

Chapter 5

PARTIAL FRACTIONS

We know that how to add two or more rational fractions into a single rational fraction. For example,

$$(i) \quad \frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$$

$$(ii) \quad \frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{5x^2 + 5x - 3}{(x+1)^2(x-2)}$$

Here we shall learn how to reverse the order (i) and (ii)

Partial Fractions

To express a single rational function as a sum of two or more single rational functions which are called Partial Fractions.

Partial Fraction Resolution

Expressing a rational function as a sum of partial fractions is called partial fraction resolution.

Rational Fraction

The quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors is called a Rational Fraction. A rational fraction is of two types.

(1) Proper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called a proper rational fraction if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example,

$$\frac{3}{x+1}, \quad \frac{2x-5}{x^2+4} \quad \text{and} \quad \frac{9x^2}{x^3-1} \quad \text{are proper rational fractions.}$$

(2) Improper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called an improper rational fraction if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator. For example

$$\frac{x}{2x-3}, \quad \frac{(x-2)(x+1)}{(x-1)(x+4)} \quad \text{and} \quad \frac{x^3 - x^2 + x + 1}{x^2 + 5}$$

are improper rational fractions.

EXERCISE 5.1

Resolve into Partial Fractions.

Q.1 $\frac{1}{x^2 - 1}$

(Gujranwala Board 2007, Lahore Board 2011)

Solution:

$$\frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)}$$

Let

$$\frac{1}{(x-1)(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} \quad (1)$$

$$\frac{1}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

$$1 = A(x-1) + B(x+1) \quad (2)$$

put $x = 1$ in equation (2), we get

$$1 = A(1-1) + B(1+1)$$

$$1 = A(0) + 2B$$

$$1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(-1-1) + B(-1+1)$$

$$1 = -2A + B(0)$$

$$-2A = 1 \Rightarrow \boxed{A = -\frac{1}{2}}$$

Put values of A and B in equation (1)

$$\frac{1}{(x+1)(x-1)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\frac{1}{2}}{(x-1)}$$

$$= \frac{-1}{2(x+1)} + \frac{1}{2(x-1)} \text{ are required partial fractions.}$$

Q.2 $\frac{x^2 (x^2 + 1)}{(x + 1) (x - 1)}$

(Lahore Board 2007, 2008)

Solution:

$$\frac{x^2 (x^2 + 1)}{(x + 1) (x - 1)} = \frac{x^4 + x^2}{x^2 - 1}$$

Dividing

$$\begin{array}{r} x^2 + 2 \\ x^2 - 1 \sqrt{x^4 + x^2} \\ + x^4 - x^2 \\ - + \\ \hline 2x^2 \\ 2x^2 - 2 \\ - + \\ \hline 2 \end{array}$$

$\Rightarrow$

$$\frac{x^4 + x^2}{x^2 - 1} = x^2 + 2 + \frac{2}{x^2 - 1} \quad (1)$$

Here

$$\frac{2}{x^2 - 1} = \frac{2}{(x + 1) (x - 1)} \quad (2)$$

$$\text{Take } \frac{2}{(x + 1) (x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$\frac{2}{(x + 1) (x - 1)} = \frac{A (x - 1) + B (x + 1)}{(x + 1) (x - 1)}$$

$$2 = A (x - 1) + B (x + 1) \quad (3)$$

Put $x = 1$ in equation (3), we get

$$2 = A (1 - 1) + B (1 + 1)$$

$$2 = A (0) + 2B$$

$$2 - 2B = 2 \Rightarrow \boxed{B = 1}$$

Put $x = -1$ in equation (3), we get

$$2 = A (-1 - 1) + B (-1 + 1)$$

$$2 = -2A + (0) B$$

$$\Rightarrow -2A = 2 \Rightarrow \boxed{A = -1}$$

Put values of A, B in equation (2)

$$\frac{2}{(x+1)(x-1)} = \frac{-1}{x-1} + \frac{1}{(x-1)}$$

equation (1) becomes

$$\frac{x^4 + x^2}{x^2 - 1} = x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$$

Hence $x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$ are required partial fractions.

Q.3
$$\frac{2x+1}{(x-1)(x+2)(x+3)}$$

Solution:

Let

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \quad (1)$$

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)}{(x-1)(x+2)(x+3)}$$

$$2x+1 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \quad (2)$$

Put $x = 1$ in equation (2)

$$2(1)+1 = A(1+2)(1+3) + B(1-1)(1+3) + C(1-1)(1+2)$$

$$3 = 12A + 0 + 0$$

$$12A = 3 \Rightarrow A = \frac{3}{12} \Rightarrow \boxed{A = \frac{1}{4}}$$

Put $x = -2$ in equation (2)

$$2(-2)+1 = A(-2+2)(-2+3) + B(-2-1)(-2+3) + C(-2-1)(-2+2)$$

$$-4+1 = A(0) + B(-3)(1) + C(0)$$

$$-3 = -3B \Rightarrow \boxed{B = 1}$$

Put $x = -3$ in equation (2)

$$2(-3)+1 = A(-3+2)(-3+3) + B(-3-1)(-3+3) + C(-3-1)(-3+2)$$

$$-6+1 = A(0) + B(0) + C(-4)(-1)$$

$$-5 = 0 + 0 + 4C$$

$$-5 = 4C \Rightarrow \boxed{C = -\frac{5}{4}}$$

Put these values in equation (1)

$$\begin{aligned} \frac{2x+1}{(x-1)(x+2)(x+3)} &= \frac{\frac{1}{4}}{x-1} + \frac{1}{x+2} - \frac{\frac{5}{4}}{x+3} \\ &= \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)} \end{aligned}$$

are required partial fractions.

Q.4
$$\frac{3x^2 - 4x - 5}{(x-2)(x^2 + 7x + 10)}$$

Solution:

As
$$\frac{3x^2 - 4x - 5}{(x-2)(x^2 + 7x + 10)} = \frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)}$$

Let

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+5)} \quad (1)$$

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2)}{(x-2)(x+2)(x+5)}$$

$$3x^2 - 4x - 5 = A(x+2)(x+5) + B(x-2)(x+5) + C(x+2)(x-2) \quad (2)$$

Put $x = 2$ in equation (2)

$$3(2)^2 - 4(2) - 5 = A(2+2)(2+5) + B(2-2)(2+5) + C(2-2)(2+2)$$

$$3(4) - 8 - 5 = A(4)(7) + 0 + 0$$

$$12 - 8 - 5 = 28A$$

$$4 - 5 = 28A \Rightarrow \boxed{A = -\frac{1}{28}}$$

Put $x = -2$ in equation (2), we get

$$\begin{aligned} 3(-2)^2 - 4(-2) - 5 &= A(-2+2)(-2+5) + B(-2-2)(-2+5) \\ &\quad + C(-2-2)(-2+2) \end{aligned}$$

$$3(4) + 8 - 5 = A(0) + B(-4)(3) + C(0)$$

$$12 + 8 - 5 = -12B$$

$$15 = -12B \Rightarrow B = -\frac{15}{12} \Rightarrow \boxed{B = -\frac{5}{4}}$$

Put $x = -5$ in equation (2), we get

$$3(-5)^2 - 4(-5) - 5 = A(-5+2)(-5+5) + B(-5-2)(-5+5) + C(-5-2)(-5+2)$$

$$3(25) + 20 - 5 = A(0) + B(0) + C(-7)(-3)$$

$$75 + 15 = 21C$$

$$90 = 21C \Rightarrow C = \frac{90}{21} \Rightarrow \boxed{C = \frac{30}{7}}$$

Put these values in equation (1)

$$\begin{aligned} \frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} &= \frac{-\frac{1}{28}}{x-2} + \frac{-\frac{5}{4}}{x+2} + \frac{\frac{30}{7}}{x+5} \\ &= \frac{1}{28(x-2)} - \frac{5}{4(x+2)} + \frac{30}{7(x+5)} \end{aligned}$$

are required partial fractions.

Q.5 $\frac{1}{(x-1)(2x-1)(3x-1)}$

Solution:

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1} \quad (1)$$

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1)}{(x-1)(2x-1)(3x-1)}$$

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \quad (2)$$

Put $x = 1$ in equation (2), we get

$$1 = A(2(1)-1)(3(1)-1) + B(1-1)(3(1)-1) + C(1-1)(2(1)-1)$$

$$1 = A(2-1)(3-1) + B(0) + C(0)$$

$$1 = A(1)(2) \Rightarrow \boxed{A = \frac{1}{2}}$$

Put $x = \frac{1}{2}$ in equation (2), we get

$$1 = A \left(2 \left(\frac{1}{2} \right) - 1 \right) \left(3 \left(\frac{1}{2} \right) - 1 \right) + B \left(\frac{1}{2} - 1 \right) \left(3 \left(\frac{1}{2} \right) - 1 \right) \\ + C \left(\frac{1}{2} - 1 \right) \left(2 \left(\frac{1}{2} \right) - 1 \right)$$

$$1 = A (1 - 1) \left(\frac{3}{2} - 1 \right) + B \left(-\frac{1}{2} \right) \left(\frac{3}{2} - 1 \right) + C \left(-\frac{1}{2} \right) (1 - 1)$$

$$1 = A (0) + B \left(-\frac{1}{2} \right) \left(\frac{1}{2} \right) + C (0)$$

$$1 = -\frac{1}{4} B \Rightarrow \boxed{B = -4}$$

Put $x = \frac{1}{3}$ in equation (2), we get

$$1 = A \left(2 \left(\frac{1}{3} \right) - 1 \right) \left(3 \left(\frac{1}{3} \right) - 1 \right) + B \left(\frac{1}{3} - 1 \right) \left(3 \left(\frac{1}{3} \right) - 1 \right) \\ + C \left(\frac{1}{3} - 1 \right) \left(2 \left(\frac{1}{3} \right) - 1 \right)$$

$$1 = A \left(\frac{2}{3} - 1 \right) (1 - 1) + B \left(-\frac{2}{3} \right) (1 - 1) + C \left(-\frac{2}{3} \right) \left(\frac{2}{3} - 1 \right)$$

$$1 = A (0) + B (0) + C \left(-\frac{2}{3} \right) \left(-\frac{1}{3} \right)$$

$$1 = \frac{2}{9} C \Rightarrow \boxed{C = \frac{9}{2}}$$

Put these values in equation (1), we get

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{-4}{2x-1} + \frac{\frac{9}{2}}{3x-1} \\ = \frac{1}{2(x-1)} - \frac{4}{(2x-1)} + \frac{9}{2(3x-1)}$$

are required partial fractions.

