

SEQUENCES AND SERIES

After studying this unit, students will be able to:

- Solve problems by analysing arithmetic sequence and series up to n terms.
- Solve problems by analysing geometric sequence and series up to n terms.
- Identify a sequence as arithmetic or geometric sequence up to n terms.
- Solve problems by analysing harmonic sequences and series up to n terms.
- Recognize sigma (Σ) notation.
- Find sum of
 - the first n natural numbers (Σn),
 - the squares of the first n natural numbers (Σn^2),
 - the cubes of the first n natural numbers (Σn^3).
- Define arithmetico-geometric series.
- Find sum to n terms of the arithmetico-geometric series.
- Solve real life problems involving arithmetic sequence, arithmetic mean and arithmetic series.

A sequence is simply an ordered list. For example, a superball dropped from the top of the tower (556 ft high) always rebounds three fourths of the distance fallen. How far (up and down) will the ball have traveled when it hits the ground for the 6th time, a sequence is being formed? When the members of a sequence are numbers, we can find their sum. Such a sum is called series.



4.1 Sequence

We encounter sequences at the very beginning of our mathematical experiences. The list of even numbers;

$$2, 4, 6, 8, 10, \dots$$

and the list of odd numbers;

$$1, 3, 5, 7, 9, \dots$$

are examples. We can 'predict' what the 20th term of each sequence will be just by using common sense.

Sequences can be either finite or infinite. For example,

$$2, 4, 6, 8, 10$$

is a finite sequence with five terms whereas,

$$5, 10, 15, 20, 25, \dots$$

continues without bound and is an infinite sequence.

The list of positive odd numbers:

$$1, 3, 5, 7, 9, \dots$$

is an example of a typical infinite sequence. We use the symbol a_n to denote the n th term of a given sequence. Thus, in the above sequence; $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and so on, the first term is $a_1 = 1$, but there is no last term.

The list of positive odd numbers less than 100 is:

$$1, 3, 5, 7, 9, \dots, 99$$

This is an example of finite sequence. The last term is 99. This sequence contains 50 terms.

There are several ways to display a sequence.

- Write out the first few terms.
- Give a formula for the general terms.
- Give a recurrence relation.

Sequences: If the domain of function f in the set of positive integers, then the element of $f(x)$ is its range can be arranged in order of increasing n :

$$f(1), f(2), f(3), \dots, f(x), \dots$$

For example if n is a positive integer, then the first several elements in the range of:

Functions whose domain are the entire set of positive integers are given a special name.

Definition:

Sequences: A sequence is a function whose domain is the set of positive integers.

Terms: Rather than using the customary function $f(x)$, a sequence usually denoted by symbol $\{a_n\}$. The terms of the sequence are formed by letters n take on the values $1, 2, 3, \dots$ in the general term a_n . Thus, $\{a_n\}$ is equivalent to:

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & \dots & a_n, \dots & \boxed{\text{elements in range}} \\ \uparrow & \uparrow & \uparrow & & \uparrow & \\ 1 & 2 & 3 & \dots & n, \dots & \boxed{\text{elements in domain}} \end{array}$$

A much better way to describe a sequence is to give a formula for the n th term a_n . This is also called a formula for the general term. For example, $a_n = 2n - 1$ is the general term for the sequence of odd numbers.

Consider the sequence $2, 4, 8, 16, \dots$

Here, first term: $a_1 = 2^1 = 2$

second term: $a_2 = 2^2 = 4$

third term: $a_3 = 2^3 = 8$

The general term is $a_n = 2^n$

This sequence can also be written as:

$$2, 4, 8, \dots, 2^n, \dots$$

Example 1: Find the first four terms and the 57th term of the sequence whose general term is

given by $a_n = \frac{(-1)^n}{n+1}$.

Solution: $a_1 = \frac{(-1)^1}{1+1} = -\frac{1}{2}$, $a_2 = \frac{(-1)^2}{2+1} = \frac{1}{3}$

$a_3 = \frac{(-1)^3}{3+1} = -\frac{1}{4}$, $a_4 = \frac{(-1)^4}{4+1} = \frac{1}{5}$

$a_{57} = \frac{(-1)^{57}}{57+1} = -\frac{1}{58}$

Note that the expression $(-1)^n$ causes the signs of the terms to alternate between positive and negative, depending on whether n is even or odd.

Example 2: For each sequence, predict the general terms.

(i) $1, 4, 9, 16, 25, \dots$ (ii) $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots$

(iii) $-1, 2, -4, 8, -16, \dots$ (iv) $2, 4, 8, 16, \dots$

Solution: (i) There are squares of consecutive positive integers.

So, the general term is n^2 i.e. $a_n = n^2$.

(ii) There are square roots of consecutive positive integers. So, the general term is $\sqrt{n}$.

(iii) There are powers of 2 starting from 0 with alternating signs.

So, the general term is $(-1)^n [2^{n-1}]$.

(iv) If we see the pattern of powers of 2, we will see 16 as the next term and gives 2^n for the general term.

Exercise 4.1

1. In each of the following, the n th term of the sequence is given. In each case find the first 4 terms; the 10th term, a_{10} and the 15th term, a_{15} .

- (i) $a_n = 3n + 1$ (ii) $a_n = 3n - 1$ (iii) $a_n = \frac{n}{n+1}$ (iv) $a_n = n^2 + 1$
 (v) $a_n = n^2 - 2n$ (vi) $a_n = \frac{n^2-1}{n^2+1}$ (vii) $a_n = \left(\frac{-1}{2}\right)^{n-1}$ (viii) $a_n = (-1)^n n^2$
 (ix) $a_n = (-1)^n (n + 3)$ (x) $a_n = (-1)^{n+1} (3n - 5)$

2. Find the indicated term of the sequence

- (i) $a_n = 4n - 3$; a_8 (ii) $a_n = 5n + 11$; a_9
 (iii) $a_n = (3n + 4)(2n - 5)$; a_7 (iv) $a_n = (-1)^{n-1} (3.4n - 17.3)$; a_{12}
 (v) $a_n = 4n^2 (11n + 31)$; a_{22} (vi) $a_n = \left(1 + \frac{1}{n}\right)^2$; a_{20}
 (vii) $a_n = \log 10^n$; a_{43} (viii) $a_n = \ln e^n$; a_{67}

3. Predict the general term or n th term, a_n , of the sequence.

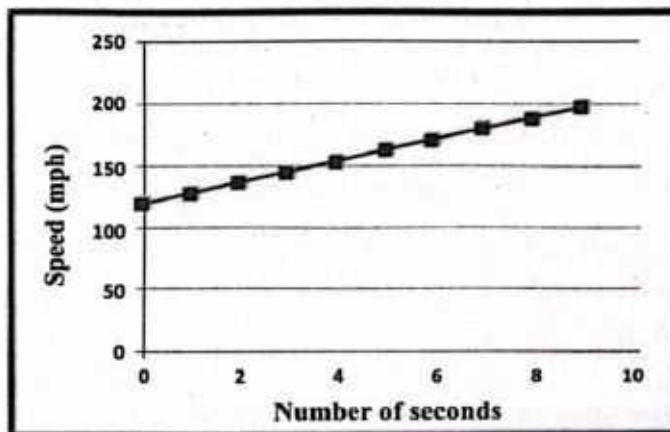
- (i) 1, 3, 5, 7, 9, ... (ii) 3, 9, 27, 81, 243, ...
 (iii) $\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}, \sqrt{10}, \dots$ (iv) 1.2, 2.3, 3.4, 4.5, ...

4.2 Arithmetic Sequence

A professional race car driver drives out of a curve. He enters the straight away at 119.9 mph. He increases his speed by 78.3 mph and after 9 seconds his speed is 198.2 mph.

The table below shows how his speed increased each second after entering the straight path.

Number of seconds	0	1	2	3	4	5	6	7	8	9
Speed in mph	119.9	128.6	137.3	146.0	154.7	163.4	172.1	180.8	189.5	198.2



From the table and graph, we observe the number and pattern. This set of numbers is an example of a **sequence**. Each number in a sequence is called a **term**. The first term is symbolized by a_1 , the second term by a_2 and so on to a_n , the n th term. The sequence shown in the table contains ten terms. Therefore, $a_1=119.9$, $a_2=128.6$ and $a_{10}=198.2$ (each term is obtained by adding 8.7 to the previous term). A sequence of this type is called an **arithmetic sequence** or **arithmetic progression (A.P.)**. The number added to find the next term of an arithmetic sequence is called the **common difference** and is symbolized by the variable d .

Explanation: In an arithmetic sequence, all terms (other than the first) can be found by adding the same number to the preceding term. For example, the sequence 2, 5, 8, 11, 14, 17, ... is arithmetic because adding 3 to any term produces the next term. In other words, the difference between any term and the preceding one is 3. Arithmetic sequences are also called arithmetic progressions (A.P.).

To find the next terms in an arithmetic sequence, first find the common difference d by subtracting any term from its succeeding term, then add the common difference to the last term to find successive terms.

Example 3: Find the next four terms of the arithmetic sequence 33, 39, 45...

Solution: Find the common difference d by subtracting two consecutive terms.

$$d = 39 - 33 = 6 \quad \text{or} \quad d = 45 - 39 = 6.$$

Now add 6 to the last term of the sequence and then continue adding until the next four terms are found.

$$\begin{aligned} a_4 &= 45 + 6 = 51, & a_5 &= 51 + 6 = 57 \\ a_6 &= 57 + 6 = 63, & a_7 &= 63 + 6 = 69 \end{aligned}$$

The next four terms of the sequence are 51, 57, 63, and 69.

