} = 1$$

$$\hat{j} \cdot \hat{k} \times \hat{i} = \hat{j} \cdot \hat{j} = 1$$

Key Facts



$(\vec{a} \cdot \vec{b}) \times \vec{c}$ is meaningless, because $\vec{a} \cdot \vec{b}$ is a scalar and will not have cross product with $\vec{c}$. Thus, there should be no confusion in writing $\vec{a} \cdot (\vec{b} \times \vec{c})$ as $\vec{a} \cdot \vec{b} \times \vec{c}$.

$$\hat{k} \cdot \hat{i} \times \hat{j} = \hat{k} \cdot \hat{k} = 1$$

Thus

$$\hat{i} \cdot \hat{j} \times \hat{k} = \hat{j} \cdot \hat{k} \times \hat{i} = \hat{k} \cdot \hat{i} \times \hat{j} = 1$$

Also

$$\hat{i} \cdot \hat{k} \times \hat{j} = \hat{i} \cdot (-\hat{i}) = -(\hat{i} \cdot \hat{i}) = -1$$

$$\hat{j} \cdot \hat{i} \times \hat{k} = \hat{j} \cdot (-\hat{j}) = -(\hat{j} \cdot \hat{j}) = -1$$

$$\hat{k} \cdot \hat{j} \times \hat{i} = \hat{k} \cdot (-\hat{k}) = -(\hat{k} \cdot \hat{k}) = -1$$

Thus

$$\hat{i} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot \hat{i} \times \hat{k} = \hat{k} \cdot \hat{j} \times \hat{i} = -1$$

Similarly,

$$\hat{i} \cdot \hat{j} \times \hat{j} = \hat{i} \cdot (\vec{0}) = 0$$

$$\hat{j} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot (-\hat{i}) = -(\hat{j} \cdot \hat{i}) = 0 \text{ etc.}$$

Key Facts

When any two vectors are same the value of their scalar triple product is zero.



3.7.4 Dot and Cross are Interchangeable in Scalar Triple Product

Here we will prove that the cross and dot product in the scalar triple product are interchangeable.

$$\text{i.e., } \vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}$$

For this let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

As already proved that

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (1)$$

$$\text{Now } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = [(a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}] \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (a_2b_3 - a_3b_2)c_1 - (a_1b_3 - a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2 + a_1b_2c_3 - a_2b_1c_3$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_2b_3c_1) + (a_3b_1c_2 - a_3b_2c_1)$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\vec{a} \times \vec{b} \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (2)$$

From (1) and (2) it is clear that:

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}$$

Example 23: If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$; $\vec{b} = -3\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = -\hat{i} + \hat{j} - \hat{k}$, find $\vec{a} \cdot \vec{b} \times \vec{c}$.

Solution:

Since $\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Then $\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 2 & -3 & 1 \\ -3 & 2 & 3 \\ -1 & 1 & -1 \end{vmatrix}$

$$\vec{a} \cdot \vec{b} \times \vec{c} = 2(-2 - 3) + 3(3 + 3) + 1(-3 + 2)$$

$$= -10 + 18 - 1$$

$$\vec{a} \cdot \vec{b} \times \vec{c} = 7$$

Example 24: Let we have three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ such that $\vec{b}$ is parallel to $\vec{c}$. Compute $\vec{a} \cdot \vec{b} \times \vec{c}$.

Solution:

Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Given that $\vec{b}$ and $\vec{c}$ are parallel; so

$$\vec{b} = \lambda\vec{c} \text{ for some scalar } \lambda.$$

$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = (\lambda c_1)\hat{i} + (\lambda c_2)\hat{j} + (\lambda c_3)\hat{k}$$

$$\Rightarrow \boxed{b_1 = \lambda c_1;}$$

$$\boxed{b_2 = \lambda c_2;}$$

$$\boxed{b_3 = \lambda c_3.}$$

Now

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{a} \cdot \vec{b} \times \vec{c} = \lambda(0) = 0$$

($\because R_2$ & R_3 are identical)