Q.6
$$\frac{x}{(x-a)(x-b)(x-c)}$$

Solution:

Let

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \dots\dots\dots (1)$$

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \quad \dots\dots\dots (2)$$

Put $x = a$ in equation (2), we get

$$a = A(a-b)(a-c) + B(a-a)(a-c) + C(a-a)(a-b)$$

$$a = A(a-b)(a-c) + 0 + 0$$

$$a = A(a-b)(a-c)$$

$$\boxed{A = \frac{a}{(a-b)(a-c)}}$$

Put $x = b$ in equation (2), we get

$$b = A(b-b)(b-c) + B(b-a)(b-c) + C(b-a)(b-b)$$

$$b = 0 + B(b-a)(b-c) + 0$$

$$\boxed{B = \frac{b}{(b-a)(b-c)}}$$

Put $x = c$ in equation (2), we get

$$c = A(c-b)(c-c) + B(c-a)(c-c) + C(c-a)(c-b)$$

$$c = 0 + 0 + C(c-a)(c-b)$$

$$\boxed{C = \frac{c}{(c-a)(c-b)}}$$

put these values in equation (1), we get

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}$$

are required partial fractions.

Q.7 $\frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1}$

Solution:

First dividing

$$\begin{array}{r} 3x + 4 \\ 2x^2 - x - 1 \overline{) 6x^3 + 5x^2 - 7} \end{array}$$

$$\begin{array}{r}
 6x^3 - 3x^2 \quad - 3x \\
 - \quad + \quad + \\
 \hline
 8x^2 + 3x - 7 \\
 8x^2 - 4x - 4 \\
 - \quad + \quad + \\
 \hline
 7x - 3
 \end{array}$$

$$\Rightarrow \frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1} = 3x + 4 + \frac{7x - 3}{2x^2 - x - 1} \quad \dots\dots\dots (1)$$

Take

$$\frac{7x - 3}{2x^2 - x - 1} = \frac{7x - 3}{(x - 1)(2x + 1)}$$

Let

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A}{(x - 1)} + \frac{B}{(2x + 1)} \quad \dots\dots\dots (2)$$

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A(2x + 1) + B(x - 1)}{(x - 1)(2x + 1)}$$

$$7x - 3 = A(2x + 1) + B(x - 1) \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (3), we get

$$7(1) - 3 = A(2(1) + 1) + B(1 - 1)$$

$$7 - 3 = A(2 + 1) + 0$$

$$\Rightarrow 4 = 3A \Rightarrow \boxed{A = \frac{4}{3}}$$

Put $x = -\frac{1}{2}$ in equation (3), we get

$$7\left(-\frac{1}{2}\right) - 3 = A\left(2\left(-\frac{1}{2}\right) + 1\right) + B\left(-\frac{1}{2} - 1\right)$$

$$\frac{7}{2} - 3 = A(-1 + 1) + B\left(-\frac{3}{2}\right)$$

$$-\frac{13}{2} = 0 - \frac{3}{2}B$$

$$13 = 3b \Rightarrow \boxed{B = \frac{13}{3}}$$

Put these values in equation (2), we get

$$\begin{aligned} \frac{7x-3}{(x-1)(2x+1)} &= \frac{\frac{4}{3}}{x-1} + \frac{\frac{13}{3}}{2x+1} \\ &= \frac{4}{3(x-1)} + \frac{13}{3(2x+1)} \end{aligned}$$

equation (1) becomes

$$\frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1} = (3x + 4) + \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

are required partial fractions.

Q.8
$$\frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x}$$

Solution:

First divide $2x^3 + x^2 - 5x + 3$ by $2x^3 + x^2 - 3x$

i.e.

$$\begin{array}{r} 1 \\ 2x^3 + x^2 - 3x \overline{) 2x^3 + x^2 - 5x + 3} \\ \underline{2x^3 + x^2 - 3x} \\ -2x + 3 \end{array}$$

$\Rightarrow$

$$\begin{aligned} \frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x} &= 1 + \frac{-2x + 3}{2x^3 + x^2 - 3x} \\ &= 1 + \frac{-2x + 3}{x(x-1)(2x+3)} \quad \dots\dots\dots (1) \end{aligned}$$

Let

$$\frac{-2x + 3}{x(x-1)(2x+3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+3} \quad \dots\dots\dots (2)$$

$$\frac{-2x + 3}{x(x-1)(2x+3)} = \frac{A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1)}{x(x-1)(2x+3)}$$

$$-2x + 3 = A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1) \dots\dots\dots (3)$$

Put $x = 0$ in equation (3), we get

$$\begin{aligned} -2(0) + 3 &= A(0-1)(2(0)+3) + B(0)(2(0)+3) \\ &\quad + C(0)(0-1) \end{aligned}$$

$$3 = A(-1)(3) + 0 + 0$$

$$3 = -3A \Rightarrow \boxed{A = -1}$$

Put $x = 1$ in equation (3), we get

$$\begin{aligned} -2(1) + 3 &= A(1-1)(2(1)+3) + B(1)(2(1)+3) \\ &\quad + C(1)(1-1) \end{aligned}$$

$$-2 + 3 = A(0) + B(2+3) + C(0)$$

$$1 = 5B \Rightarrow \boxed{B = \frac{1}{5}}$$

Put $x = -\frac{3}{2}$ in equation (3), we get

$$\begin{aligned} -2\left(-\frac{3}{2}\right) + 3 &= A\left(-\frac{3}{2}-1\right)\left[2\left(-\frac{3}{2}\right)+3\right] + B\left(-\frac{3}{2}\right)\left[2\left(-\frac{3}{2}\right)+3\right] \\ &\quad + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right) \end{aligned}$$

$$3 + 3 = A(0) + B(0) + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)$$

$$6 = \frac{15}{4}C \Rightarrow C = \frac{6 \times 4}{15} = \frac{8}{5}$$

$$\Rightarrow \boxed{C = \frac{8}{5}}$$

Put these values in equation (2), we get

$$\begin{aligned} \frac{-2x+3}{x(x-1)(2x+3)} &= \frac{-1}{x} + \frac{\frac{1}{5}}{x-1} + \frac{\frac{8}{5}}{(2x+3)} \\ &= -\frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)} \end{aligned}$$

equation (1) becomes

$$\frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x} = 1 - \frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)}$$

are required Partial formula.

Q.9
$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$$

Solution:

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = \frac{x^3 - 9x^2 + 23x - 15}{x^3 - 12x^2 + 44x - 48}$$

By division

$$\begin{array}{r} 1 \\ x^3 - 12x^2 + 44x - 48 \quad \sqrt{x^3 - 9x^2 + 23x - 15} \\ \underline{x^3 - 12x^2 + 44x - 48} \\ - + \quad - + \end{array}$$

$$3x^2 - 21x + 33$$

$$\begin{aligned} \frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} &= 1 + \frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} \\ &= 1 + \frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} \quad \dots\dots\dots (1) \end{aligned}$$

Now Let

$$\frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2} + \frac{B}{x-4} + \frac{C}{x-6} \quad \dots\dots\dots (2)$$

$$\frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} = \frac{A(x-4)(x-6) + B(x-2)(x-6) + C(x-2)(x-4)}{(x-2)(x-4)(x-6)}$$

$$3x^2 - 21x + 33 = A(x-4)(x-6) + B(x-2)(x-6) + C(x-2)(x-4) \quad \dots\dots\dots (3)$$

Put $x = 2$ in equation (3), we get

$$3(2)^2 - 21(2) + 33 = A(2-4)(2-6) + B(0) + C(0)$$

$$3(4) - 42 + 33 = A(-2)(-4)$$

$$12 - 42 + 33 = 8A$$

$$3 = 8A \Rightarrow \boxed{A = \frac{3}{8}}$$

Put $x = 4$ in equation (3), we get

$$3(4)^2 - 21(4) + 33 = A(0) + B(4 - 2)(4 - 6) + C(0)$$

$$3(16) - 84 + 33 = B(2)(-2)$$

$$48 - 84 + 33 = -4B$$

$$-3 = -4B \Rightarrow \boxed{B = \frac{3}{4}}$$

Put $x = 6$ in equation (3), we get

$$3(6)^2 - 21(6) + 33 = 0 + 0 + C(6 - 2)(6 - 4)$$

$$3(36) - 126 + 33 = C(4)(2)$$

$$108 - 126 + 33 = 8C$$

$$15 = 8C \Rightarrow \boxed{C = \frac{15}{8}}$$

Put these values in equation (2), we get

$$\begin{aligned} \frac{3x^2 - 21x + 33}{(x - 2)(x - 4)(x - 6)} &= \frac{\frac{3}{8}}{(x - 2)} + \frac{\frac{3}{4}}{(x - 4)} + \frac{\frac{15}{8}}{(x - 6)} \\ &= \frac{3}{8(x - 2)} + \frac{3}{4(x - 4)} + \frac{15}{8(x - 6)} \end{aligned}$$

$\Rightarrow$ equation (1) becomes

$$\frac{x^3 - 9x^2 + 23x - 15}{x^3 - 12x^2 + 44x - 48} = 1 + \frac{3}{8(x - 2)} + \frac{3}{4(x - 4)} + \frac{15}{8(x - 6)}$$

are required Partial fractions.

Q.10 $\frac{1}{(1 - ax)(1 - bx)(1 - cx)}$

(Lahore Board 2007)

Solution:

Let

$$\frac{1}{(1 - ax)(1 - bx)(1 - cx)} = \frac{A}{(1 - ax)} + \frac{B}{(1 - bx)} + \frac{C}{(1 - cx)} \quad \dots\dots\dots (1)$$

$$\frac{1}{(1 - ax)(1 - bx)(1 - cx)} = \frac{A(1 - bx)(1 - cx) + B(1 - ax)(1 - cx) + C(1 - ax)(1 - bx)}{(1 - ax)(1 - bx)(1 - cx)}$$

$$1 = A(1 - bx)(1 - cx) + B(1 - ax)(1 - cx) + C(1 - ax)(1 - bx) \quad \dots\dots\dots (2)$$

Put $x = \frac{1}{a}$ in equation (2), we get

$$1 = A \left(1 - b \cdot \frac{1}{a} \right) \left(1 - c \cdot \frac{1}{a} \right) + B (0) + C (0)$$

$$1 = \left(1 - \frac{b}{a} \right) \left(1 - \frac{c}{a} \right)$$

$$1 = A \left(\frac{a-b}{a} \right) \left(\frac{a-c}{a} \right)$$

$$1 = \frac{A (a-b) (a-c)}{a^2} \Rightarrow \boxed{A = \frac{a^2}{(a-b) (a-c)}}$$

Put $x = \frac{1}{b}$ in equation (2), we get

$$1 = A (0) + B \left(1 - a \cdot \frac{1}{b} \right) \left(1 - c \cdot \frac{1}{b} \right) + C (0)$$

$$1 = 0 + B \left(1 - a \cdot \frac{1}{b} \right) \left(1 - c \cdot \frac{1}{b} \right) + 0$$

$$1 = B \frac{(b-a) (b-c)}{b^2} \Rightarrow \boxed{B = \frac{b^2}{(b-a) (b-c)}}$$

Put $x = \frac{1}{c}$ in equation (2), we get

$$1 = 0 + 0 + C \left(1 - a \cdot \frac{1}{c} \right) \left(1 - b \cdot \frac{1}{c} \right)$$

$$1 = C \left(1 - \frac{a}{c} \right) \left(1 - \frac{b}{c} \right)$$

$$1 = C \left(\frac{c-a}{c} \right) \left(\frac{c-b}{c} \right)$$

$$1 = \frac{C (c-a) (c-b)}{c^2} \Rightarrow \boxed{C = \frac{c^2}{(c-a) (c-b)}}$$

Put these values in equation (1), we get

$$\begin{aligned} \frac{1}{(1-ax) (1-bx) (1-cx)} &= \frac{\frac{a^2}{(a-b) (a-c)}}{(1-ax)} + \frac{\frac{b^2}{(b-a) (b-c)}}{(1-bx)} + \frac{\frac{c^2}{(c-a) (c-b)}}{(1-cx)} \\ &= \frac{a^2}{(a-b) (a-c) (1-ax)} + \frac{b^2}{(b-a) (b-c) (1-bx)} \\ &\quad + \frac{c^2}{(c-a) (c-b) (1-bx)} \end{aligned}$$

are required Partial fractions.