In this way terms of an arithmetic sequence are formed.

4.2.1 Formula for the n th Term of an Arithmetic Sequence (A.P)

The n th term a_n , of an arithmetic sequence with first term a_1 and common difference d is given by

$$a_n = a_1 + (n - 1) d$$

This is known as recursive formula.

Note that the coefficient of d in each case is 1 less than subscript.

Example 4: Suppose a race car driver increases speed at constant rate. What will his speed be after 15 seconds, if his initial speed is 85 mph and his rate of acceleration is 4.5 mph per second?

Solution: $a_1 = 85$ and $d = 4.5$, $a_{16} = ?$ (After 15 sec means we have to find a_{16} term)

Find a_{16} using $a_n = a_1 + (n - 1) d$

$$a_{16} = 85 + (16 - 1)(4.5)$$

$$a_{16} = 152.5$$

His speed will be 152.5 mph after 15 seconds.

Example 5: The third term of an arithmetic sequence is 8, and the sixteenth term is 47. Find a_1 , d and construct the sequence. Also find a_{15} .

Solution: We know that $a_3 = 8$ and $a_{16} = 47$. We need first term a_1 and d (common difference).

$$a_3 = 8 \text{ (Here } n = 3\text{)}$$

$$\text{So, } a_3 = a_1 + (3 - 1)d \Rightarrow 8 = a_1 + 2d \quad \text{(i)}$$

$$\text{and } a_{16} = 47 \text{ (Here } n = 16\text{)}$$

$$a_{16} = a_1 + (16 - 1)d \Rightarrow 47 = a_1 + 15d \quad \text{(ii)}$$

Solving (i) and (ii), we have

$$a_1 = 2, \quad d = 3$$

$$\text{So, } a_1 = 2, \quad a_2 = a_1 + 1.d = 2 + 3 = 5, \quad a_3 = a_1 + 2.d = 2 + 6 = 8,$$

$$a_4 = a_1 + 3.d = 2 + 9 = 11$$

The sequence is 2, 5, 8, 11, ...

$$\text{Now, } a_{15} = a_1 + (15 - 1)d = a_1 + 14d = 2 + 14(3)$$

$$a_{15} = 44$$

4.2.2 Arithmetic Mean

To find arithmetic mean between two numbers a and b , we use formula

$$A.M = \frac{a+b}{2}$$

A number A is said to be arithmetic mean (A.M) between two numbers a and b if a , A , b are in A.P.

If d is the common difference, then

$$d = A - a = b - A$$

$$2A = a + b$$

$$A = \frac{a+b}{2}$$

Example 6: Find the four arithmetic means between 19 and 54.

Solution: We can use the n th term formula to find the common difference.

In the sequence 19, _____, _____, _____, _____, 54; we have, $a_1 = 19$ and $a_6 = 54$.

$$\text{To find } d, \text{ use } a_6 = a_1 + 5d$$

$$54 = 19 + 5d \Rightarrow d = 7$$

Use $a_1 = 19$ and $d = 7$ to find the four arithmetic means

$$a_2 = a_1 + d = 19 + 7 = 26$$

$$a_3 = a_1 + 2d = 19 + 2(7) = 33$$

$$a_4 = a_1 + 3d = 19 + 3(7) = 40$$

$$a_5 = a_1 + 4d = 19 + 4(7) = 47$$

The four arithmetic means are 26, 33, 40 and 47.

Example 7: Find the A.M between 6 and 18.

Solution: We have $a_1 = 6$, $b = 18$, then

$$A.M = \frac{a+b}{2} = \frac{6+18}{2} = 12$$

Challenge

Find three numbers that have a sum of 27, a product of 288 and form an arithmetic sequence.

Example 8: Find the 7 A.Ms between 7 and 20.

Solution: Let $A_1, A_2, A_3, \dots, A_7$ be the required A.Ms between 7 and 20. Then

7, $A_1, A_2, A_3, A_4, A_5, A_6, A_7, 20$ are in A.P.

$$a_1 = 7, \quad n = 9, \quad a_9 = 20$$

$$a_1 + 8d = 20 \Rightarrow 7 + 8d = 20 \Rightarrow d = \frac{11}{8}$$

$$A_1 = a_1 + d = 7 + \frac{11}{8} = \frac{67}{8}$$

$$A_2 = a_1 + 2d = 7 + 2\left(\frac{11}{8}\right) = \frac{33}{4}$$

$$\text{Similarly, } A_3 = \frac{103}{8}, A_4 = \frac{57}{4}, A_5 = \frac{21}{2}, A_6 = \frac{68}{5}, A_7 = \frac{147}{10}$$

Check Point

Show that sum of n A. Ms between a & b is equal to n times A. M between a & b .

Exercise 4.2

- Find the first four terms of each arithmetic sequence.
 - $a_1 = 4, d = 3$
 - $a_1 = 7, d = 5$
 - $a_1 = 16, d = -2$
 - $a_1 = 38, d = -4$
 - $a_1 = \frac{3}{4}, d = \frac{1}{4}$
 - $a_1 = \frac{3}{8}, d = \frac{5}{8}$
- Find the next three terms of each arithmetic sequence.
 - 5, 9, 13, ...
 - 11, 14, 17, ...
 - $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$
 - 0.07, 0.12, 0.17, ...
- Find the 11th term of the arithmetic sequence 0.07, 0.12, 0.17, ...
- The third term of an arithmetic sequence is 14 and the ninth term is -1. Find the first four terms of the sequence.
- Find an arithmetic sequence for $a_{17} = -40$ and $a_{28} = -73$, find a_1 and d . Write first five terms of the sequence.
- The fifth term of an arithmetic sequence is 19 and 11th term is 43. Find the first term and 87th term.
- Which term of the sequence -6, -2, 2, ... is 70? 8. Which term of the sequence $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ is $-\frac{105}{2}$?
- If $\frac{1}{a}, b, \frac{1}{c}$ are in A.P. Show that the common difference is $\frac{a-c}{2ac}$.
- During a free fall, a sky diver falls 16 feet in the first second, 48 feet in the 2nd second and 80 feet in the third second. If he continues to fall at this rate, how many feet will he fall during the 8th second?
- If Rs. 1000 is saved on August 1, Rs. 3000 on August 2, Rs. 5000 on August 3 and so on. How much is saved till August 20?
- A gardener is making a triangular planting, with 35 plants in the first row, 31 in the second row, 27 in the third row and so on. If the pattern is consistent, how many plants will there be in the eighth row?
- Find A.M. between
 - 7 and 17
 - $3 + 3\sqrt{2}$ and $7 - 3\sqrt{2}$
 - $7\sqrt{5}$ and $\sqrt{5}$
 - $2y + 5$ and $5y + 3$
- Find 'b' if 10 is A.M between b and 20.
- Find x and y if 2 and 13 are two arithmetic means between x and y .
- Find the two arithmetic means between 5 and 17. 17. Find three arithmetic means between 2 and -18.

4.3 Arithmetic Series

A sky driver falls freely covering the distance in the following pattern. These free-fall distances form an arithmetic sequence. 16, 48, 80, 112, 144, 176, ...

To find out what the total distance covered by the sky diver is, we would add the terms in the sequence. $16 + 48 + 80 + 112 + 144 + 176$

The indicated sum of the terms of a sequence is called a **series**. Above series is called an **arithmetic series**.

Definition: Arithmetic Series: An arithmetic series is the sum of the terms in an arithmetic sequence. An arithmetic sequence is a sequence of numbers in which the difference between consecutive terms is constant.

Following are the examples of arithmetic sequences and their corresponding arithmetic series.

Arithmetic Sequence

$$2, 4, 6, 8, 10$$

$$-8, -2, 4$$

$$\frac{4}{5}, \frac{8}{5}, \frac{12}{5}, \frac{16}{5}$$

$$a_1, a_2, a_3, a_4, \dots, a_n$$

Arithmetic Series

$$2 + 4 + 6 + 8 + 10$$

$$-8 + (-2) + 4$$

$$\frac{4}{5} + \frac{8}{5} + \frac{12}{5} + \frac{16}{5}$$

$$a_1 + a_2 + a_3 + a_4 + \dots + a_n$$

The symbol S_n is used to represent the sum of the first n -terms of a series. For example, S_4 means the sum of the first four terms of a series. For example, the sum of series $3 + 6 + 9 + 12$ is 30. If a series has a large number of terms, it is not convenient to list all the terms and then find their sum.