3.7.5 Volume of Parallelopiped and a Tetrahedron

i. Volume of a Parallelopiped

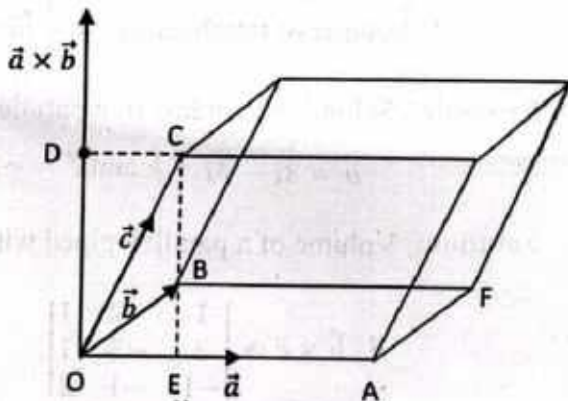
Consider a parallelopiped with adjacent sides as $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Volume of the parallelopiped

= (area of base)(perpendicular height)

$$= |\vec{a} \times \vec{b}| \cdot |\overrightarrow{OD}| \quad (1)$$

$\vec{a} \times \vec{b}$ is the vector perpendicular to both $\vec{a}$ and $\vec{b}$. So $\overrightarrow{OD}$ is in the direction of $\vec{a} \times \vec{b}$.



$|\overrightarrow{OD}|$ is the projection of $\vec{c}$ along $\vec{a} \times \vec{b}$. Therefore

$$|\overrightarrow{OD}| = \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} = \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{|\vec{a} \times \vec{b}|}$$

Putting in equation (1), we get:

$$\begin{aligned} \therefore \text{Volume of parallelepiped} &= |\vec{a} \times \vec{b}| \left(\frac{\vec{a} \cdot \vec{b} \times \vec{c}}{|\vec{a} \times \vec{b}|} \right) \\ &= \vec{a} \cdot \vec{b} \times \vec{c} \end{aligned}$$

ii. Volume of Tetrahedron

Consider a tetrahedron with its three coterminal edges $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Volume of tetrahedron

$$= \frac{1}{3} (\text{area of base})(\text{perpendicular height})$$

$$= \frac{1}{3} (\text{area of triangle OAB})(|\overrightarrow{OD}|)$$

$$= \frac{1}{3} \left(\frac{1}{2} |\vec{a} \times \vec{b}| \right) (|\overrightarrow{OD}|)$$

$$= \frac{1}{6} |\vec{a} \times \vec{b}| |\overrightarrow{OD}|$$

(1)

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. So $|\overrightarrow{OD}|$ is in the direction of $\vec{a} \times \vec{b}$.

Also $|\overrightarrow{OD}|$ is the projection of $\vec{c}$ on $\vec{a} \times \vec{b}$. Therefore

$$|\overrightarrow{OD}| = \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} = \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{|\vec{a} \times \vec{b}|}$$

Putting value in equation (1), we get:

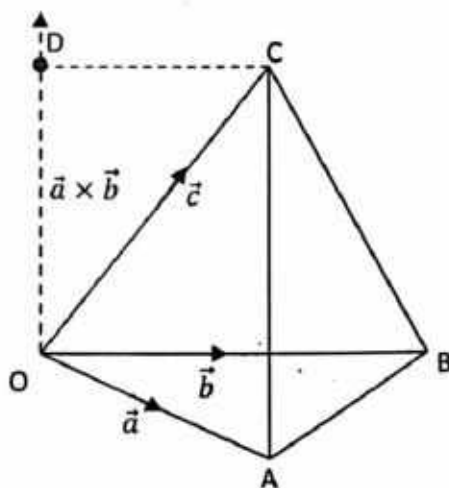
$$\therefore \text{Volume of tetrahedron} = \frac{1}{6} |\vec{a} \times \vec{b}| \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{|\vec{a} \times \vec{b}|} = \frac{1}{6} (\vec{a} \cdot \vec{b} \times \vec{c})$$

Example 25: Find the volume of a parallelepiped with adjacent sides $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$,

$$\vec{b} = 3\hat{i} - 3\hat{j} + \hat{k} \text{ and } \vec{c} = -\hat{i} - \hat{j} + 2\hat{k}.$$

Solution: Volume of a parallelepiped with adjacent sides $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 1 & -2 & 1 \\ 3 & -3 & 1 \\ -1 & -1 & 2 \end{vmatrix}$$



$$= 1(-6 + 1) + 2(6 + 1) + 1(-3 - 3) = -5 + 14 - 6$$

$$= 3 \text{ cubic units}$$

Example 26: Find the volume of a tetrahedron with vertices A(0, 0, 0), B(1, 3, -1), C(2, 2, 1) and D(1, 6, 5).