Q.11 $\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$

Solution:

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$$

Let $x^2 = y$ and neglecting the square of each term

We have

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)} = \frac{y + a}{(y + b)(y + c)(y + d)}$$

Let

$$\frac{y + a}{(y + b)(y + c)(y + d)} = \frac{A}{(y + b)} + \frac{B}{(y + c)} + \frac{C}{(y + d)} \quad \dots\dots\dots (1)$$

$$\frac{y + a}{(y + b)(y + c)(y + d)} = \frac{A(y + c)(y + d) + B(y + b)(y + d) + C(y + b)(y + c)}{(y + b)(y + c)(y + d)}$$

$$y + a = A(y + c)(y + d) + B(y + b)(y + d) + C(y + b)(y + c) \quad \dots\dots\dots (2)$$

Put $y = -b$ in equation (2), we get

$$-b + a = A(-b + c)(-b + d) + 0 + 0$$

$$a - b = A(c - b)(d - b)$$

$$\boxed{A = \frac{(a - b)}{(c - b)(d - b)}}$$

Put $y = -c$ in equation (2), we get

$$-c + a = 0 + B(-c + b)(-c + d) + 0$$

$$a - c = B(b - c)(d - c)$$

$$\boxed{B = \frac{(a - c)}{(b - c)(d - c)}}$$

Put $y = -d$ in equation (2), we get

$$-d + a = 0 + 0 + C(-d + b)(-d + c)$$

$$a - d = C(b - d)(c - d)$$

$$C = \frac{(a-d)}{(b-d)(c-d)}$$

Put these values in equation (1), we get

$$\frac{y+a}{(y+b)(y+c)(y+d)} = \frac{(a-b)}{(c-b)(d-b)} \cdot \frac{1}{y+b} + \frac{(a-c)}{(b-c)(d-c)} \cdot \frac{1}{y+c} + \frac{(a-d)}{(b-b)(c-d)} \cdot \frac{1}{y+d}$$

$$\frac{a}{b+c+d} = \frac{a-b}{(c-b)(d-b)(y+b)} + \frac{a-c}{(b-c)(d-c)(y+c)} + \frac{a-d}{(b-d)(c-d)(y+d)}$$

Replacing the neglecting squares, we get

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)} + \frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(x^2+d^2)}$$

are required Partial fractions.

EXERCISE 5.2

Resolve the following into Partial fraction.

Q.1 $\frac{2x^2 - 3x + 4}{(x-1)^3}$

Solution:

Let

$$\frac{2x^2 - 3x + 4}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad \dots\dots\dots (1)$$

$$\frac{2x^2 - 3x + 4}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$2x^2 - 3x + 4 = A(x-1)^2 + B(x-1) + C \quad \dots\dots\dots (2)$$

$$2x^2 - 3x + 4 = A(x^2 + 1 - 2x) + Bx - B + C$$

$$2x^2 - 3x + 4 = Ax^2 + A - 2Ax + Bx - B + C \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$2(1)^2 - 3(1) + 4 = A(1-1)^2 + B(1-1) + C$$

$$2 - 3 + 4 = 0 + 0 + C$$

$$3 = C \Rightarrow \boxed{C = 3}$$

Equating coefficients of x^2 , x in equation (3), we get

$$x^2 ; \quad \boxed{A = 2}$$

$$x ; \quad -2A + B = -3$$

$$\Rightarrow -2(2) + B = -3$$

$$\Rightarrow -4 + B = -3$$

$$B = 4 - 3$$

$$\boxed{B = 1}$$

Put values of A, B and C in equation (1)

$$\frac{2x^2 - 3x + 4}{(x-1)^3} = \frac{2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{3}{(x-1)^3}$$

are required partial fractions.

Q.2
$$\frac{5x^2 - 2x + 3}{(x+2)^2}$$

(Lahore Board 2005)

Solution:

Let

$$\frac{5x^2 - 2x + 3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \quad \dots\dots\dots (1)$$

$$\frac{5x^2 - 2x + 3}{(x+2)^2} = \frac{A(x+2)^2 + B(x+2) + C}{(x+2)^3}$$

$$5x^2 - 2x + 3 = A(x+2)^2 + B(x+2) + C \quad \dots\dots\dots (2)$$

$$5x^2 - 2x + 3 = A(x^2 + 4 + 4x) + Bx + 2B + C$$

Put $x = -2$ in equation (2), we get

$$5(-2)^2 - 2(-2) + 3 = A(-2+2)^2 + B(-2+2) + C$$

$$5(4) + 4 + 3 = 0 + 0 + C$$

$$20 + 7 = C \Rightarrow \boxed{C = 27}$$

Equating coefficients of x^2 , x , we have

$$x^2 ; \quad \boxed{A = 5}$$

$$x ; \quad 4A + B = -2$$

$$4(5) + B = -2$$

$$\Rightarrow A = 5$$

$$20 + B = -2$$

$$B = -2 - 20$$

$$\boxed{B = -22}$$

Put values of A, B and C in equation (1) we get

$$\begin{aligned}\frac{5x^2 - 2x + 3}{(x+2)^2} &= \frac{5}{x+2} + \frac{-22}{(x+2)^2} + \frac{27}{(x+2)^3} \\ &= \frac{5}{x+2} - \frac{22}{(x+2)^2} + \frac{27}{(x+2)^3}\end{aligned}$$

are required partial fractions.

Q.3 $\frac{4x}{(x+1)^2(x-1)}$ (Lahore Board 2009, Gujranwala Board 2003, 2005)

Solution:

Let

$$\frac{4x}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} \quad \dots\dots\dots (1)$$

$$\frac{4x}{(x+1)^2(x-1)} = \frac{A(x+1)(x-1) + B(x-1) + C(x+1)^2}{(x+1)^2(x-1)}$$

$$4x = A(x+1)(x-1) + B(x-1) + C(x+1)^2 \quad \dots\dots\dots (2)$$

$$4x = A(x^2 - 1) + Bx - B + C(x^2 + 1 + 2x)$$

$$4x = Ax^2 - A + Bx - B + Cx^2 + C + 2Cx \quad \dots\dots\dots (3)$$

Put $x = -1$ in equation (2), we get

$$4(-1) = A(0) + B(-1-1) + C(0)$$

$$-4 = -2B \Rightarrow \boxed{B = 2}$$

Put $x = 1$ in equation (2), we get

$$4(1) = A(0) + B(0) + C(1+1)^2$$

$$4 = C(2)^2 \Rightarrow 4 = 4C \Rightarrow \boxed{C = 1}$$

Equating coefficient of x^2 in equation (3) we get,

$$x^2 ; \quad A + C = 0 \quad \rightsquigarrow \quad C = 1$$

$$A + 1 = 0$$

$$\boxed{A = -1}$$

Put values of A, B and C in equation (1), we have

$$\frac{4x}{(x+1)^2(x-1)} = \frac{-1}{(x+1)} + \frac{2}{(x+1)^2} + \frac{1}{(x-1)}$$

are required partial fractions.

Q.4 $\frac{9}{(x+2)^2(x-1)}$

Solution:

Let

$$\frac{9}{(x+2)^2(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)} \quad \dots\dots\dots (1)$$

$$\frac{9}{(x+2)^2(x-1)} = \frac{A(x+2)(x-1) + B(x-1) + C(x+2)^2}{(x+2)^2(x-1)}$$

$$9 = A(x+2)(x-1) + B(x-1) + C(x+2)^2 \quad \dots\dots\dots (2)$$

$$9 = A(x^2 + x - 2) + Bx - B + C(x^2 + 4 + 4x)$$

$$9 = Ax^2 + Ax - 2A + Bx - B + Cx^2 + 4C + 4Cx \quad \dots\dots\dots (3)$$

Put $x = -2$ in equation (2), we get

$$9 = A(0) + B(-2-1) + C(0)$$

$$9 = -3B \Rightarrow \boxed{B = -3}$$

Put $x = 1$ in equation (2), we get

$$9 = A(0) + B(0) + C(1+2)^2$$

$$9 = 0 + 0 + C(3)^2$$

$$9 = 9C \Rightarrow \boxed{C = 1}$$

Equating co-efficient of x^2 in equation (3) we get

$$A + C = 0$$

$$A + 1 = 0 \quad \quad \quad \Rightarrow \quad C = 1$$

$$\boxed{A = -1}$$

Put values of A, B and C in equation (1) we get

$$\frac{9}{(x+2)^2(x-1)} = \frac{-1}{(x+2)} - \frac{3}{(x+2)^2} + \frac{1}{(x-1)}$$

are required partial fractions.