4.3.1 Sum of n th term of A.P

To develop a general formula for the sum of any arithmetic series, let's consider the series of sky diving distances.

$$16 + 48 + 80 + 112 + 144 + 176$$

We write S_6 in two different orders and find the sum.

$$S_6 = 16 + 48 + 80 + 112 + 144 + 176$$

$$+ S_6 = 176 + 144 + 112 + 80 + 48 + 16$$

$$2S_6 = 192 + 192 + 192 + 192 + 192 + 192 \quad \dots \quad 6 \text{ times } 192 \text{ (6 sums of 192)}$$

$$= 6 [192] \Rightarrow S_6 = \frac{6}{2} [192] \quad (\text{Divide each side by 2})$$

Here, 6 represents n , 192 represents the sum of the first and last terms ($16 + 176$) i.e. $a_1 + a_n$. We can replace the equation with the formula:

$$S_n = \frac{n}{2} [a_1 + a_n] \quad (i)$$

We have learnt that in an arithmetic sequence, $a_n = a_1 + (n - 1)d$. Using this formula(i), we get another version for the sum of an arithmetic sequence.

$$S_n = \frac{n}{2} [a_1 + a_n]; \quad \text{replace } a_n \text{ with } a_1 + (n - 1)d$$

$$S_n = \frac{n}{2} [a_1 + (a_1 + (n - 1)d)]$$

$$S_n = \frac{n}{2} [2a_1 + (n - 1)d]$$

The sum S_n of the first n -terms of an arithmetic series is given by:

$$S_n = \frac{n}{2} [a_1 + a_n] = S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

Example 9: Find the sum of the first 100 positive integers.

Solution:

1st Method: In this series, $a_1 = 1$ and $a_n = a_{100} = 100$

$$S_n = \frac{n}{2} [a_1 + a_n]$$

$$S_{100} = \frac{100}{2} [1 + 100] = 5050$$

2nd Method Sum is $1 + 2 + 3 + \dots + 100$ term

$$a_1 = 1, d = 1, n = 100$$

$$S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

$$S_{100} = \frac{100}{2} [2(1) + (100-1)(1)]$$

$$S_{100} = 50 [101] = 5050$$

Example 10:

Find the sum of the first 50 terms of an arithmetic series where $a_1 = 5$ and $d = 25$.

Solution: Given

$$a_1 = 5, d = 25, n = 50$$

$$S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

$$S_{50} = \frac{50}{2} [2(5) + (50-1)(25)] \quad (\text{substituting values})$$

$$S_{50} = 25 [10 + (49)(25)] = 30875$$

Example 11: Theaters are often built with more seats per row as the rows move towards the back. Suppose the main floor of a theater has 28 seats in the first row, 32 in the second, 36 in the third and so on for 50 rows. How many seats are on the main floor?

Solution: From the given information, 1st row = 28, 2nd row = 32, 3rd row = 36. The series is:

$$28 + 32 + 36 + \dots \quad (50 \text{ rows})$$

$$a_1 = 28, d = 4, n = 50$$

$$S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

$$S_{50} = \frac{50}{2} [2(28) + (50-1)(4)] \quad (\text{substituting values})$$

$$S_{50} = 25 [56 + 196] = 6300$$

Example 12:

Find the first three terms of an arithmetic series where $a_1 = 17$, $a_n = 101$ and $S_n = 472$.

Solution: First, find 'n'.

$$S_n = \frac{n}{2} [a_1 + a_n];$$

$$472 = \frac{n}{2} [17 + 101] \Rightarrow 944 = 18n \Rightarrow n = 8$$

Next, find 'd'.

$$a_n = a_1 + (n - 1) d$$

$$101 = 17 + (8 - 1) d$$

$$84 = 7d \Rightarrow d = 12$$

Now we have:

$$a_2 = a_1 + d = 17 + 12 = 29$$

$$a_3 = a_1 + 2d = 17 + 2(12) = 41$$

Thus, the first three terms are 17, 29 and 41.

Exercise 4.3

Find the sum of each series (1 – 7).

- $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25$
- $a_1 = 2, a_n = 200, n = 100$
- $a_1 = 5, a_n = 100, n = 200$
- $a_1 = 4, n = 15, d = 3$
- $a_1 = 50, n = 20, d = -4$
- $-3 + (-7) + (-11) + \dots + a_{10}$
- $9 + 11 + 13 + 15 + \dots$ for $n = 12$
- Find the sum of the even numbers from 2 to 100.
- Find the sum of the odd numbers from 1 to 99.
- Find the sum of all multiples of 4 that are between 14 and 523.

Find S_n for each arithmetic series.

- $a_1 = 3, a_n = -38, n = 8$
- $a_1 = 85, n = 21, a_n = 25$
- $a_1 = 34, n = 9, a_n = 2$
- $a_1 = 5, d = \frac{1}{2}, n = 13$
- $a_1 = 91, d = -4, a_n = 15$
- $d = -4, n = 9, a_n = 27$

Find sum of the arithmetic series.

- $6 + 12 + 18 + \dots + 96$
- $34 + 30 + 26 + \dots + 2$
- $10 + 4 + (-2) + \dots + (-50)$

Find the first three terms of each arithmetic series.

- $a_1 = 7, a_n = 139, S_n = 876$
- $n = 14, a_n = 53, S_n = 378$
- $a_1 = 6, a_n = 306, S_n = 1716$
- A formation of a marching band has 14 marches in the front row, 16 in the second row, 18 in the third row and so on, for 25 rows. How many marchers are in the last row? How many marchers are there altogether?

24. How many poles will be in a pile of telephone poles if there are 50 in the first layer, 49 in the second and so on, until there are 6 in the last layer?
25. A family saves money in an arithmetic sequence: Rs. 6000 in the first year, Rs. 70,000 in second year and so on, for 20 years. How much do they save in all?
26. Mr. Saleem saves Rs. 500 on October 1, Rs. 550 on October 2, and Rs. 600 on October 3 and so on. How much is saved during October? (October has 31 days)

4.4 Geometric Sequence

Iodine is used medically as a tracer isotope in monitoring the activity of the thyroid gland. A patient is given a compound containing the radioactive iodine. The amount of iodine retained by this gland is a measure of its ability to function.

Iodine has a half-life of about 8 days. That means approximately every 8 days, half the mass of iodine decays into another element. Then in the next 8 days, half of the remaining iodine decays, and so on.

Suppose a container hold a mass of 64 milligrams of iodine. To find the remaining mass of iodine after each half-life, 64, 32, 16, 8, 4, 2, 1, and 0.5, are what type of patterns do you suggest?

The pattern of masses forms a sequence of numbers known as a **geometric sequence** or **geometric progression**. The terms in this example are 64, 32, 16, 8, 4, 2, 1, and 0.5.

Explanation: In an arithmetic sequence, we added a certain number to each term to get the next term. With the kind of sequence we consider now, each term is multiplied by the certain number to get the next term. Then are called geometric sequence or geometric progressions (G.P).

Consider the sequence: 2, 6, 18, 54, 162, ...

If we multiply each term by 3, we obtain the next term. Sequence in which term can be multiplied by a certain number in order to get the next term are called geometric. We call this multiplier the 'common ratio' because it is found by dividing any term by the preceding term.

In any geometric sequence, the common ratio r is found by dividing any term by the previous term.

Example 13: Find the next two terms of the geometric sequence 4, 12, and 36.

Solution: To find the common ratio, find the quotient of any two consecutive terms.

$$\frac{12}{4} = \frac{36}{12} = 3; \text{ the common ratio is } 3.$$

The fourth term = $36(3) = 108$

The fifth term = $108(3) = 324$

$\therefore$ The next two terms of the geometric sequence are 108 and 324.

4.4.1 Formula for the n th Term of a Geometric Sequence

Successive terms of a geometric sequence are usually expressed in the product of r and the previous term. Thus, a geometric sequence is also a recursive sequence. Each succeeding term in a GP contains a factor of r , each term can be expressed as a product of r .

We derive the formula for GP using previous example. Observe the following table:

a_1	a_2	a_3	a_4	.	.	.	.	.	.	a_n
4	$4(3) = 12$	$4(3^2) = 36$	$4(3^3) = 108$	.	.	.	.	.	.	$4(r^{n-1})$
a	ar	ar^2	ar^3	.	.	.	.	.	.	ar^{n-1}

The n th term a_n of a geometric sequence with first term a_1 and the common ratio r is given by formula:

$$a_n = a_1 r^{n-1}$$

Example 14: Write the first five terms of a geometric sequence in which $a_1 = 5$ and $r = 2$.

Solution: Given $a_1 = 5$. Write next term using formula; $a_n = a_1 r^{n-1}$

$$a_2 = a_1 r^{2-1} = a_1 r = (5)(2) = 10 \quad (\text{Substituting values } a_1 = 5, r = 2 \text{ and } n = 2, 3, 4, 5)$$

$$a_3 = a_1 r^{3-1} = a_1 r^2 = (5)(2)^2 = 20$$

$$a_4 = a_1 r^{4-1} = a_1 r^3 = (5)(2)^3 = 40$$

$$a_5 = a_1 r^{5-1} = a_1 r^4 = (5)(2)^4 = 80$$

$\therefore$ The first five terms of a sequence are 5, 10, 20, 40, and 80.

Example 15: Find the seventh term, a_7 , of a geometric sequence in which $a_3 = 96$ and $r = 4$.

Solution: The general form of the third term of a sequence is $a_1 r^2$ ($a_1 r^{3-1}$).

We have $a_3 = 96$, $r = 4$, $a_7 = ?$

$$a_3 = a_1 r^2$$

$$96 = a_1 (4)^2; \quad (\text{we need } a_1 \text{ to find } a_7)$$

$$96 = a_1 (16) \Rightarrow a_1 = 6$$

Thus, $a_7 = a_1 r^6 = (6)(4)^6 = 24,576$.

Example 16:

Mr. Khalid saves Rs. 1000 on the first day. Then each day thereafter, saves double the amount he saved the day before. Find the amount he should save the 20th day of the month.