Solution:

Let the sides of tetrahedron ABCD are :

$$\vec{a} = \overrightarrow{AB} = (1 - 0)\hat{i} + (3 - 0)\hat{j} + (-1 - 0)\hat{k} = \hat{i} + 3\hat{j} - \hat{k}$$

$$\vec{b} = \overrightarrow{AC} = (2 - 0)\hat{i} + (2 - 0)\hat{j} + (1 - 0)\hat{k} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{c} = \overrightarrow{AD} = (1 - 0)\hat{i} + (6 - 0)\hat{j} + (5 - 0)\hat{k} = \hat{i} + 6\hat{j} + 5\hat{k}$$

$$\therefore \text{Volume of tetrahedron} = \frac{1}{6} (\vec{a} \cdot \vec{b} \times \vec{c})$$

$$= \frac{1}{6} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 1 & 6 & 5 \end{vmatrix}$$

$$= \frac{1}{6} [1(10 - 6) - 3(10 - 1) - 1(12 - 2)]$$

$$= \frac{1}{6} (4 - 27 - 10) = \frac{-33}{6} = \frac{-11}{2}$$

Since volume is a non-negative quantity, so

$$\text{Volume of tetrahedron} = \frac{11}{2} \text{ cubic units}$$

3.7.6 Coplanar Vectors and Condition for the Coplanarity of Three Vectors

Coplanar Vectors

Two or more vectors lying in the same plane are known as coplanar vectors.

Condition for the Coplanarity of Three Vectors

Consider three coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

$\vec{a} \times \vec{b}$ is the vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then $\vec{a} \times \vec{b}$ is also perpendicular to vector $\vec{c}$, then

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$$

$\therefore \vec{a} \cdot \vec{b} \times \vec{c} = 0$ is the condition for the three vectors to be coplanar.

Example 27: Find the value of λ so that vectors

$$\vec{a} = \hat{i} - \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j} + \hat{k} \text{ and } \vec{c} = -\hat{i} + \lambda\hat{j} + 2\hat{k} \text{ are coplanar.}$$

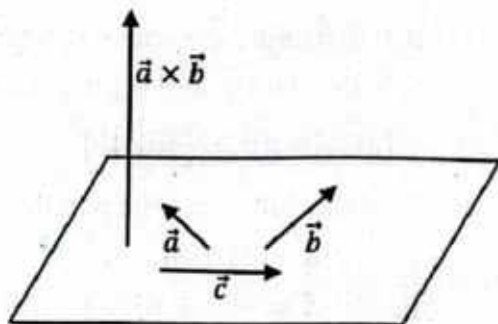
Solution:

$\vec{a}$, $\vec{b}$ and $\vec{c}$ will be coplanar if

$$\vec{a} \cdot \vec{b} \times \vec{c} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & \lambda & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1(8 - \lambda) + 1(16 + 1) + 1(2\lambda + 1) = 0$$