Q.5 $\frac{1}{(x-3)^2(x+1)}$

(Lahore Board 2007)

Solution:

Let

$$\frac{1}{(x-3)^2(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x+1)} \quad \dots\dots\dots (1)$$

$$\frac{1}{(x-3)^2(x+1)} = \frac{A(x-3)(x+1) + B(x+1) + C(x-3)^2}{(x-3)^2(x+1)}$$

$$1 = A(x-3)(x+1) + B(x+1) + C(x-3)^2 \quad \dots\dots\dots (2)$$

$$1 = A(x^2 - 2x - 3) + Bx + B + C(x^2 + 9 - 6x)$$

$$1 = Ax^2 - 2Ax - 3A + Bx + B + Cx^2 + 9C - 6Cx \quad \dots\dots\dots (3)$$

Put $x = 3$ in equation (2), we get

$$1 = A(0) + B(3+1) + C(0)$$

$$1 = 0 + 4B + 0$$

$$1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(0) + B(0) + C(-1-3)^2$$

$$1 = 0 + 0 + C(-4)^2$$

$$1 = 16C \Rightarrow \boxed{C = \frac{1}{16}}$$

Equating coefficients of x^2 in equation (3) we get

$$A + C = 0$$

$$A + \frac{1}{16} = 0 \quad \quad \quad \therefore C = \frac{1}{16}$$

$$\boxed{A = -\frac{1}{16}}$$

Put values of A, B and C in equation (1), we get

$$\begin{aligned} \frac{1}{(x-3)^2(x+1)} &= \frac{-\frac{1}{16}}{x-3} + \frac{\frac{1}{4}}{x-3} + \frac{\frac{1}{16}}{x+1} \\ &= \frac{1}{16(x-3)} + \frac{1}{4(x-3)^2} + \frac{1}{16(x+1)} \end{aligned}$$

Q.6 $\frac{x^2}{(x-2)(x-1)^2}$

(Lahore Board 2011)

Solution:

Let

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \quad \dots\dots\dots (1)$$

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{A(x-1)^2 + B(x-2)(x-1) + C(x-2)}{(x-2)(x-1)^2}$$

$$x^2 = A(x-1)^2 + B(x-2)(x-1) + C(x-2) \quad \dots\dots\dots (2)$$

$$x^2 = A(x^2 + 1 - 2x) + B(x^2 - 3x - 2) + Cx - 2C$$

$$x^2 = Ax^2 + A - 2Ax + Bx^2 - 3Bx - 2B + Cx - 2C \quad \dots\dots\dots (3)$$

Put $x = 2$ in equation (2), we get

$$(2)^2 = A(2-1)^2 + B(0) + C(0)$$

$$4 = A(3)^2$$

$$4 = 9A \Rightarrow \boxed{A = \frac{4}{9}}$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A(0) + B(0) + C(1-2)$$

$$1 = 0 + 0 + C(-1)$$

$$1 = -C \Rightarrow \boxed{C = -1}$$

Equating coefficients of x^2 in equation (3) we get

$$A + B = 1$$

$$\frac{4}{9} + B = 1$$

$$B = 1 - \frac{4}{9}$$

$$\boxed{B = \frac{5}{9}}$$

Put values of A , B and C in equation (1), we get

$$\begin{aligned} \frac{x^2}{(x-2)(x-1)^2} &= \frac{\frac{4}{9}}{(x-2)} + \frac{\frac{5}{9}}{(x-1)} + \frac{-1}{(x-1)^2} \\ &= \frac{4}{9(x-2)} + \frac{5}{9(x-1)} - \frac{1}{(x-1)^2} \end{aligned}$$

Q.7 $\frac{1}{(x-1)^2(x+1)}$

Solution:

Let

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} \quad \dots\dots\dots (1)$$

$$\frac{1}{(x-1)^2(x+1)} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \quad \dots\dots\dots (2)$$

$$1 = A(x^2 - 1) + Bx + B + C(x^2 + 1 - 2x)$$

$$1 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$1 = A(0) + 2B + 0$$

$$1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(0) + B(0) + C(-1-1)^2$$

$$1 = 0 + 0 + C(-2)^2$$

$$1 = 4C \Rightarrow \boxed{C = \frac{1}{4}}$$

Equating coefficients of x^2 in equation (3), we get

$$A + C = 0$$

$$A + \frac{1}{4} = 0 \quad \quad \quad \Rightarrow \quad C = \frac{1}{4}$$

$$\boxed{A = -\frac{1}{4}}$$

Put values of A, B and C in equation (1) we get

$$\begin{aligned} \frac{1}{(x-1)^2(x+1)} &= \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1} \\ &= \frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)} \end{aligned}$$

are required partial fraction.

Q.8 $\frac{x^2}{(x-1)^3(x+1)}$

(Gujranwala Board 2003)

Solution:

Let

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+1)} \quad \dots\dots\dots (1)$$

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3}{(x-1)^3(x+1)}$$

$$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \quad \dots\dots\dots (2)$$

$$x^2 = A(x^3 - x^2 - x + 1) + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D$$

$$x^2 = Ax^3 - Ax^2 - Ax + A + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A(0) + B(0) + C(1+1) + D(0)$$

$$1 = 0 + 0 + 2C + 0$$

$$2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$(-1)^2 = A(0) + B(0) + C(0) + D(-1-1)^3$$

$$1 = 0 + 0 + 0 + D(-2)^3$$

$$1 = -8D \Rightarrow \boxed{D = -\frac{1}{8}}$$

Equating coefficients of x^3, x^2 in equation (3), we get

$$x^3 ; \quad A + D = 0$$

$$A - \frac{1}{8} = 0 \quad \quad \quad \simeq \quad D = -\frac{1}{8}$$

$$\boxed{A = \frac{1}{8}}$$

$$x^2 ; \quad -A + B - 3D = 1$$

$$-\frac{1}{8} + B - \left(-\frac{1}{8}\right) = 1 \quad \quad \quad \simeq \quad A = \frac{1}{8}, D = -\frac{1}{8}$$

$$-\frac{1}{8} + B + \frac{3}{8} = 1$$

$$B = 1 + \frac{1}{8} - \frac{3}{8}$$

$$B = \frac{8 + 1 - 3}{8}$$

$$B = \frac{6}{8} \Rightarrow \boxed{B = \frac{3}{4}}$$

Put values of A, B, C and D in equation (1) we get

$$\begin{aligned} \frac{x^2}{(x-1)^3(x+1)} &= \frac{\frac{1}{8}}{(x-1)} + \frac{\frac{3}{4}}{(x-1)^2} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{-\frac{1}{8}}{(x+1)} \\ &= \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^2} - \frac{1}{8(x+1)} \end{aligned}$$

are required partial fractions.

Q.9 $\frac{x-1}{(x-2)(x+1)^3}$ **(Gujranwala Board 2006)**

Solution:

Let

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \quad \dots\dots\dots (1)$$

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)}{(x-2)(x+1)^3}$$

$$(x-1) = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2) \quad \dots\dots\dots (2)$$

$$x-1 = A(x^3 + 1 + 3x^2 + 3x) + B(x^3 - 3x - 2) + C(x^2 - x + 2) + D(x-2)$$

$$x-1 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 - 3Bx - 2B + Cx^2 - Cx + 2C + Dx - 2D \quad \dots\dots\dots (3)$$

Put $x = 2$ in equation (2), we get

$$2-1 = A(2+1)^3 + 0 + 0 + 0$$

$$1 = A(3)^3$$

$$1 = 27A \Rightarrow \boxed{A = \frac{1}{27}}$$

Put $x = -1$ in equation (2), we get

$$-1-1 = 0 + 0 + 0 + D(-1-2)$$

$$-2 = D(-3)$$

$$\boxed{D = \frac{2}{3}}$$

$$x^3 ; A + B = 0$$

$$\frac{1}{27} + B = 0 \quad \Rightarrow \quad A = \frac{1}{27}$$

$$\boxed{B = -\frac{1}{27}}$$

$$x^2 ; 3A + C = 0$$

$$3 \cdot \frac{1}{27} + C = 0$$

$$\frac{1}{9} + C = 0 \Rightarrow \boxed{C = -\frac{1}{9}}$$

Put values of A, B, C and D in equation (1) we get

$$\begin{aligned} \frac{x-1}{(x-2)(x+1)^3} &= \frac{\frac{1}{27}}{(x-2)} + \frac{-\frac{1}{27}}{(x+1)} + \frac{-\frac{1}{9}}{(x+1)^2} + \frac{\frac{2}{3}}{(x+1)^3} \\ &= \frac{1}{27(x-2)} - \frac{1}{27(x+1)} - \frac{1}{9(x+1)^2} + \frac{2}{3(x+1)^3} \end{aligned}$$

are required partial fractions.

Q.10 $\frac{4x^3}{(x^2-1)(x+1)^2}$ **(Gujranwala Board 2005, Lahore Board 2005)**

Solution:

$$\frac{4x^3}{(x^2-1)(x+1)^2} = \frac{4x^3}{(x+1)(x-1)(x+1)^2} = \frac{4x^3}{(x-1)(x+1)^3}$$

Let

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \quad \dots\dots\dots (1)$$

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1)}{(x-1)(x+1)^3}$$

$$4x^3 = A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1) \quad \dots\dots\dots (2)$$

$$4x^3 = A(x^3 + 1 + 3x^2 + 3x) + B(x^3 + x^2 - x - 1) + C(x^2 - 1) + D(x - 1)$$

$$4x^3 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 + Bx^2 - Bx - B + Cx^2 - C + Dx - D \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$4(1)^3 = A(1+1)^3 + 0 + 0 + 0$$

$$4 = (2)^3 A$$

$$4 = 8A \Rightarrow \boxed{A = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$4(-1)^3 = 0 + 0 + 0 + D(-1-1)$$

$$-4 = -2D \Rightarrow \boxed{D = 2}$$

Equating coefficients of x^3, x^2 in equation (3) we get

$$x^3 ; \quad A + B = 4$$

$$\frac{1}{2} + B = 4 \qquad \qquad \qquad \Rightarrow A = \frac{1}{2}$$

$$B = 4 - \frac{1}{2}$$

$$\boxed{B = \frac{7}{2}}$$

$$x^2 ; \quad 3A + B + C = 0$$

$$3 \cdot \frac{1}{2} + \frac{7}{2} + C = 0$$

$$\frac{3}{2} + \frac{7}{2} + C = 0$$

$$\frac{10}{2} + C = 0$$

$$5 + C = 0 \Rightarrow \boxed{C = -5}$$

Put values of A, B, C and D in equation (1), we get

$$\begin{aligned} \frac{4x^3}{(x-1)(x+1)^3} &= \frac{\frac{1}{2}}{(x-1)} + \frac{\frac{7}{2}}{(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3} \\ &= \frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3} \end{aligned}$$

are required partial fractions.