Solution:

In this sequence, $a_1 = 1000$. Since the amount of money is twice that of day before, so $r = 2$.

$$a_n = a_1 r^{n-1}; \quad a_{20} = ?$$

$$a_{20} = a_1 r^{20-1} = a_1 r^{19} = (1000)(2)^{19}$$

$$a_{20} = 524288000$$

On the 20th day, Khalid should save Rs. 524,288,000.

4.5 Geometric Mean

If a, G, b is in a geometric sequence, then G is called the geometric mean of a and b .

From geometric sequence a, G, b , we have :

$$\text{Common ratio: } r = \frac{a}{G} \quad (\text{i}), \quad r = \frac{G}{b} \quad (\text{ii})$$

Form (i) and (ii)

$$\frac{a}{G} = \frac{G}{b} \Rightarrow G^2 = ab$$

$$G = \pm \sqrt{ab}$$

Thus the geometric means of two numbers is the square root of their product.



- The positive square root is chosen, if both the numbers are positive.
- The negative square root is chosen, if both the numbers are negative.
- The mean is imaginary, if two numbers have opposite signs.

Example 17: Find the geometric mean of each of the following pairs of numbers.

(i) 9 and 4

(ii) $-\frac{3}{2}$ and $-\frac{27}{8}$

Solution: (i) Here $a = 9$ and $b = 4$. So,

$$G = \sqrt{ab} \quad (\text{both are positive})$$

$$G = \sqrt{9 \times 4} = 6$$

(ii) Given $a = -\frac{3}{2}$, $b = -\frac{27}{8}$

$$G = \sqrt{ab} \quad (\text{both are negative})$$

$$= \sqrt{-\frac{3}{2} \times -\frac{27}{8}} = \pm \sqrt{\frac{81}{16}}$$

$$= \pm \frac{9}{4}$$

Example 18: Find two geometric means between 81 and 3.

Solution: The sequence is 81, _____, _____, 3.

Use the general formula for the n th term to find the value of r .

Since $a_1 = 81$, $a_4 = 3$, $n = 4$.

So, $a_4 = a_1 r^{n-1}$ becomes $a_4 = a_1 r^3$ or $3 = 81 (r)^3$

$$r^3 = \frac{1}{27} \Rightarrow (r)^3 = \left(\frac{1}{3}\right)^3 \quad (\text{taking cube})$$

$$\Rightarrow r = \frac{1}{3}$$

$$a_2 = a_1 r = 81 \left(\frac{1}{3}\right) = 27$$

$$a_3 = a_1 r^2 = 81 \left(\frac{1}{3}\right)^2 = 9$$

The missing geometric means are 27 and 9.

Example 19: A vacuum pump removes $\frac{1}{5}$ of the air from a sealed container on each stroke of its piston. What percent of the air remains after five stroke of the piston?

Solution:

Let 1 represent the original amount of air. After the first stroke, $1 - \frac{1}{5}$ or $\frac{4}{5}$ of the air remains.

The second stroke removes $\frac{1}{5}$ of the remaining air.

Thus the amount that remain after two strokes is $\frac{4}{5} \left(1 - \frac{1}{5}\right) = \frac{4}{5} \times \frac{4}{5}$ or $\frac{16}{25}$

This pattern can be expressed as a geometric sequence.

Number of strokes	0	1	2	3	4	5
Sequence	1	$\frac{4}{5}$	$\frac{16}{25}$			
Terms	a_1	a_2	a_3	a_4	a_5	a_6

Now we use the formula $a_n = a_1 r^{n-1}$ to find a_6 , the amount of air left after five strokes

$$a_n = a_1 r^{n-1} \quad (\text{substituting the values; } a_1 = 1 \text{ and } r = \frac{4}{5})$$

$$a_6 = 1 \cdot \left(\frac{4}{5}\right)^5 \text{ or } \frac{4^5}{5^5}$$

$$a_6 = \frac{1024}{3125} \text{ or } 0.32768$$

Exercise 4.4

Determine whether each sequence is geometric. If so, find the common ratio.

1. 5, 20, 100, 500, ...

2. 2, 4, 6, 8, ...

3. $\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \dots$

4. 7, 14, 21, 28, ...

Find the first four terms of the geometric sequence.

5. $a_1 = 3, r = -2$

6. $a_1 = 27, r = -\frac{1}{3}$

7. $a_1 = 12, r = \frac{1}{2}$

Find the next two terms of each geometric sequence.

8. 90, 30, 10, ...

9. 2, 6, 18, ...

10. 20, 30, 45, ...

11. 729, 243, 81, ...

12. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, \dots$

13. $\frac{1}{4}, \frac{1}{2}, 1, \dots$

Find the n th term of each geometric sequence.

14. $a_1 = 4, n = 3, r = 5$

15. $a_1 = 2, n = 5, r = 2$

16. $a_1 = 7, n = 4, r = 2$

17. $a_1 = 243, n = 5, r = -\frac{1}{3}$

18. $a_1 = 32, n = 6, r = -\frac{1}{2}$

19. $a_1 = 16, n = 8, r = \frac{1}{2}$

Find the missing geometric means.

20. 3, _____, _____, _____, 48

21. 1, _____, _____, 8

22. 8, _____, _____, _____, $\frac{1}{4}$

23. 3, _____, 75

24. 5, _____, _____, _____, 80

25. 7, _____, _____, _____, 112

26. A Ping-Pong ball is dropped from a height of 16 ft and always rebounds one-fourth of the distance fallen. How high does it rebound the 6th time?

27. A city has a current population of 100,000 and the population is increasing by 3% each year. What will the population be in 15th years?

28. A super ball dropped from the top of the tower (556 ft high) always rebounds three-fourths of the distance fallen. How far (up and down) will the ball have travelled when it hits the ground for the 6th time?
29. The teaching staff of high school informs its members of school cancellation by telephone. The principal calls 2 teachers, each of whom in turn calls 2 other teachers, and so on. In order to inform the entire staff, 6 rounds of calls are made. Counting the principal, find how many people are in staff at high school?
30. A 5-day rain caused the river to rise. After the first day, the river rose one inch. Each day the rise in the river tripled. How much had the river risen after 5 days?

4.6 Geometric Series

The sum of the terms of a geometric sequence is called a **geometric series**.

4.6.1 Sum of the First n Terms of a Geometric Sequence

We want to find a formula for S_n when sequence is geometric as given below.

$$a_1, a_1r^1, a_1r^2, a_1r^3, \dots, a_1r^{n-1}$$

The geometric series S_n (sum of n terms) is given by:

$$S_n = a_1 + a_1r^1 + a_1r^2 + a_1r^3 + \dots + a_1r^{n-2} + a_1r^{n-1} \quad (1)$$

If we multiply both sides of equation (1) by r , we have

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^{n-1} + a_1r^n \quad (2)$$

Subtracting corresponding sides of equation (2) from equation (1), we get:

$$S_n - rS_n = a_1 - a_1r^n$$

$$\text{or } S_n(1 - r) = a_1(1 - r^n)$$

Dividing on both sides by $1 - r$ gives the following formula:

The formula for finding the sum of n terms of geometric series:

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{for any } r \neq 1$$

Note: When $r = 1$, the denominator becomes zero. Therefore, the formula is applicable when $r \neq 1$.

$$\text{For } |r| > 1, \text{ we use: } S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$\text{For } |r| < 1, \text{ we use: } S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Example 20:

Find the sum of the first 7 terms of the geometric sequence 3, 15, 75, 375, ...

Solution: First we note that:

$$a_1 = 3, n = 7, r = \frac{15}{3} \text{ or } 5$$

Using the formula for the sum of geometric series:

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \quad \text{For } |r| > 1$$

$$S_7 = \frac{3(5^7 - 1)}{5 - 1} \quad (\text{Substituting values of } a_1, n \text{ and } r.)$$

$$= \frac{3(78,125 - 1)}{4} = 58,593.$$

Key Facts



Another form of the formula for S_n can be developed and used, when we don't have number of terms.

$$a_n = a_1 r^{n-1}$$

$$a_n \cdot r = a_1 r^{n-1} \cdot r \quad (\text{Multiplying by 'r'})$$

$$a_n r = a_1 r^n \quad (i)$$

We have,

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_n = \frac{a_1 - a_n r}{1 - r} \quad (\text{Substituting value of } a_1 r^n)$$

Example 21: Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$ and $r = -\frac{1}{2}$.

Solution: Since we don't know n ,

$$S_n = \frac{a_1 - a_n r}{1 - r}$$

$$S_n = \frac{48 - 3(-\frac{1}{2})}{1 - (-\frac{1}{2})} \quad (\text{Substituting } a_1 = 48, a_n = 3 \text{ and } r = -\frac{1}{2})$$

$$= \frac{48 + \frac{3}{2}}{1 + \frac{1}{2}} = 33$$

Example 22: Find a_1 in a geometric series where $S_7 = 3279$ and $r = 3$.

Solution: Now, Here $S_7 = 3279$, $r = 3$, $a_1 = ?$

$$\therefore S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$S_7 = \frac{a_1(r^7 - 1)}{r - 1} \quad (\text{Taking } n = 7 \text{ to get } S_7).$$

$$3279 = \frac{a_1(3^7 - 1)}{3 - 1} \quad (\text{Substituting } r = 3)$$

$$3279 = \frac{a_1(2187 - 1)}{2}$$

$$= \frac{3279(2)}{2186} = a_1 \quad (\text{Solve for } a_1)$$

$$\therefore a_1 = 3$$

4.5.2 Infinite Geometric Series

The first swing of a pendulum measures 25cm. The lengths of the successive swings of the pendulum form the geometric sequence 25, 20, 16, 12.8,...