$$\Rightarrow 8 - \lambda + 17 + 2\lambda + 1 = 0$$

$$\Rightarrow \lambda + 26 = 0$$

$$\Rightarrow \lambda = -26$$

Exercise 3.4

- For the given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$; prove that $\vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a} = \vec{c} \cdot \vec{a} \times \vec{b}$
 - $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$; $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$; $\vec{c} = -\hat{i} + 2\hat{j} - 3\hat{k}$
 - $\vec{a} = -2\hat{i} + 7\hat{j} + \hat{k}$; $\vec{b} = 4\hat{i} + 2\hat{j} + \hat{k}$; $\vec{c} = 2\hat{j} + \hat{k}$
- For the given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$; prove that $\vec{a} \cdot \vec{b} \times \vec{c} = -\vec{b} \cdot \vec{a} \times \vec{c} = -\vec{a} \cdot \vec{c} \times \vec{b}$
 - $\vec{a} = \hat{i} + \hat{j}$; $\vec{b} = \hat{j} + \hat{k}$; $\vec{c} = \hat{i} + \hat{k}$
 - $\vec{a} = 7\hat{i} - 2\hat{j} + \hat{k}$; $\vec{b} = \hat{i} + \hat{j}$; $\vec{c} = \hat{j} - \hat{k}$
- Show that the vectors $\vec{a} = -4\hat{i} - 6\hat{j} - 54\hat{k}$, $\vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - 3\hat{k}$ are coplanar.
 - Find the value of λ so that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = -\hat{i} + \lambda\hat{j} + 2\hat{k}$ are coplanar.
- Find the value of λ if the points $A(-1, 4, -3)$; $B(3, \lambda, -5)$; $C(-3, 8, -5)$ and $D(-3, 2, 1)$ are coplanar.
 - If the vectors $\vec{a} = \alpha\hat{i} + \hat{j} + \hat{k}$; $\vec{b} = \hat{i} + \beta\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + \gamma\hat{k}$ are coplanar, then prove that $\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 1$ where $\alpha, \beta, \gamma \neq 1$
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then show that $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar.
 - If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$; then prove that $[\vec{a} \ \vec{b} \ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$
- Find the volume of the parallelepiped with given three coterminal edges:
 - $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$; $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$; $\vec{c} = 3\hat{i} + \hat{j} + \hat{k}$
 - $\vec{a} = -3\hat{i} + 6\hat{j} + \hat{k}$; $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$; $\vec{c} = -\hat{i} + 2\hat{j} + 6\hat{k}$
- Find the volume of the tetrahedron with given vertices:
 - $A(2, 1, 0)$; $B(-1, 2, 6)$; $C(2, 0, 3)$; $D(1, -1, 0)$
 - $A(0, 1, 0)$; $B(2, 0, 1)$; $C(3, 1, 2)$; $D(5, 6, -1)$

3.8 Application of Vectors in Real World

Vectors can be used by **air-traffic** controllers when tracking planes, by meteorologists when describing wind conditions, and by computer programmers when they are designing virtual worlds. In this section, we will present some applications of vectors that are commonly used in the study of physics: work, torque, and magnetic force.

3.8.1 Projectile Motion

A projectile (stone) thrown with an initial speed u at angle ϕ with the horizontal, has a vertical component of $(u \sin \phi - g t)$ and the horizontal component of $u \cos \phi$ under components of vector.

3.8.2 Sharpening wooden pencil with a blade

We cut the pencil at an angle. The component of force in the direction perpendicular to the pencil cuts the pencil. The component of force in the direction parallel to the pencil removes the thin wooden part.

3.8.3 Earth's magnetic field

Earth's magnetic field has two components B and H which are perpendicular to Earth's surface and parallel to the surface.

Pendulum

The tension in the string has two components to balance the weight and to give the centripetal force.

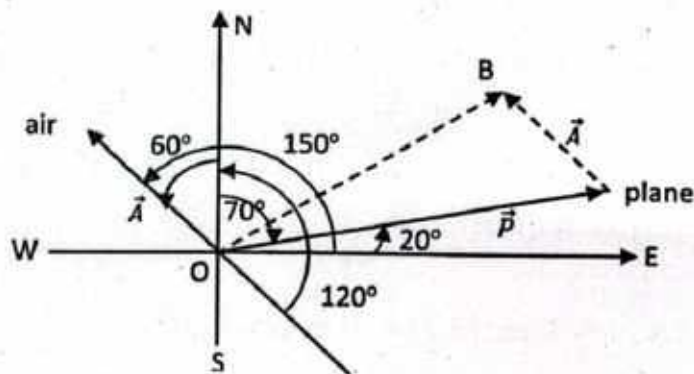
3.8.4 Digital graphics: Vector art can be defined as digital graphics using mathematical formulas to construct shapes and lines. Vector images maintain their quality irrespective of size. This adaptability makes vector file formats flexible, resilient, and always looking sharp. Vector artwork is digital art produced with vector design software like Linearity Curve (formerly Vectornator), Adobe Illustrator, and Sketch. These vector graphics editors generate simple shapes between points instead of pixels.

Programing: A vector, in programming, is a type of array that is one dimensional. Vectors are a logical element in programming languages that are used for storing a sequence of data elements of the same basic type. Members of a vector are called components.

GPS Unit: When you use your GPS unit to get from point A to point B. The GPS unit will give you a distance (magnitude) and a direction. A vector is, therefore, a directed quantity: a number with a direction.

Example 28: An air-plane is flying with an airspeed of 475 km/h on heading of 70° . If an 80 km/h wind is blowing from a true heading of 120° . Determine the velocity and direction of plane relative to the ground.