Q.11 $\frac{2x+1}{(x+3)(x-1)(x+2)^2}$

Solution:

Let

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} \quad \dots\dots\dots (1)$$

$$\begin{aligned} & \frac{2x+1}{(x+3)(x-1)(x+2)^2} \\ &= \frac{A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x+3)(x-1)(x+2) + D(x+3)(x-1)}{(x+3)(x-1)(x+2)^2} \end{aligned}$$

$$(2x+1) = A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x+3)(x-1)(x+2) + D(x+3)(x-1) \quad (2)$$

$$2x+1 = A(x^3+3x^2-4) + B(x^3+7x^2+16x+12) + C(x^3+4x^2+x-6) + D(x^2+2x-3)$$

$$\begin{aligned} 2x+1 = & Ax^3+3Ax^2-4A+Bx^3+7Bx^2+16Bx+12B+Cx^3+4Cx^2+Cx-6C \\ & +Dx^2+2Dx-3D \end{aligned} \quad (3)$$

Put $x = -3$ in equation (2), we get

$$\begin{aligned} 2(-3)+1 &= A(-3-1)(-3+2)^2+0+0+0 \\ -6+1 &= A(-4)(-1)^2 \end{aligned}$$

$$-5 = -4A \Rightarrow \boxed{A = \frac{5}{4}}$$

Put $x = 1$ in equation (2), we get

$$2(1)+1 = 0+B(1+3)(1+2)^2+0+0$$

$$2+1 = B(4)(3)^2$$

$$3 = B(4)(9)$$

$$3 = 36B \Rightarrow B = \frac{3}{36} \Rightarrow \boxed{B = \frac{1}{12}}$$

Put $x = -2$ in equation (2), we get

$$2(-2)+1 = 0+0+0+D(-2+3)(-2-1)$$

$$-4+1 = D(1)(-3)$$

$$-3 = -3D \Rightarrow \boxed{D = 1}$$

Equating coefficients of x^3 in equation (3) we get

$$A+B+C = 0$$

$$\frac{5}{4} + \frac{1}{12} + C = 0$$

$$\frac{15+1}{12} + C = 0$$

$$\frac{16}{12} + C = 0$$

$$\frac{4}{3} + C = 0 \Rightarrow \boxed{C = -\frac{4}{3}}$$

Put values of A, B, C and D in equation (1), we get

$$\begin{aligned} \frac{2x+1}{(x+3)(x-1)(x+2)^2} &= \frac{\frac{5}{4}}{(x+3)} + \frac{\frac{1}{12}}{(x-1)} + \frac{-\frac{4}{3}}{(x+2)} + \frac{1}{(x+2)^2} \\ &= \frac{5}{4(x+3)} + \frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{1}{(x+2)^2} \end{aligned}$$

are required partial fraction.

Q.12 $\frac{2x^4}{(x-3)(x+2)^2}$

Solution:

$$\frac{2x^4}{(x-3)(x+2)^2} = \frac{2x^4}{(x-3)(x^2+4+4x)} = \frac{2x^4}{x^3+x^2-8x-12}$$

$$\begin{array}{r} 2x-2 \\ x^3+x^2-8x-12 \sqrt{2x^4} \\ \underline{2x^4+2x^3-16x^2-24x} \\ - \quad - \quad + \quad + \\ \hline -2x^3+16x^2+24x \\ -2x^3-2x^2+16x+24 \\ + \quad + \quad - \quad - \\ \hline \end{array}$$

$$\begin{aligned} \frac{2x^4}{x^3+x^2-8x-12} &= 2x-2 + \frac{18x^2+8x-24}{x^3+x^2-8x-12} \\ &= 2x-2 + \frac{18x^2+8x-24}{(x-3)(x+2)^2} \end{aligned} \quad \dots\dots\dots (1)$$

Let $\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{A}{(x-3)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \quad \dots\dots\dots (2)$

$$\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{A(x+2)^2+B(x-3)(x+2)+C(x-3)}{(x-3)(x+2)^2}$$

$$18x^2 + 8x - 24 = A(x+2)^2 + B(x-3)(x+2) + C(x-3) \quad \dots\dots\dots (3)$$

$$18x^2 + 8x - 24 = A(x^2 + 4x + 4) + B(x^2 - x - 6) + C(x - 3)$$

$$18x^2 + 8x - 24 = Ax^2 + 4Ax + 4A + Bx^2 - Bx - 6B + Cx - 3C \quad \dots\dots\dots (4)$$

Put $x = 3$ in equation (3), we get

$$18(3)^2 + 8(3) - 24 = A(3+2)^2 + 0 + 0$$

$$18(9) + 24 - 24 = A(5)^2$$

$$162 = 25A \Rightarrow \boxed{A = \frac{162}{25}}$$

Put $x = -2$ in equation (3), we get

$$18(-2)^2 + 8(-2) - 24 = 0 + 0 + C(-2-3)$$

$$-36 - 16 - 24 = C(-5)$$

$$32 = -5C \Rightarrow \boxed{B = -\frac{32}{5}}$$

Equating coefficients of x^2 in equation (4), we get

$$A + B = 18$$

$$B = 18 - A$$

$$B = 18 - \frac{162}{25} \Rightarrow \boxed{B = \frac{288}{25}}$$

Put values of A, B, C in equation (2), we get

$$\frac{18x^2 + 8x - 24}{(x-3)(x+2)^2} = \frac{162}{25(x-3)} + \frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

equation (1) becomes

$$\frac{4x^3}{(x-3)(x+2)^2} = (2x-2) + \frac{162}{25(x-3)} + \frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

are required partial fractions.

EXERCISE 5.3

Resolve the following into partial fractions.

Q.1 $\frac{9x-7}{(x^2+1)(x+3)}$ **(Lahore Board 2004, 2010)**

Solution:

Let

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3} \quad \dots\dots\dots (1)$$

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{(Ax+B)(x+3) + C(x^2+1)}{(x^2+1)(x+3)}$$

$$9x-7 = (Ax+B)(x+3) + C(x^2+1) \quad \dots\dots\dots (2)$$

$$9x - 7 = Ax^2 + 3Ax + Bx + 3B + Cx^2 + C \dots\dots\dots (3)$$

Put $x = -3$ in equation (2), we get

$$9(-3) - 7 = 0 + C [(-3)^2 + 1]$$

$$-27 - 7 = 0 + C (9 + 1)$$

$$-34 = 10C \Rightarrow \boxed{C = \frac{-17}{5}}$$

Equating coefficients of x^2, x in equation (3) we get

$$x^2 ; \quad A + C = 0$$

$$A - \frac{17}{5} = 0$$

$$\Rightarrow \boxed{A = \frac{17}{5}}$$

$$x ; \quad 3A + B = 9$$

$$3\left(\frac{17}{5}\right) + B = 9$$

$$\frac{51}{5} + B = 9$$

$$B = 9 - \frac{51}{5}$$

$$= \frac{45 - 51}{5}$$

$$\boxed{B = \frac{-6}{5}}$$

Put values of A, B, C in equation (1) we get

$$\begin{aligned} \frac{9x - 7}{(x^2 + 1)(x + 3)} &= \frac{\frac{17}{5}x - \frac{6}{5}}{x^2 + 1} - \frac{\frac{17}{5}}{x + 3} \\ &= \frac{17x - 6}{5(x^2 + 1)} - \frac{17}{5(x + 3)} \end{aligned}$$

are required partial fractions.

Q.2 $\frac{1}{(x^2 + 1)(x + 1)}$

(Lahore Board 2003, 2006)

Solution:

Let

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1} \dots\dots\dots (1)$$

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{(Ax + B)(x + 1) + C(x^2 + 1)}{(x^2 + 1)(x + 1)}$$

$$1 = (Ax + B)(x + 1) + C(x^2 + 1) \dots\dots\dots (2)$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2 + C \dots\dots\dots (3)$$

Put $x = -1$ in equation (2), we get

$$1 = 0 + C((-1)^2 + 1)$$

$$1 = C(1 + 1)$$

$$1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$$

Equating coefficients of x^2 and x in equation (3) we get

$$x^2 ; \quad A + C = 0$$

$$A = -C$$

$$\boxed{A = -\frac{1}{2}}$$

Equating coefficient of x

$$x ; \quad A + B = 0$$

$$B = -A$$

$$\boxed{B = \frac{1}{2}}$$

Put values of A , B and C in equation (1) we get

$$\begin{aligned} \frac{1}{(x^2 + 1)(x + 1)} &= \frac{-\frac{1}{2}x + \frac{1}{2}}{(x^2 + 1)} + \frac{\frac{1}{2}}{(x + 1)} \\ &= \frac{-x + 1}{2(x^2 + 1)} + \frac{1}{2(x + 1)} \end{aligned}$$

are required partial fractions.

Q.3 $\frac{3x + 7}{(x^2 + 4)(x + 3)}$

Solution:

Let

$$\frac{3x + 7}{(x^2 + 4)(x + 3)} = \frac{Ax + B}{(x^2 + 4)} + \frac{C}{(x + 3)} \quad \dots\dots\dots (1)$$

$$\frac{3x + 7}{(x^2 + 4)(x + 3)} = \frac{(Ax + B)(x + 3) + C(x^2 + 4)}{(x^2 + 4)(x + 3)}$$

$$3x + 7 = (Ax + B)(x + 3) + C(x^2 + 4) \quad \dots\dots\dots (2)$$

$$3x + 7 = Ax^2 + 3Ax + Bx + 3B + Cx^2 + 4C \quad \dots\dots\dots (3)$$

Put $x = -3$ in equation (2), we get

$$3(-3) + 7 = 0 + C((-3)^2 + 4)$$

$$-9 + 7 = C(9 + 4)$$

$$-2 = 13C \Rightarrow \boxed{C = -\frac{2}{13}}$$

Equating coefficients of x^2 and x in equation (3) we get

$$x^2 ; A + C = 0$$

$$A = -C$$

$$\boxed{A = \frac{2}{13}}$$

$$x ; 3A + B = 3$$

$$B = 3 - 3A$$

$$= 3 - 3 \cdot \frac{2}{13}$$

$$= 3 - \frac{6}{13}$$

$$= \frac{39 - 6}{13}$$

$$\boxed{B = \frac{33}{13}}$$

Put values of A , B and C in equation (1) we get

$$\begin{aligned} \frac{3x + 7}{(x^2 + 4)(x + 3)} &= \frac{\frac{2}{13}x + \frac{33}{13}}{x^2 + 4} + \frac{-\frac{2}{13}}{x + 3} \\ &= \frac{2x + 33}{13(x^2 + 4)} - \frac{2}{13(x + 3)} \end{aligned}$$

are required partial fractions.

Q.4
$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)}$$

Solution:

Let

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x - 1} \quad \dots\dots\dots (1)$$

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{(Ax + B)(x - 1) + C(x^2 + 2x + 5)}{(x^2 + 2x + 5)(x - 1)}$$

$$x^2 + 15 = (Ax + B)(x - 1) + C(x^2 + 2x + 5) \quad \dots\dots\dots (2)$$

$$x^2 + 15 = Ax^2 - Ax + Bx - B + Cx^2 + 2Cx + 5C \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 + 15 = 0 + C ((1)^2 + 2(1) + 5)$$

$$1 + 15 = C(1 + 2 + 5)$$

$$16 = 8C \Rightarrow \boxed{C = 2}$$

Equating coefficients of x^2 , and x in equation (3), we get

$$x^2 ; A + C = 1$$

$$A = 1 - C$$

$$A = 1 - 2$$

$$\boxed{A = 1}$$

$$x ; -A + B + 2C = 0$$

$$B = A - 2C$$

$$B = -1 - 2(2) = -1 - 4$$

$$\boxed{B = -5}$$

Put values of A , B and C in equation (1) we get

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{-x - 5}{x^2 + 2x + 5} + \frac{2}{(x - 1)}$$

are required partial fraction.