Suppose the pendulum continues to swing back and forth indefinitely then the sequence shown above becomes an infinite geometric sequence.

The total distance the pendulum travels can be expressed as the infinite geometric series

$$25 + 20 + 16 + 12 + \dots$$

In the series, $a_1 = 25$ and $r = \frac{20}{25} = 0.8$

So, the series can be expressed as:

$$25 + 25(0.8)^1 + 25(0.8)^2 + 25(0.8)^3 + 25(0.8)^4 + \dots$$

Look for a pattern in the values of $(0.8)^n$ as n increases.

$$(0.8)^1 = 0.8, (0.8)^{10} = 0.107374, (0.8)^{50} = 0.0000143$$

In an infinite geometric series where $|r| < 1$, as the value of n increases infinitely, the value of r^n approaches 0. Therefore, substituting value of r^n in the formula:

$$S_n = \frac{a_1(1 - r^n)}{1 - r},$$

we get: $S_\infty = \frac{a_1}{1 - r}$. This is formula for the sum of an infinite geometric series.

Sum of an Infinite Geometric Series

The sum, S_∞ , of an infinite geometric series where $-1 < r < 1$ is given by the following formula:

$$S_\infty = \frac{a_1}{1 - r}$$

Key Facts



An infinite geometric series in which $|r| > 1$ does not have a sum. For example, consider the series $1 + 2 + 4 + 8 + \dots$ where $a_1 = 1$ and $r = 2$. The terms of this series keep increasing, so the sum becomes greater with each additional term and never approaches to any point or number.

Example 23: Find the total distance travelled by the pendulum before coming to rest, if its successive swings form the geometric series:

$$25 + 20 + 16 + 12.8 + \dots$$

Solution: Sum of the infinite geometric series is given by:

$$S = 25 + 20 + 16 + 12.8 + \dots$$

Here $a_1 = 25$ and $r = 0.8$

$$S = \frac{a_1}{1 - r} = \frac{25}{1 - 0.8} = 125$$

Thus, the pendulum travels 125 cm.

Example 24: Find the sum of the infinite geometric series $\frac{4}{3} - \frac{2}{3} + \frac{1}{3} - \frac{1}{6} + \dots$

Solution: To find the value of r , divide any term by its preceding term,

$$r = \frac{-2/3}{4/3} = -\frac{1}{2}$$

Since $|r| < 1$, we have $S_{\infty} = \frac{a_1}{1-r}$

$$S_{\infty} = \frac{4/3}{1-(-\frac{1}{2})} = \frac{8}{9}$$

Example 25: Find fractional notation for 0.63636363...

Solution: We can express this decimal as:

$$0.63636363\dots = 0.63 + 0.0063 + 0.000063 + \dots$$

This is an infinite geometric series, where $a_1 = 0.63$ and $r = 0.01$. Since $|r| < 1$, this series has a sum:

$$S_{\infty} = \frac{a_1}{1-r} = \frac{0.63}{1-0.01} = \frac{0.63}{0.99} = \frac{63}{99}$$

Thus, the fractional notation for 0.63636363... is $\frac{63}{99}$ or $\frac{7}{11}$.

Exercise 4.5

Find the sum of each geometric series.

1. $16 + 16 + 16 + \dots$ to 11 terms
2. $75 + 15 + 3 + \dots$ to 10 terms
3. $a_1 = 5$, $r = 3$, $n = 12$
4. $a_1 = 256$, $r = 0.75$, $n = 9$
5. $a_1 = 7$, $r = 2$, $n = 14$
6. $a_1 = 12$, $a_5 = 972$, $r = -3$
7. $a_1 = 16$, $r = -\frac{1}{2}$, $n = 10$
8. $a_1 = 243$, $r = -\frac{2}{3}$, $n = 5$
9. $a_1 = 343$, $a_4 = -1$, $r = -\frac{1}{7}$
10. $a_3 = \frac{3}{4}$, $a_6 = \frac{3}{32}$, $n = 6$

Find a_1 for each geometric series:

11. $S_n = 244$, $r = -3$, $n = 5$
12. $S_n = 32$, $r = 2$, $n = 6$
13. $a_n = 324$, $r = 3$, $S_n = 484$

14. Find fractional notation for the infinite geometric series.

- | | | |
|----------------|-------------------|--------------------|
| (i) 0.444... | (ii) 9.99999... | (iii) 0.5555... |
| (iv) 0.6666... | (v) 0.15151515... | (vi) 0.12121212... |

15. To test its elasticity, a rubber ball is dropped into a 30ft hollow tube that is calibrated so that the scientist can measure the height of each subsequent bounce. The scientist found that on each bounce, the ball rises to a height $\frac{2}{5}$ the height of the previous bounce. How far will the ball travel before it stops bouncing?
16. A hot-air balloon rises 80ft in the first minute of flight. If in each succeeding minutes the balloon rises only 90% as far as in the previous minute, what will be its maximum altitude if it is allowed to rise without limit?

4.7 Harmonic Sequence

A sequence of numbers is called a harmonic sequence or harmonic progression (H.P) if the reciprocals of its terms are in arithmetic progression.

For example, the sequence; $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \dots$ is a harmonic sequence because the reciprocals of its terms are 1, 4, 7, 10, . . . which form an arithmetic sequence.

4.7.1 The n th Term of a Harmonic Sequence

The sequence:

$$\frac{1}{a_1}, \frac{1}{a_1 + d}, \frac{1}{a_1 + 2d}, \dots \text{ is H.P.}$$

Because reciprocals of the terms are:

$$a_1, a_1 + d, a_1 + 2d, \dots \text{ in A.P.}$$

We know that general term of A.P. is

$$a_n = a_1 + (n - 1) d$$

The reciprocal of the term:

$$\boxed{\frac{1}{a_n} = \frac{1}{a_1 + (n-1)d} \text{ (in H.P.)}}$$

where a_1 and d are the first term and common difference of the corresponding A.P.

Example 26: Find the 9th term of the H.P. $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \dots$

Solution:

$$\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \dots \text{ is H.P.}$$

The reciprocals of the terms 2, 7, 12, 17, . . . are in A.P.

We have $a_1 = 2, d = 5, n = 9$

$$a_n = a_1 + (n - 1) d$$

$$a_9 = 2 + (9 - 1) 5$$

$$= 2 + 40 = 42 \text{ in A.P.}$$

Thus, the 9th term of the H.P. is $\frac{1}{42}$.

Example 27: Find the harmonic sequence, whose fourth term is $\frac{1}{13}$ and eleventh term is $\frac{1}{25}$.

Solution: The fourth and eleventh terms of H.P. are $\frac{1}{13}$ and $\frac{1}{25}$ respectively.

The reciprocals are in A.P. So,

$$\text{Fourth term (A.P.)} = a_4 = 13,$$

$$\text{and eleventh term (A.P.)} = a_{11} = 25$$

$$a_4 = 13 \Rightarrow a_1 + 3d = 13 \quad \text{(i)}$$

$$a_{11} = 25 \Rightarrow a_1 + 10d = 25 \quad \text{(ii)}$$

Solving (i) and (ii), we have

$$a_1 = 7 \text{ and } d = 2$$

Here, $a_1 = 7$
 $a_2 = a_1 + d = 7 + 2 = 9$
 $a_3 = a_1 + 2d = 7 + 2(2) = 11$
 $a_4 = a_1 + 3d = 7 + 3(2) = 13$

The arithmetic sequence is

$$7, 9, 11, 13, \dots$$

So, harmonic sequence is

$$\frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \dots$$

4.7.2 Harmonic Mean

A number H is said to be the harmonic mean (H.M.) between two numbers a and b if a, H, b are in H.P.

As $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$$\text{Common difference} = \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\frac{1}{H} + \frac{1}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{2}{H} = \frac{a+b}{ab}$$

$$\Rightarrow \boxed{H = \frac{2ab}{a+b} \text{ (Harmonic Mean)}}$$

Which is the formula for H.M. between a and b .

Example 28: Find the harmonic mean between 15 and 7.

Solution: Here $a = 15$ and $b = 7$, therefore

$$\begin{aligned} \text{H.M.} &= \frac{2(15)(7)}{15+7} \\ &= \frac{210}{22} = \frac{105}{11} \end{aligned}$$

4.7.3 Relations between Arithmetic, Geometric and Harmonic Means

(i) If A, G, H are the arithmetic, geometric and harmonic mean between two positive numbers a and b , then show that

$$A > G > H$$

We know that

$$A = \frac{a+b}{2} \quad (\text{Arithmetic Mean})$$

$$G = \sqrt{ab} \quad (\text{Geometric Mean})$$

$$H = \frac{2ab}{a+b} \quad (\text{Harmonic Mean})$$

$$\diamond A > G \quad \text{if} \quad \frac{a+b}{2} > \sqrt{ab}$$

$$\text{We have} \quad a+b > 2\sqrt{ab}$$

$$a+b-2\sqrt{ab} > 0$$

Key Facts

If a and b are negative real numbers then

$A < G < H$

We can write: $(\sqrt{a} + \sqrt{b} - 2\sqrt{ab}) > 0$

$$(\sqrt{a} - \sqrt{b})^2 > 0, \text{ always true}$$

$$\therefore A > G$$

❖ $G > H$ if $\sqrt{ab} > \frac{2ab}{a+b}$

We can write: $a + b > \frac{2ab}{\sqrt{ab}}$

$$a + b > 2\sqrt{ab}$$

So, $a + b - 2\sqrt{ab} > 0 \Rightarrow (\sqrt{a} - \sqrt{b})^2 > 0, \text{ always true}$

$$\therefore G > H$$

Therefore, we have

$$A > G > H$$

(ii) $A \times H = G^2$

L.H.S. = $A \times H$

$$= \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$= ab = (\sqrt{ab})^2 = G^2 = \text{R.H.S.}$$

$$\therefore A \times H = G^2$$

Example 29: Find the arithmetic, geometric and harmonic means of 24 and 16.