Solution:



$$\vec{P} = 475 \cos 20^\circ \hat{i} + 475 \sin 20^\circ \hat{j}$$

$$\vec{A} = 80 \cos 150^\circ \hat{i} + 80 \sin 150^\circ \hat{j}$$

$$\vec{P} = 446.35\hat{i} + 162.46\hat{j}$$

$$\vec{A} = -69.28\hat{i} + 40\hat{j}$$

$$\vec{OB} = \vec{P} + \vec{A} = 377.07\hat{i} + 202.46\hat{j}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1}(0.536) = 28.23^\circ$$

$$OB = \sqrt{377.07^2 + 202.46^2} = 428 \text{ km/h}$$

Review Exercise

1. Choose the correct option.

i. The vector in the direction of $\hat{i} + 2\hat{j} - 2\hat{k}$ and having magnitude 12 is:

- a. $\frac{1}{12}(\hat{i} + 2\hat{j} - 2\hat{k})$ b. $12(\hat{i} + 2\hat{j} - 2\hat{k})$
c. $\frac{1}{4}(\hat{i} + 2\hat{j} - 2\hat{k})$ d. $4(\hat{i} + 2\hat{j} - 2\hat{k})$

ii. The position vectors of three vertices of triangle are $2\hat{i} + \hat{j} - \hat{k}$, $3\hat{i} - 2\hat{j} + 4\hat{k}$ and $\hat{i} + 4\hat{j} - 3\hat{k}$. The triangle is:

- a. isosceles b. right angled c. scalene d. equilateral

iii. Given two vectors $\hat{i} - \hat{j}$ and $\hat{i} + 2\hat{j}$, then the unit vector coplanar with two vectors and $\perp$ to first is:

- a. $\pm \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$ b. $\pm \frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$
c. $\pm \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ d. $\pm \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$

iv. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ then:

- a. $|\vec{a}| = |\vec{b}|$ b. $\vec{a} \perp \vec{b}$ c. $\vec{a} \parallel \vec{b}$ d. $\vec{a} = \vec{b} = 0$

v. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors; the value of $|\hat{a} + \hat{b} + \hat{c}|$ is:

- a. 1 b. $\sqrt{2}$ c. $\sqrt{3}$ d. 2

vi. If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$ then angle between $\vec{a}$ and $\vec{b}$ is:

- a. $\frac{\pi}{6}$ b. $\frac{2\pi}{3}$ c. $\frac{5\pi}{3}$ d. $\frac{\pi}{3}$

vii. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$; $\vec{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$ then the value of $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ is:

- a. 74 b. -74 c. 52 d. -52

viii. If $|\vec{a} \times \vec{b}| = 4$ and $|\vec{a} \cdot \vec{b}| = 2$; then $|\vec{a}|^2 |\vec{b}|^2$ is:

- a. 6 b. 20 c. 2 d. 8

- ix. If θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$ and $|\vec{a} \times \vec{b}| = |\vec{a} \cdot \vec{b}|$ then value of θ is:
 a. 0 b. $\frac{\pi}{6}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{2}$
- x. The value of $[\vec{a} - \vec{b} \quad \vec{b} - \vec{c} \quad \vec{c} - \vec{a}]$ where $|\vec{a}| = 1, |\vec{b}| = 5, |\vec{c}| = 3$ is:
 a. 0 b. 1 c. 6 d. 15
2. Find the value of λ so that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{b} = \hat{i} - \lambda\hat{j} + 3\lambda\hat{k}$ are:
 i. parallel ii. perpendicular
3. If $\vec{a} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 4\hat{k}$ then find the component of $\vec{a} + \vec{b}$ along $\vec{a} - \vec{b}$.
4. Find $|\vec{u}|$ if $\hat{v}$ is unit vector and $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = 18$.
5. The scalar product of $\hat{i} + \hat{j} - \hat{k}$ with the unit vector along the sum of the vectors $2\hat{i} - 3\hat{j} + \hat{k}$ and $\lambda\hat{i} - 2\hat{j} + 3\hat{k}$ is -1 . Find the value of λ .
6. For any vector $\vec{a}$; prove that $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$.
7. With usual notations for a triangle ABC ; prove that $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$ by vector method.
8. Suppose an airplane has a velocity relative to the air with a speed of 200 km/h and a direction of 60° . Suppose the wind is blowing from the west at 40 km/h. Calculate the ground speed and the true course for the plane.
9. A pilot wants to have a true course of 100° with a ground speed of 250 km/h. If the wind has a velocity vector $(r, \theta) = (20, 30^\circ)$, what should be the speed and direction of airplane with respect to air.