Q.5
$$\frac{x^2}{(x^2 + 4)(x + 2)}$$

Solution:

Let

$$\frac{x^2}{(x^2 + 4)(x + 2)} = \frac{Ax + B}{(x^2 + 4)} + \frac{C}{(x + 2)} \quad \dots\dots\dots (1)$$

$$\frac{x^2}{(x^2 + 4)(x + 2)} = \frac{(Ax + B)(x + 2) + C(x^2 + 4)}{(x^2 + 4)(x + 2)}$$

$$x^2 = (Ax + B)(x + 2) + C(x^2 + 4) \quad \dots\dots\dots (2)$$

$$x^2 = Ax^2 + 2Ax + Bx + 2B + Cx^2 + 4C \quad \dots\dots\dots (3)$$

Put $x = -2$ in equation (2), we get

$$(-2)^2 = 0 + C((-2)2 + 4)$$

$$4 = 0 + C(4 + 4)$$

$$4 = 8C \Rightarrow \boxed{C = \frac{1}{2}}$$

Equating coefficients of x^2 and x in equation (3), we get

$$x^2 ; A + C = 1$$

$$A = 1 - C = 1 - \frac{1}{2}$$

$$\boxed{A = \frac{1}{2}}$$

$$x ; 2A + B = 0$$

$$2\left(\frac{1}{2}\right) + B = 0$$

$$1 + B = 0$$

$$\boxed{B = -1}$$

Put values of A , B and C in equation (1) we get

$$\begin{aligned} \frac{x^2}{(x^2 + 4)(x + 2)} &= \frac{\frac{1}{2}x - 1}{x^2 + 4} + \frac{\frac{1}{2}}{x + 2} \\ &= \frac{x - 2}{2(x^2 + 4)} + \frac{1}{2(x + 2)} \end{aligned}$$

are required partial fraction.

Q.6 $\frac{x^2 + 1}{x^3 + 1}$

(Lahore Board 2003)

Solution:

Let

$$\frac{x^2 + 1}{x^3 + 1} = \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} = \frac{A}{(x + 1)} + \frac{Bx + C}{(x^2 - x + 1)} \quad \dots\dots\dots (1)$$

$$\Rightarrow \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} = \frac{A(x^2 - x + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 - x + 1)}$$

$$x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \dots\dots\dots (2)$$

$$x^2 + 1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C \quad \dots\dots\dots (3)$$

Put $x = -1$ in equation (2), we get

$$(-1)^2 + 1 = A((-1)^2 - (-1) + 1)$$

$$1 + 1 = A(1 + 1 + 1)$$

$$2 = 3A \Rightarrow \boxed{A = \frac{2}{3}}$$

Equating coefficients of x^2 and x in equation (3), we get

$$x^2 \quad ; \quad A + B = 1$$

$$\frac{2}{3} + B = 1$$

$$B = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\boxed{B = \frac{1}{3}}$$

$$x \quad ; \quad -A + B + C = 0$$

$$-\frac{2}{3} + \frac{1}{3} + C = 0$$

$$-\frac{1}{3} + C = 0$$

$$\boxed{C = \frac{1}{3}}$$

Put values of A , B and C in equation (1), we get

$$\begin{aligned} \frac{x^2 + 1}{x^3 + 1} &= \frac{\frac{2}{3}}{x + 1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2 - x + 1} \\ &= \frac{2}{3(x + 1)} + \frac{x + 1}{3(x^2 - x + 1)} \end{aligned}$$

are required partial fraction.

Q.7
$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)}$$

Solution:

Let

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} = \frac{Ax + B}{(x^2 + 3)} + \frac{C}{(x + 1)} + \frac{D}{(x - 1)} \quad \dots\dots\dots (1)$$

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} = \frac{(Ax + B)(x + 1)(x - 1) + C(x^2 + 3)(x - 1) + D(x^2 + 3)(x + 1)}{(x^2 + 3)(x + 1)(x - 1)}$$

$$x^2 + 2x + 2 = (Ax + B)(x + 1)(x - 1) + C(x^2 + 3)(x - 1) + D(x^2 + 3)(x + 1) \dots\dots (2)$$

$$x^2 + 2x + 2 = Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + 3Cx - 3C + Dx^3 + Dx^2 + 3Dx + 3D$$

Put $x = -1$ in equation (2), we get

$$(-1)^2 + 2(-1) + 2 = 0 + C((-1)^2 + 3)(-1 - 1) + 0$$

$$1 - 2 + 2 = C(1 + 3)(-2)$$

$$1 = -8C \Rightarrow \boxed{C = -\frac{1}{8}}$$

Put $x = 1$ in equation (2), we get

$$(1)^2 + 2(1) + 2 = 0 + 0 + D((1)^2 + 3)(1 + 1)$$

$$1 + 2 + 2 = D(1 + 3)(2)$$

$$5 = 8D \Rightarrow \boxed{D = \frac{5}{8}}$$

Equating coefficients of x^3 and x^2 in equation (3), we get

$$x^3 \quad ; \quad A + C + D = 0$$

$$A = -C - D$$

$$= \frac{1}{8} - \frac{5}{8} = \frac{-4}{8}$$

$$\boxed{A = -\frac{1}{2}}$$

$$x^2 \quad ; \quad B + D - C = 1$$

$$B = 1 - D + C = 1 - \frac{5}{8} - \frac{1}{8}$$

$$\boxed{B = \frac{1}{4}}$$

Put values of A, B, C, D in equation (1), we get

$$\begin{aligned} \frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} &= \frac{-\frac{1}{2}x + \frac{1}{4}}{(x^2 + 3)} + \frac{-\frac{1}{8}}{(x + 1)} + \frac{\frac{5}{8}}{(x - 1)} \\ &= \frac{1 - 2x}{4(x^2 + 3)} - \frac{1}{8(x + 1)} + \frac{5}{8(x - 1)} \end{aligned}$$

are required partial fraction.

Q.8 $\frac{1}{(x-1)^2 (x^2+2)}$

(Gujranwala Board 2006)

Solution:

Let

$$\frac{1}{(x-1)^2 (x^2+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+2)} \quad \dots\dots\dots (1)$$

$$\frac{1}{(x-1)^2 (x^2+2)} = \frac{A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2}{(x-1)^2 (x^2+2)}$$

$$1 = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \quad \dots\dots\dots (2)$$

$$1 = A(x^3 - x^2 + 2x - 2) + B(x^2 + 2) + (Cx + D)(x^2 + 1 - 2x)$$

$$1 = Ax^3 - Ax^2 + 2Ax - 2A + Bx^2 + 2B + Cx^3 + Cx - 2Cx^2 + Dx^2 + D - 2Dx \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we ge

$$1 = 0 + B((1)^2 + 2) + 0$$

$$1 = B(1 + 2)$$

$$1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$$

Equating coefficients of x^3 , x^2 , and x in equation (3), we get

$$1 = 0 + B((1)^2 + 2) + 0$$

$$1 = B(1 + 2)$$

$$1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$$

Equating coefficients of x^3 , x^2 , and x in equation (3), we get

$$x^3 \quad ; \quad A + C = 0 \quad \dots\dots\dots (i)$$

$$x^2 \quad ; \quad -A + B - 2C + D = 0 \quad \rightsquigarrow \quad B = \frac{1}{3}$$

$$-A - 2C + D = -\frac{1}{3} \quad \dots\dots\dots (ii)$$

$$x \quad ; \quad 2A + C - 2D = 0 \quad \dots\dots\dots (iii)$$

from (i) $A = -C$

Put in (ii) and (iii).

$$-(-C) - 2C + D = \frac{1}{3}$$

$$-C + D = \frac{1}{3} \quad \dots\dots\dots \text{(iv)}$$

and $2(-C) + C - 2D = 0$

$$-2C + C - 2D = 0$$

$$-C - 2D = 0 \quad \dots\dots\dots \text{(v)}$$

Subtracting (iv) from (v)

$$-C - 2D = 0$$

$$-C + D = -\frac{1}{3}$$

$$+ \quad - \quad +$$

$$-3D = 0 + \frac{1}{3}$$

$$-3D = \frac{1}{3} \Rightarrow \boxed{D = -\frac{1}{9}}$$

Put $D = -\frac{1}{9}$ in equation (v)

$$-C - 2\left(-\frac{1}{9}\right) = 0$$

$$-C + \frac{2}{9} = 0 \Rightarrow \boxed{C = \frac{2}{9}}$$

Put this value in equation (i) we get

$$\boxed{A = -C = \frac{-2}{9}}$$

Putting values of A, B, C and D in equation (1) we get

$$\begin{aligned} \frac{1}{(x-1)^2(x^2+2)} &= \frac{-\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\frac{2}{9}x - \frac{1}{9}}{x^2+2} \\ &= \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)} \end{aligned}$$

are required partial fraction.

Q.9 $\frac{x^4}{1-x^4}$

Solution:

$$\frac{x^4}{1-x^4} = \frac{-x^4}{x^4-1}$$

By division

$$\begin{array}{r} -1 \\ x^4-1 \overline{\sqrt{-x^4}} \\ -x^4+1 \\ + \\ \hline -1 \end{array}$$

$$\Rightarrow \frac{-x^4}{x^4-1} = -1 - \frac{1}{x^4-1} \quad \dots\dots\dots (1)$$

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(x^2+1)}$$

Let

$$\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \quad \dots\dots\dots (2)$$

$$\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)}{(x+1)(x-1)(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1) \quad \dots\dots\dots (3)$$

$$1 = A(x^3 - x^2 + x - 1) + B(x^3 + x^2 + x + 1) + (Cx + D)(x^2 - 1)$$

$$1 = Ax^3 - Ax^2 + Ax - A + Bx^3 + Bx^2 + Bx + B + Cx^3 - Cx - Cx + Dx^2 - D \quad \dots\dots\dots (4)$$

Put $x = 1$ in equation (3), we get

$$1 = A(-1-1)((-1)^2+1) + 0 + 0$$

$$1 = A(-2)(1+1)$$

$$1 = 4A \Rightarrow \boxed{A = -\frac{1}{4}}$$

Put $x = -1$ in equation (3), we get

$$1 = 0 + B(1+1)((1)^2+1) + 0$$

$$1 = B(2)(1+1)$$

$$1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$$

Equating coefficients of x^3 and x^2 in equation (4) we get

$$x^3 ; A + B + C = 0$$

$$-\frac{1}{4} + \frac{1}{4} + C = 0$$

$$\Rightarrow \boxed{C = 0}$$

$$x^2 ; -A + B + D = 0$$

$$\frac{1}{4} + \frac{1}{4} + D = 0$$

$$\boxed{D = -\frac{1}{2}}$$

Put values of A, B, C, D in equation (2) we get

$$\begin{aligned} \frac{1}{(1-x)(1+x)(1+x^2)} &= \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{(0)x - \frac{1}{2}}{x^2+1} \\ &= \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)} \end{aligned}$$

Equation (1) becomes

$$\begin{aligned} \frac{-x^4}{x^4-1} &= -1 - \left[\frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)} \right] \\ &= -1 + \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)} \end{aligned}$$

are required partial fractions.