Also show that $AH = G^2$.

Solution: Here $a = 24, b = 16$

$$A = \frac{a+b}{2} = \frac{24+16}{2} = 20 \text{ (A.M.)}$$

$$G = \sqrt{ab} = \sqrt{24 \times 16} = 8\sqrt{6} \text{ (G.M.)}$$

$$H = \frac{2ab}{a+b} = \frac{2(24)(16)}{24+16} = \frac{96}{5} \text{ (H.M.)}$$

We have $AH = G^2$

$$\text{L.H.S.} = A \times H = 20 \times \frac{96}{5} = 384$$

$$\text{R.H.S.} = G^2 = (8\sqrt{6})^2 = 64 \times 6 = 384$$

$$\therefore A \times H = G^2$$

Exercise 4.6

Find the indicated term of the harmonic progression (Q. 1-6).

1. $\frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \dots$ 7th term

2. $\frac{1}{11}, \frac{1}{9}, \frac{1}{7}, \dots$ 10th term

3. $\frac{1}{18}, \frac{1}{13}, \frac{1}{8}, \dots$ 20th term

4. $\frac{1}{4}, \frac{1}{9}, \frac{1}{14}, \dots$ nth term

5. $\frac{1}{27}, \frac{1}{20}, \frac{1}{13}, \dots$ nth term

6. $\frac{1}{2}, \frac{1}{2\frac{1}{2}}, \frac{1}{3}, \frac{1}{3\frac{1}{2}}, \dots$ nth term

7. Find the 14th term of H.P. $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \dots$

8. 7, 4, 1, ... is arithmetic sequence, find the 17th term of H.P.

9. Find the 8th term of H.P.

$$-\frac{1}{2}, -\frac{1}{5}, -\frac{1}{8}, \dots$$

10. Find H.M. between 9 and 11. Also find A, H, G and show that $AH = G^2$.

11. Find H.M. between $\frac{2}{3}$ and $\frac{4}{7}$.

12. Find four H.Ms. between $\frac{1}{3}$ and $\frac{1}{11}$.

Note: Sum of Harmonic Progression Formula

Sum of n terms in H.P.

For $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \dots + \frac{1}{a+(n-1)d}$

$$S_n \approx \frac{1}{d} \ln \left(\frac{2a + (2n-1)d}{2a-d} \right) \quad (\text{This is approximated sum for harmonic series})$$

Where: ' a ' is the first term of A.P., ' d ' is the common difference of A.P., and " $\ln$ " is the natural logarithm

4.8 Miscellaneous Series

A sequence is simply an ordered list. For example, when a baseball coach writes a batting order, a sequence is being formed. When the members of a sequence are numbers, we can find their sum. Such a sum is called a **series**.

4.8.1 Sigma Notation

When the general term of a sequence is known, the Greek letter Σ (Sigma) can be used to write a series. For example, the sum of the first four terms of the sequence 3, 5, 7, 9, ..., $2k+1$, can be named as follows, using sigma notation or summation notation;

$$\sum_{k=1}^4 (2k+1)$$

This is read as, "the sum as k goes from 1 to 4 of $(2k+1)$." The letter k is called the **index** of summation.

Example 30: Find the following sums.

a) $\sum_{k=1}^5 k^2$

b) $\sum_{k=1}^4 (-1)^k (2k)$

c) $\sum_{k=0}^3 (2^k + 5)$

Solution:

$$a) \sum_{k=1}^5 k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

Evaluate k^2 for all integers from 1 to 5 and then add.

$$\therefore 1 + 4 + 9 + 16 + 25 = 55$$

$$b) \sum_{k=1}^4 (-1)^k (2k) = (-1)^1 (2 \cdot 1) + (-1)^2 (2 \cdot 2) + (-1)^3 (2 \cdot 3) + (-1)^4 (2 \cdot 4) \\ = -2 + 4 - 6 + 8 = 4$$

$$c) \sum_{k=0}^3 (2^k + 5) = (2^0 + 5) + (2^1 + 5) + (2^2 + 5) + (2^3 + 5) \\ = 6 + 7 + 9 + 13 = 35$$

Example 31: Write sigma notation for the sum.

$$a) 1 + 4 + 9 + 16 + 25$$

$$b) -1 + 3 - 5 + 7$$

$$c) 3 + 9 + 27 + 81 + \dots$$

Solution:

$$a) 1 + 4 + 9 + 16 + 25$$

This is a sum of squares i.e. $1^2 + 2^2 + 3^2 + 4^2 + 5^2$. So, the general term is k^2 and its sigma notation is,

$$\sum_{k=1}^5 k^2$$

$$b) -1 + 3 - 5 + 7$$

Except for the alternating signs, this is the sum of the first four positive odd numbers.

Note that $2k - 1$ is a formula for the k th positive odd number and $(-1)^k = 1$, when k is even and $(-1)^k = -1$, when k is odd.

The general term is thus $(-1)^k (2k - 1)$, beginning with $k = 1$.

So, its sigma notation is:

$$\sum_{k=1}^4 (-1)^k (2k - 1)$$

$$c) 3 + 9 + 27 + 81 + \dots$$

This is the sum of powers of 3, and it is also an infinite series. We use the symbol ∞ to represent infinity and name the infinite series using sigma notation as follows:

$$\sum_{k=1}^{\infty} 3^k$$

4.8.2 Some Important Results

The sum of the first n natural numbers, the sum of squares of the first n natural numbers and the sum of the cubes of the first n natural numbers are expressed in sigma notation as:

$$\sum_{k=1}^n k = 1 + 2 + 3 + 4 + \dots + n$$

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2$$

$$\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3$$

We evaluate $\sum_{k=1}^n [k^m - (k-1)^m]$ for any positive integer m and shall use this result to find out formulas for three expressions stated above.

$$\begin{aligned}\sum_{k=1}^n [k^m - (k-1)^m] &= (1^m - 0^m) + (2^m - 1^m) + (3^m - 2^m) + \dots \\ &\quad + [(n-1)^m - (n-2)^m] + [n^m - (n-1)^m] \\ &= 1^m - 0^m + 2^m - 1^m + 3^m - 2^m + \dots + (n-1)^m - (n-2)^m + n^m - (n-1)^m \\ &= n^m \quad \text{[only } n^m \text{ will left, all other terms will be cancelled out]}\end{aligned}$$

Thus,

$$\boxed{\sum_{k=1}^n [k^m - (k-1)^m] = n^m} \quad (i)$$

If $m = 1$, the equation (i) will become

$$\sum_{k=1}^n [k^1 - (k-1)^1] = n^1$$

$$\sum_{k=1}^n [k - k + 1] = n$$

$$\boxed{\sum_{k=1}^n 1 = n}$$

[Means; $1 + 1 + 1 + \dots + 1 = n$]

(n times)

When $m = 2$, the equation (i) will become

$$\sum_{k=1}^n [k^2 - (k-1)^2] = n^2$$

$$\sum_{k=1}^n [k^2 - k^2 + 2k - 1] = n^2$$

$$\sum_{k=1}^n [2k - 1] = n^2$$

$$2 \sum_{k=1}^n k - \sum_{k=1}^n 1 = n^2$$

$$2 \sum_{k=1}^n k - n = n^2 \quad (\because \sum_{k=1}^n 1 = n)$$

$$2 \sum_{k=1}^n k = n^2 + n$$

$$\sum_{k=1}^n k = \frac{n^2 + n}{2}$$

$$\boxed{\sum_{k=1}^n k = \frac{n(n+1)}{2}}$$

Key Facts



$$\Sigma(a+b) = \Sigma a + \Sigma b$$

$$\Sigma 3a = 3 \Sigma a$$

Taking $m = 3$ in equation (i), we have

$$\sum_{k=1}^n [k^3 - (k-1)^3] = n^3$$

$$\sum_{k=1}^n [3k^2 - 3k + 1] = n^3$$

$$3 \sum_{k=1}^n k^2 - 3 \sum_{k=1}^n k + \sum_{k=1}^n 1 = n^3$$

We have, $\sum_{k=1}^n 1 = n$; $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

$$3 \sum_{k=1}^n k^2 - 3 \frac{n(n+1)}{2} + n = n^3$$

$$3 \sum_{k=1}^n k^2 = n^3 - n + \frac{3n(n+1)}{2}$$

$$\begin{aligned}
 &= \frac{2n^3 - 2n + 3n^2 + 3n}{2} \\
 &= \frac{2n^3 + 3n^2 + n}{2} \\
 &= \frac{n(2n^2 + 3n + 1)}{2} \\
 &= \frac{n(2n^2 + 2n + n + 1)}{2} \\
 &= n \left[\frac{2n^2 + 2n + n + 1}{2} \right] \\
 &= n \left[\frac{2n(n+1) + 1(n+1)}{2} \right] \\
 3 \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{2}
 \end{aligned}$$

$$\boxed{\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Similarly, we can prove that

$$\sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Example 32: Find the sum of the n terms of the series

$$1.2 + 2.3 + 3.4 + \dots$$

Solution: We know that the general term of

$$1 + 2 + 3 + \dots \text{ is } k.$$

$$2 + 3 + 4 + \dots \text{ is } k + 1.$$

If T_k is the k th term or general term of the series, then:

$$T_k = k(k+1)$$

$$T_k = k^2 + k$$

To find sum, taking summation both sides:

$$\begin{aligned}
 \sum_{k=1}^n T_k &= \sum_{k=1}^n k^2 + k \\
 &= \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\
 &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
 &= \frac{n(n+1)}{2} \left[\frac{2n+1}{3} + 1 \right] \\
 &= \frac{n(n+1)(2n+4)}{6} \\
 \sum_{k=1}^n T_k &= \frac{n(n+1)(n+2)}{3}
 \end{aligned}$$

$$\boxed{\left[\sum k = \frac{n(n+1)}{2} \right] \text{ and}}$$

$$\boxed{\sum k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Example 33: Find the sum to n terms of the series whose n th term is $n^2 + 4n + 1$.