Q.10 $\frac{x^2 - 2x + 3}{x^4 + x^2 + 1}$

Solution:

$$\frac{x^2 - 2x + 3}{x^4 + x^2 + 1} = \frac{x^2 - 2x + 3}{(x^2 + x + 1)(x^2 - x + 1)}$$

Let

$$\frac{x^2 - 2x + 3}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{Ax + B}{(x^2 + x + 1)} + \frac{Cx + D}{(x^2 - x + 1)} \quad \dots\dots\dots (1)$$

$$\frac{x^2 - 2x + 3}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{(Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)}{(x^2 + x + 1)(x^2 - x + 1)}$$

$$x^2 - 2x + 3 = (Ax + B)(x^2 + x + 1) + (Cx + D)(x^2 - x + 1)$$

$$x^2 - 2x + 3 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D \quad \dots\dots\dots (2)$$

Equating coefficients of x^3, x^2, x, x and constant term in equation (2), we get

$$x^3 \quad ; \quad A + C = 0 \quad (i)$$

$$x^2 \quad ; \quad -A + B + C + D = 1 \quad (ii)$$

$$x \quad ; \quad A - B + C + D = -2 \quad (iii)$$

$$\text{cons} \quad ; \quad B + D = 3 \quad (iv)$$

Put $A + C = 0$ in equation (iii), we get

$$-B + D = -2 \quad (v)$$

adding equation (iv) and (v), we get

$$2D = 1 \Rightarrow \boxed{D = \frac{1}{2}}$$

Put $D = \frac{1}{2}$ in equation (iv), we get

$$B + \frac{1}{2} = 3$$

$$B = 3 - \frac{1}{2} = \frac{6-1}{2}$$

$$\boxed{B = \frac{5}{2}}$$

Put $B + D = 3$ in equation (ii), we get

$$-A + C + 3 = 1$$

$$-A + C = -2 \quad (vi)$$

adding equation (vi) and (i)

$$2C = -2$$

$$\boxed{C = -1}$$

Put $C = -1$ in equation (i), we get

$$A - 1 = 0$$

$$\boxed{A = 1}$$

Put values of $A, B, C,$ and D in equation (1), we get

$$\begin{aligned} \frac{x^2 - 2x + 3}{(x^2 - x + 1)(x^2 + x + 1)} &= \frac{x + \frac{5}{2}}{x^2 + x + 1} + \frac{-x + \frac{1}{2}}{x^2 - x + 1} \\ &= \frac{(2x + 5)}{2(x^2 + x + 1)} - \frac{2x - 1}{2(x^2 - x + 1)} \end{aligned}$$

are required partial fraction.

EXERCISE 5.4

Resolve the following partial fractions.

Q.1 $\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2}$

Solution:

Let

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)}{(x^2 + x + 1)} + \frac{(Cx + D)}{(x^2 + x + 1)^2} \quad \dots\dots\dots (1)$$

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)(x^2 + x + 1) + (Cx + D)}{(x^2 + x + 1)^2}$$

$$x^3 + 2x + 2 = (Ax + B)(x^2 + x + 1) + (Cx + D)$$

$$x^3 + 2x + 2 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx + D \quad \dots\dots\dots (2)$$

Equating coefficients of x^3 , x^2 , x and constant term in equation (2), we get

$$x^3 \quad ; \quad \boxed{A = 1}$$

$$x^2 \quad ; \quad A + B = 0$$

$$1 + B = 0 \Rightarrow \boxed{B = -1}$$

$$x \quad ; \quad A + B + C = 2$$

$$1 - 1 + C = 2 \Rightarrow \boxed{C = 2}$$

$$\text{Cons} \quad ; \quad B + D = 2$$

$$-1 + D = 2 \Rightarrow \boxed{D = 3}$$

Put values of A, B, C, D in equation (1) we get

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{x - 1}{(x^2 + x + 1)} + \frac{2x + 3}{(x^2 + x + 1)^2}$$

are required partial fraction.

Q.2 $\frac{x^2}{(x^2 + 1)^2 (x - 1)}$

Solution:

Let

$$\frac{x^2}{(x - 1)(x^2 + 1)^2} = \frac{A}{(x - 1)} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \quad (1)$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)}{(x-1)(x^2+1)^2}$$

$$x^2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \quad \dots\dots\dots (2)$$

$$x^2 = A(x^4+1+2x^2) + (Bx+C)(x^3+x-x^2-1) + (Dx+E)(x-1)$$

$$x^2 = Ax^4 + A + 2Ax^2 + Bx^4 + Bx^2 - Bx^3 - Bx + Cx^3 + Cx - Cx^2 - C + Dx^2 - Dx + Ex - E \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A((1)^2+1)^2 + 0 + 0$$

$$1 = A(1+1)^2$$

$$1 = A(2)^2 \Rightarrow \boxed{A = \frac{1}{4}}$$

Equating coefficients of x^4, x^3, x^2 and constant term in equation (3), we get

$$x^4 \quad ; \quad A + B = 0$$

$$B = -A$$

$$\boxed{B = -\frac{1}{4}}$$

$$x^3 \quad ; \quad -B + C = 0$$

$$C = B$$

$$\boxed{C = -\frac{1}{4}}$$

$$x^2 \quad ; \quad 2A + B - C + D = 1$$

$$D = 1 - 2A - B + C$$

$$= 1 - 2\left(\frac{1}{4}\right) + \frac{1}{4} - \frac{1}{4}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{D = \frac{1}{2}}$$

$$x \quad ; \quad -B + C - D + E = 0$$

$$\frac{1}{4} - \frac{1}{4} - \frac{1}{2} + E = 0$$

$$\boxed{E = \frac{1}{2}}$$

Put values of A, B, C, D and E in equation (1) we get

$$\begin{aligned} \frac{x^2}{(x^2+1)^2(x-1)} &= +\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2} \\ &= \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2} \end{aligned}$$

are required partial fraction.

Q.3 $\frac{2x-5}{(x^2+2)^2(x-2)}$

Solution:

Let

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \dots\dots\dots (1)$$

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2)}{(x-2)(x^2+2)^2}$$

$$2x-5 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2) \quad \dots\dots\dots (2)$$

$$2x-5 = A(x^4+4+4x^2) + (Bx+C)(x^3-2x^2+2x-4) + (Dx+E)(x-2)$$

$$\begin{aligned} 2x-5 &= Ax^4+4A+4Ax^2+Bx^4-2Bx^3+2Bx^2-4Bx+Cx^3-2Cx^2+2Cx-4C \\ &\quad + Dx^2-2Dx+Ex-2E \quad \dots\dots\dots (3) \end{aligned}$$

Put $x = 2$ in equation (2), we get

$$2(2)-5 = A((2)^2+2)^2+0+0$$

$$4-5 = A(4+2)^2$$

$$-1 = A(6)^2 \Rightarrow \boxed{A = -\frac{1}{36}}$$

Equating coefficients of x^4 , x^3 , x^2 and x in equation (3), we get

$$x^4 ; \quad A+B = 0$$

$$B = -A = -\left(-\frac{1}{36}\right)$$

$$\boxed{B = -\frac{1}{36}}$$

$$x^3 \quad ; \quad -2B + C = 0$$

$$-2\left(-\frac{1}{36}\right) + C = 0$$

$$\boxed{C = \frac{1}{18}}$$

$$x^2 \quad ; \quad 4A + 2B - 2C + D = 0$$

$$4\left(-\frac{1}{36}\right) + 2\left(-\frac{1}{36}\right) - 2\left(-\frac{1}{18}\right) + D = 0$$

$$-\frac{1}{9} + \frac{1}{18} - \frac{1}{9} + D = 0$$

$$D = \frac{1}{9} + \frac{1}{9} - \frac{1}{18}$$

$$= \frac{2+2-1}{18} = \frac{3}{18}$$

$$\boxed{D = \frac{1}{6}}$$

$$x \quad ; \quad -4B + 2C - 2D + E = 0$$

$$E = 2 + 4B - 2C + 2D$$

$$= 2 + 4\left(\frac{1}{36}\right) - \left(\frac{1}{18}\right) + 2\left(\frac{1}{6}\right)$$

$$= 2 + \frac{1}{9} - \frac{1}{9} + \frac{1}{3}$$

$$= 2 + \frac{1}{3} = \frac{6+1}{3}$$

$$\boxed{E = \frac{7}{3}}$$

Put values of A, B, C, D, E in equation (1) we get

$$\begin{aligned} \frac{2x-5}{(x^2+2)^2(x-2)} &= \frac{-\frac{1}{36}}{(x-2)} + \frac{\frac{1}{36}x + \frac{1}{18}}{x^2+2} + \frac{\frac{1}{6}x + \frac{7}{3}}{(x^2+2)^2} \\ &= \frac{-1}{36(x-2)} + \frac{x+2}{36(x^2+2)} + \frac{x+14}{6(x^2+2)^2} \end{aligned}$$

Q.4 $\frac{8x^2}{(x^2 + 1)^2 (1 - x^2)}$

Solution:

As

$$\frac{8x^2}{(1 - x^2) (x^2 + 1)^2} = \frac{8x^2}{(1 - x) (1 + x) (x^2 + 1)^2}$$

Let

$$\frac{8x^2}{(1 - x) (1 + x) (x^2 + 1)^2} = \frac{A}{1 - x} + \frac{B}{1 + x} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + F}{(x^2 + 1)^2} \quad \dots\dots\dots (1)$$

$$\begin{aligned} \frac{8x^2}{(1 - x) (1 + x) (x^2 + 1)^2} \\ = \frac{A (1 + x) (x^2 + 1)^2 + B (1 - x) (x^2 + 1)^2 + (Cx + D) (x^2 + 1) (1 - x) (1 + x) + (Ex + F) (1 - x) (1 + x)}{(1 - x) (1 + x) (x^2 + 1)^2} \end{aligned}$$

$$\begin{aligned} 8x^2 = A (1 + x) (x^2 + 1)^2 + B (1 - x) (x^2 + 1)^2 + (Cx + D) (x^2 + 1) (1 - x) (1 + x) \\ + (Ex + F) (1 - x) (1 + x) \quad \dots\dots\dots (2) \end{aligned}$$

$$\begin{aligned} 8x^2 = A (x^5 + x^4 + 2x^3 + 2x^2 + x) + B (-x^5 + x^4 - 2x^3 + 2x^2 - x) + C (-x^5 + x) \\ + D (-x^4 + 1) + (Ex + F) (1 - x^2) \end{aligned}$$

$$\begin{aligned} 8x^2 = Ax^5 + Ax^4 + 2Ax^3 + 2Ax^2 + Ax - Bx^5 + Bx^4 - 2Bx^3 + 2Bx^2 - Bx \\ - Cx^5 + Cx - Dx^4 + D + Ex - Ex^3 + F - Fx^2 \quad \dots\dots\dots (3) \end{aligned}$$

Put $x = 1$ in equation (2), we get

$$8 (1)^2 = A (1 + 1) ((1)^2 + 1)^2$$

$$8 = A (2) (1 + 1)^2$$

$$8 = A (2) (4) \Rightarrow \boxed{A = 1}$$

Put $x = -1$ in equation (2), we get

$$8 (-1)^2 = 0 + B (+1 + 1) ((-1)^2 + 1)^2$$

$$8 = B (+2) (1 + 1)^2$$

$$8 = B (8)$$

$$\boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3 in equation (3) we get

$$x^5 \quad ; \quad A - B - C = 0$$

$$C = A - B = 1 - 1 = 0$$

$$\boxed{C = 0}$$

$$x^4 \quad ; \quad A + B - D = 0$$

$$D = A + B = 1 + 1$$

$$\boxed{D = 2}$$

$$x^3 \quad ; \quad 2A - 2B - E = 0$$

$$E = 2A - 2B$$

$$= 2(1) - 2(1)$$

$$\boxed{E = 0}$$

Put values of A, B, C, D and E in equation (1) we get

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2} = \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{x^2+1} - \frac{4}{(x^2+1)^2}$$

are required partial fractions.