Solution: Replace n with k .

$$T_k = k^2 + 4k + 1$$

Taking summation

$$\sum T_k = \sum (k^2 + 4k + 1)$$

$$\begin{aligned}
 &= \sum k^2 + 4 \sum k + \sum 1 \\
 &= \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} + n \\
 &= n \left[\frac{(n+1)(n+2)}{6} + 2(n+1) + 1 \right] \\
 &= n \left[\frac{n^2 + 2n + n + 2 + 12n + 12 + 6}{6} \right] \\
 \sum T_k &= n \left[\frac{n^2 + 15n + 20}{6} \right]
 \end{aligned}$$

4.9 Arithmetico-Geometric Series

In mathematics, arithmetico-geometric sequence is the result of term-by-term multiplication of a geometric progression with the corresponding terms of arithmetic progression. The n th term of an arithmetico-geometric sequence is the product of the n th term of an arithmetic sequence and the n th term of a geometric sequence. Arithmetico-geometric sequence arise in various applications such as the computation of expected values in statistics and other fields. For instance the sequence

$$\frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \dots$$

is an arithmetico-geometric sequence. The arithmetic component appears in the numerator and geometric one in the denominator.

The summation of this infinite sequence is known as **arithmetico-geometric series**.

4.9.1 Terms of the Sequence

The first few terms of an arithmetico-geometric sequence composed of an arithmetic progression with common difference d and initial value a and geometric progression with initial value b and common ratio r are given by:

$$\begin{aligned}
 T_1 &= ab = A_1 G_1 \\
 T_2 &= (a + d) br = A_2 G_2 \\
 T_3 &= (a + 2d) br^2 = A_3 G_3 \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

$$T_n = \underbrace{[a + (n-1)d]}_{A_n} \underbrace{br^{n-1}}_{G_n} = A_n G_n$$

For example, in the sequence $\frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \dots$ $d = 1$, $a = 0$, $b = 1$ and $r = \frac{1}{2}$.

Then n th term is:

$$\begin{aligned}
 T_n &= [0 + (n-1)1] 1 \cdot \left(\frac{1}{2}\right)^{n-1} \\
 T_n &= (n-1) \left(\frac{1}{2}\right)^{n-1} \\
 T_n &= \frac{n-1}{2^{n-1}}
 \end{aligned}$$

4.9.2 Sum of the n Terms

The sum of the first n terms of an arithmetico-geometric sequence has the form:

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k = \sum_{k=1}^n [a + (k-1)d] br^{k-1} \\ &= ab + (a+d)br + (a+2d)br^2 + \dots + [a + (n-1)d] br^{n-1} \quad (i) \\ S_n &= A_1G_1 + A_2G_2 + A_3G_3 + \dots + A_nG_n \end{aligned}$$

This sum can be written in closed form.

Proof:

Equation (i), is written as by putting $b = 1$

$$S_n = a + (a+d)r + (a+2d)r^2 + \dots + [a + (n-1)d] r^{n-1} \quad (ii)$$

Multiplying both sides of equation (ii) by r .

$$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + (a+3d)r^4 + \dots + [a + (n-1)d] r^n \quad (iii)$$

Subtracting rS_n from S_n and using the technique of telescope, we get:

$$\begin{aligned} S_n - rS_n &= [a + (a+d)r + (a+2d)r^2 + \dots + [a + (n-1)d] r^{n-1}] \\ &\quad - [ar + (a+d)r^2 + (a+2d)r^3 + (a+3d)r^4 + \dots + [a + (n-1)d] r^n] \\ &= a + ar + dr + ar^2 + 2dr^2 + \dots + ar^{n-1} + (n-1)dr^{n-1} \\ &\quad - ar - ar^2 - dr^2 - ar^3 - 2dr^3 - \dots - ar^n - (n-1)dr^n \end{aligned}$$

After cancelling like terms, we have:

$$\begin{aligned} &= a + d(r + r^2 + r^3 + \dots + r^{n-1}) - [a + (n-1)d] r^n \\ &= a + d(r + r^2 + r^3 + \dots + r^{n-1}) - ar^n - ndr^n + dr^n \\ S_n - rS_n &= a + d(r + r^2 + r^3 + \dots + r^{n-1} + r^n) - (a + nd)r^n \\ (1-r)S_n &= a + dr(1 + r + r^2 + \dots + r^{n-1}) - (a + nd)r^n \\ (1-r)S_n &= a + dr \frac{(1-r^n)}{1-r} - (a + nd)r^n \end{aligned}$$

Note

To generate the formula for finding the sum of the n th term " S_n ", we take $b = 1$.

$$S_n = \frac{a}{1-r} + dr \frac{(1-r^n)}{(1-r)^2} - \frac{(a + nd)r^n}{1-r} \quad (iv)$$

Hence, a is first term and d is common difference of arithmetic series and r is common ratio of geometric series.

4.9.3 Sum to Infinite Terms of Arithmetico-Geometric Series

Let $|r| < 1$

We know that $r^n \rightarrow 0$ as $n \rightarrow \infty$, then equation (iv) will become

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

This is sum to infinity of arithmetico-geometric series.

Example 34: Find the sum of $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \dots$ to n terms.

Solution: We know that the sum of arithmetico-geometric series formula for n terms is

$$S_n = \frac{a}{1-r} + dr \frac{(1-r^n)}{(1-r)^2} - \frac{(a+nd)r^n}{1-r} \quad (a)$$

We need the value of a (1st term) and d (common difference) for arithmetic series and r (common ratio) for geometric series.

Given series is:

$$1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \dots \text{ to } n \text{ terms}$$

We can rearrange as:

$$1, \frac{1}{2} + 3, \frac{1}{2} + 5, \frac{1}{4} + 7, \frac{1}{8} + \dots \text{ to } n \text{ terms}$$

It can be guessed that 1, 3, 5, 7, ... is arithmetic sequence with $a = 1$, $d = 2$, and $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ is geometric sequence with $r = \frac{1}{2}$.

Substituting the value of $a = 1$, $d = 2$ and $r = \frac{1}{2}$, we get:

$$\begin{aligned} S_n &= \frac{1}{1-\frac{1}{2}} + 2 \cdot \frac{1}{2} \cdot \frac{(1-\frac{1}{2^n})}{(1-\frac{1}{2})^2} - \frac{(1+2n)\frac{1}{2^n}}{1-\frac{1}{2}} \\ &= \frac{1}{\frac{1}{2}} + 1 \cdot \frac{(1-\frac{1}{2^n})}{\frac{1}{4}} - \frac{(1+2n)\frac{1}{2^n}}{\frac{1}{2}} \\ &= 2 + 4 \left(1 - \frac{1}{2^n}\right) - 2(1+2n)\frac{1}{2^n} \\ &= 2 + 4 - \frac{4}{2^n} - 2(1+2n)\frac{1}{2^n} \\ &= 6 - \frac{1}{2^n}(4 + 2 + 4n) \\ &= 6 - \frac{2}{2^n}(3 + 2n) \\ S_n &= 6 - \frac{2n+3}{2^{n-1}} \end{aligned}$$

Example 35: Find the sum to infinity of the arithmetic-geometric series:

$$1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \dots$$

Solution: Given arithmetic-geometric series can be written as:

$$1 \times 1 + 2 \times \frac{1}{3} + 3 \times \frac{1}{9} + 4 \times \frac{1}{27} + 5 \times \frac{1}{81} + \dots$$

The numbers 1, 2, 3, 4, 5, ... are in A.P. with $a = 1$ and $d = 1$.

Similarly, the numbers $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ are in G.P. with first term as 1 and $r = \frac{1/3}{1} = \frac{1}{3}$.

Thus, sum to infinity of the arithmetico-geometric series for

$$1 + 2 \times \frac{1}{3} + 3 \times \frac{1}{9} + 4 \times \frac{1}{27} + 5 \times \frac{1}{81} + \dots \text{ is:}$$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$\text{Here } a = 1, d = 1, r = \frac{1}{3}$$

We have, $S_{\infty} = \frac{1}{1-\frac{1}{3}} + \frac{1}{\left(1-\frac{1}{3}\right)^2}$

$$S_{\infty} = \frac{1}{\frac{2}{3}} + \frac{1}{\frac{4}{9}} = \frac{3}{2} + \frac{9}{4} = \frac{9}{4}$$

Exercise 4.7

Evaluate the sum:

1. $\sum_{k=1}^5 \frac{1}{2k}$

2. $\sum_{k=1}^6 \frac{1}{2k+1}$

3. $\sum_{k=0}^5 2^k$

4. $\sum_{k=0}^9 \pi k$

5. $\sum_{k=1}^8 \frac{k}{k+1}$

6. $\sum_{k=1}^7 (-1)^k 4^{k+1}$

7. $\sum_{k=0}^5 (k^2 - 2k + 3)$

8. $\sum_{k=1}^{10} \frac{1}{k(k+1)}$

Rewrite the sum using sigma notation:

9. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \dots$

10. $3 + 6 + 9 + 12 + 15$

11. $-2 + 4 - 8 + 16 - 32 + 64$

12. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots$

13. Prove that $\sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2$

Find the sum to n terms of the series whose n th terms are given:

14. $n + 1$

15. $n^2 + 2n$

16. $3n^2 + 2n + 1$

Sum the following series up to n terms:

17. $2^2 + 5^2 + 8^2 + \dots$

18. $2^2 + 4^2 + 6^2 + \dots$

19. $1^3 + 3^3 + 5^3 + \dots$

20. $2 + 5 + 10 + 17 + \dots$ to n terms

21. $1 \times 4 + 2 \times 7 + 3 \times 10 + \dots$

22. $1 \times 3 \times 5 + 3 \times 5 \times 7 + 5 \times 7 \times 9 + \dots$ to n terms

Sum to n terms of the following series (arithmetico-geometric series):

23. $1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \dots$

24. $1 + 4y + 7y^2 + 10y^3 + \dots$

$$25. 1 + \frac{4}{7} + \frac{7}{7^2} + \frac{10}{7^3} + \dots$$

$$26. 1 + \frac{7}{2} + \frac{13}{4} + \frac{19}{8} + \frac{25}{16} + \dots$$

Find sum to infinity of the following series:

$$27. 5 + \frac{7}{3} + \frac{9}{9} + \frac{11}{27} + \dots$$

$$28. 1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \dots$$

$$29. 1 + 4x + 7x^2 + 10x^3 + \dots$$

$$30. 3 + \frac{6}{10} + \frac{9}{100} + \frac{12}{1000} + \dots$$

4.10 Applications of Sequence and Series

Sequences and series have their own importance in many areas of Mathematics such as finance, statistics, population growth and physics. Most of the society and reality around us is based upon sequence after sequence, changing and repeating themselves over and over again. Common examples of this are time and calendrical system. Time (seconds, minutes, hours) always follow the same sequence, which always contains the same number of elements. Our lives are ruled over by sequences such as the routines that we follow every day without knowing leading to their great importance in the structure and function of the modern world.

Example 36: Khalid is saving for a new car. He deposits Rs. 100,000 into his account and then each month he deposits in Rs. 10,000 more than the month before. If the price of the car is Rs. 1,260,000; find:

- The amount Khalid has saved in four months.
- The time in which Khalid reaches his goal of Rs. 1,260,000.

Solution:

- Since Khalid deposits same amount every month, therefore, we will use arithmetic series.

$$\text{Let } a_1 = 100000, d = 10000, S_4 = ?$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_4 = \frac{4}{2}[2(a) + (4-1)d]$$

$$S_4 = 2[2(100000) + (3)10000]$$

$$S_4 = 2[200000 + 30000]$$

$$S_4 = 460000$$

Therefore, amount saved in 4 months = Rs. 460,000

- Let $S_n = 1260000, d = 10000, a = 100000, n = ?$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$126,0000 = \frac{n}{2} [2(100000) + (n-1)10000]$$

$$= n[190000 n + 100000]$$

$$2520000 = 190000 n + 10000n^2$$

$$n^2 + 19n - 252 = 0$$

$$n^2 + 28n - 9n - 252 = 0$$

$$n(n+28) - 9(n+28) = 0$$

$$(n-9)(n+28) = 0$$

$$n = 9, n = -28 \text{ (n cannot be negative)}$$

$\therefore$ Khalid will reach Rs. 1,200,000 in the 9th month.

Example 37: A new virus is on a remote area. On day one, there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

- Find the expected number of infected people on the 7th day.
- Find the expected number of infected people during week (7 days).

Solution:

- As the infection is increasing in percentage, therefore it is the problem of geometric sequence series.

Let $a_1 = 10$, $r = 1.4$ [40% increasing so $r = (100 + 40)\% = 140\% = 1.4$], $a_7 = ?$

Formula for nth term of a geometric progression.

$$a_n = ar^{n-1}$$

$$a_7 = ar^6$$

$$a_7 = 10(1.4)^6$$

$$a_7 = 75.29$$

$\therefore$ Expected number of new infections = 75 after seven days

- Total infected people after one week are S_7 .

$$S_n = \frac{a(r^n - 1)}{r - 1}; r > 1$$

$$S_7 = \frac{a(r^7 - 1)}{r - 1}$$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1}$$

$$S_7 = 238.53$$

$\therefore$ Expected number of total infections = 239

Exercise 4.8

1. A rocket rises 20 feet in the first second, 60 feet in the 2nd second and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second?
2. On the results declaration day, the school wants to invite parents as well as students. Auditorium has 21 seats in the first row and each of the other rows has one more seat than the one in front of it. There are 30 rows of seats in total. If they anticipate that 1200 people will come that day, will there be a seat for everyone? Justify your answer.
3. Majid retired after 30 years of employment. If his salary was Rs. 4500 in the first year and he received an increment of Rs. 820 at the end of each year of service. What was his total salary after 30 years?
4. You save Rs. 1 in the first day. Then each day thereafter, save double the amount you saved the day before. Find the amount you should save in the 20th day of your plan.
5. A vacuum pump removes $\frac{1}{5}$ of the air from a sealed container on each stroke of its piston. What percent of the air remains after five strokes of the piston?
6. Aslam borrows Rs. 20000 at 11% interest compounded annually. If he pays off the loan in full at the end of four years, how much does he pay?
7. A property dealer estimates that a piece of land will increase its value at a rate of 10% each year. If the original value of land is Rs. 450000, what will be its value in 8 years?
8. A man deposits in a bank Rs. 2000 in the first year, Rs. 4000 in the second year, Rs. 8000 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.
9. The number of bacteria in a culture increased geometrically from 16000 to 1215000 in 5 days. Find the daily rate of increase assuming the rate to be constant.
10. A car loan is in the amount of Rs. 600000 from the bank. Interest is 9% compounded annually and the entire amount is to be paid after 10 years. How much is to be paid back?
11. Zain bought a new car and got policy from insurance company. He will pay 5000 the first year, 6125 the second year, 7250 the third year and so on, for 10 years. How much he will pay to insurance company for vehicle?
12. Naveed takes a vehicle from bank after paying down payment. He deposits Rs. 13000 in a bank in first month, Rs. 14500 in the second month, Rs. 16000 in the third month and so on. Find how much total amount he has to deposit in the bank at the end of two years.
13. A man borrows a loan Rs. 1000000 for leasing a car and agrees to repay with a total 20 installments. Each installment is less than the preceding by Rs. 2000. What is his first installment?
14. Sara pays her first installment Rs. 8000 to insurance company for the vehicle. Each installment will increase by 5%. What total amount she will pay in 24 installments?

(xiv) In $\frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \frac{1}{10 \cdot 13} + \dots$, the n th term is:

a. $\frac{1}{(3n-1)(3n+1)}$

b. $\frac{1}{(3n-2)(3n+4)}$

c. $\frac{1}{(3n+1)(3n+4)}$

d. $\frac{1}{(n+2)(n+6)}$

(xv) The sum $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$ represents:

a. 1

b. $\frac{3}{4}$

c. $\frac{4}{3}$

d. $\frac{1}{4}$

(xvi) Sum of the series $1 + 3 + 5 + 7 + 9 + 11 + \dots$ is:

a. n

b. n^2

c. $n(n+1)$

d. $(n-1)$

(xvii) $\sum_{k=1}^{10} 3$ is equal to:

a. 10

b. 103

c. 300

d. 30

2. The sum of four numbers in A.P. is 24 and their product is 945. Find the numbers.

3. Find four numbers in A.P., whose sum is 6 and sum of whose square is 14.

4. Insert 20 A.Ms. between 2 and 86.

5. Evaluate $3 + 33 + 333 + \dots$ up to n terms.

6. If the product of three numbers in G.P. be 216 and their sum be 19, then find the numbers.

7. Insert 4 G.Ms. between 2 and 486.

8. Find n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the harmonic means between a and b .

9. Find the H.P., whose 3rd and 14th terms are $\frac{6}{7}$ and $\frac{1}{3}$ respectively.

10. Evaluate the sum:

(i) $\sum_{k=1}^{10} \frac{2^k}{2^{k+1}}$

(ii) $\sum_{k=1}^8 (-1)^{k+1} 3^k$

Sum to n terms of the following (arithmetico-geometric series):

11. $4 + 14 + 30 + 52 + 82 + \dots$

12. $1 + 4 + 10 + 21 + 39 + \dots$