Q.5
$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2}$$

Solution:

Let

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)} + \frac{Dx+E}{(x^2+x+1)^2} \quad \dots\dots\dots (1)$$

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1)}{(x-1)(x^2+x+1)^2}$$

$$4x^4 + 3x^3 + 6x^2 + 5x = A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1) \quad \dots\dots\dots (2)$$

$$4x^4 + 3x^3 + 6x^2 + 5x = Ax^4 + 2Ax^3 + 3Ax^2 + 2Ax + A + Bx^4 - Bx + Cx^3 - C + Dx^2 - Dx + Ex - E \quad \dots\dots\dots (3)$$

Put $x = 1$ in equation (2), we get

$$4(1)^4 + 3(1)^3 + 6(1)^2 + 5(1) = A((1)^2 + 1 + 1)^2 + 0 + 0$$

$$4 + 3 + 6 + 5 = A(1 + 1 + 1)^2$$

$$18 = 9A \Rightarrow \boxed{A = 2}$$

$$x^4 ; \quad A + B = 4$$

$$B = 4 - A = 4 - 2$$

$$\boxed{B = 2}$$

$$x^3 ; \quad 2A + C = 3$$

$$C = 3 - 2A = 3 - 2(2) = 3 - 4$$

$$\boxed{C = -1}$$

$$x^2 ; \quad 3A + D = 6$$

$$D = 6 - 3A = 6 - 3(2) = 6 - 6 = 0$$

$$\boxed{D = 0}$$

$$x ; \quad 2A - B - D + E = 5$$

$$E = 5 - 2A + B + D$$

$$= 5 - 2(2) + 2 + 0$$

$$\boxed{E = 3}$$

Put values of A, B, D, D and E in equation (1)

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{2}{(x-1)} + \frac{2x-1}{(x^2+x+1)} + \frac{3}{(x^2+x+1)^2}$$

are required partial fractions.

Q.6
$$\frac{2x^4 - 3x^3 - 4x}{(x^2 + 2)^2 (x + 1)^2}$$

Solution:

Let

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2}$$

$$= \frac{A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2(x+2) + (Ex+F)(x+1)^2}{(x+1)^2 (x^2+2)^2}$$

$$2x^4 - 3x^3 - 4x = A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2(x+2) + (Ex+F)(x+1)^2 \dots\dots (2)$$

$$\begin{aligned}
2x^4 - 3x^3 - 4x &= A(x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4) + B(x^4 + 4 + 4x^2) \\
&\quad + C(x^5 + 2x^4 + 3x^3 + 4x^2 + 2x) + D(x^4 + 2x^3 + 3x^2 + 4x + 2) + E(x^3 + 2x^2 + x) \\
&\quad + F(x^2 + 2x + 1) \\
2x^4 - 3x^3 - 4x &= Ax^5 + Ax^4 + 4Ax^3 + 4Ax^2 + 4Ax + 4A + Bx^4 + 4B + 4Bx^2 \\
&\quad + Cx^5 + 2Cx^4 + 3Cx^3 + 4Cx^2 + 2Cx + Dx^4 + 2Dx^3 + 3Dx^2 + 4Dx \\
&\quad + 2D + Ex^3 + 2Ex^2 + Ex + Fx^2 + 2Fx + F \quad (3)
\end{aligned}$$

Put $x = -1$ in equation (2), we get

$$2(-1)^4 - 3(-1)^3 - 4(-1) = 0 + B((-1)2 + 2)^2 + 0 + 0$$

$$2 + 3 + 4 = B(1 + 2)^2$$

$$9 = B(3)^2$$

$$9 = 9B \Rightarrow \boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3, x^2, x and constant in equation (3), we get

$$x^5 ; \quad A + C = 0 \quad (i)$$

$$x^4 ; \quad A + B + 2C + D = 2 \quad (ii)$$

$$x^3 ; \quad 4A + 3C + 2D + E = -3 \quad (iii)$$

$$x^2 ; \quad 4A + 4B + 4C + 3D + 2E + F = 0 \quad (iv)$$

$$x ; \quad 4A + 2C + 4D + E + 2F = -4 \quad (v)$$

$$\text{cons} ; \quad 4A + 4B + 2D + F = 0 \quad (vi)$$

from (i)

$$C = -A$$

from (ii)

$$D = 2 - A - B - 2C = 2 - A - 1 + 2A = 1 + A \quad (vii)$$

from (iii)

$$E = -3 - 4A - 3C - 2D = -3 - 4A + 3A - 2 - 2A = -3A - 5 \quad (viii)$$

Subtracting (vi) from (iv), we get

$$4C + D + 2E = 0$$

$$\Rightarrow -4A + 1 + A + 2(-3A - 5) = 0$$

$$-9A = 9$$

$$\Rightarrow \boxed{A = -1}$$

Put this value of A in (i), (vii), (viii) we get

$$(i) \Rightarrow C = -(-1) = 1 \Rightarrow \boxed{C = -1}$$

$$(vii) \Rightarrow D = 1 + (-1) = 0 \Rightarrow \boxed{D = 0}$$

$$(viii) \Rightarrow E = -3(-1) - 5 = 3 - 5 = -2 \\ \Rightarrow \boxed{E = -2}$$

from equation (vi)

$$4(-1) + 4(1) + 2(0) + F = 0$$

$$-4 + 4 + 0 + F = 0$$

$$\boxed{F = 0}$$

Put values of A, B, C, D, E and F in equation (1), we get

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2(x^2+2)^2} = \frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}$$

are required partial fractions.

PARTIAL FRACTIONS

MULTIPLE CHOICE QUESTIONS

- (1) The function of the form $f(x) = \frac{p(x)}{q(x)}$ where $q(x) \neq 0$ and $p(x)$, $q(x)$ are polynomials is called:

(a) Identity	(b) Equation
(c) Fraction	(d) Algebraic relation

[Lahore Board 2005]

- (2) Partial fractions of $\frac{1}{x^2 - 1}$ is equivalent to:

(a) $\frac{Ax + B}{x^2 - 1}$	(b) $\frac{A}{x + 1} + \frac{B}{x - 1}$
(c) $\frac{A}{x + 1}$	(d) $\frac{B}{x - 1}$

[Lahore Board 2005-07-08]

- (3) The fraction $\frac{x^2 + 7x + 3}{x + 1}$ is:

(a) Improper	(b) Proper
(c) Equivalent	(d) None of these

[Gujranwala Board 2005]

- (4) Partial fractions of $\frac{1}{x^3 + 1}$ will be of the form:

(a) $\frac{A}{x - 1} + \frac{B}{x^2 - x + 1}$	(b) $\frac{A}{x + 1} + \frac{B}{x^2 - x + 1}$
(c) $\frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$	(d) $\frac{A}{x - 1} + \frac{B}{x^2 + 1}$

[Gujranwala Board 2005]

- (5) A fraction in which the degree of numerator is greater than or equal to the degree of denominator is called:
- (a) A proper fraction (b) An improper fraction
(c) An equation (d) An algebraic relation
- [Gujranwala Board 2006]**
- (6) $\frac{x^3 + 1}{(x - 1)(x + 2)}$ is:
- (a) Proper fraction (b) Improper fraction
(c) Identity (d) None of these
- [Gujranwala Board 2006]**
- (7) Partial fraction of $\frac{7x^2 + 25}{(x + 3)(x + 4)}$ will be of the form:
- (a) $\frac{A}{x + 3} + \frac{B}{x + 4}$ (b) $1 + \frac{A}{x + 3} + \frac{B}{x + 4}$
(c) $7 + \frac{A}{x + 3} + \frac{B}{x + 4}$ (d) None of these
- [Lahore Board 2014, Lahore Board 2006]**
- (8) The fraction $\frac{2x^2 + 5}{x - 3}$ is:
- (a) Proper (b) Improper
(c) Irrational (d) None of these
- [Lahore Board 2006]**
- (9) Partial fraction of $\frac{1}{(x + 1)(x^2 - 1)}$ will be of the form:
- (a) $\frac{A}{x + 1} + \frac{Bx + C}{x^2 - 1}$ (b) $\frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{x + 1}$
(c) $\frac{A}{x - 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 1}$ (d) $\frac{A}{x - 1} + \frac{Bx + C}{(x + 1)^2}$
- [Gujranwala Board 2007]**
- (10) Partial fraction of $\frac{x^2 + 1}{(x + 1)(x - 1)}$ are of the form:
- (a) $\frac{A}{x + 1} + \frac{B}{x - 1}$ (b) $1 + \frac{A}{x + 1} + \frac{B}{x - 1}$
(c) $1 + \frac{A}{x + 1} + \frac{Bx + C}{x - 1}$ (d) $\frac{Ax + B}{x + 1} + \frac{C}{x - 1}$
- [Gujranwala Board 2007]**
- (11) Types of rational fractions are:
- (a) Three (b) Two
(c) Four (d) Infinite
- [Lahore Board 2007]**

(12) $\frac{1}{(x^2 + 1)(x + 1)}$ equals:

(a) $\frac{A}{x^2 + 1} + \frac{Bx + C}{x + 1}$

(c) $\frac{Ax + B}{x^2 + 1}$

(b) $\frac{A}{x^2 + 1} + \frac{B}{x + 1}$

(d) $\frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1}$

[Lahore Board 2008]

(13) $\frac{x^4}{1 - x^4}$ is:

(a) Proper fraction

(c) Decimal

(b) Improper fraction

(d) Equation

[Lahore Board 2008]

(14) Partial fractions of $\frac{1}{x^3 - 1}$ are of the form:

(a) $\frac{A}{x - 1} + \frac{B}{x^2 + x + 1}$

(c) $\frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$

(b) $\frac{A}{x + 1} + \frac{B}{x^2 - x + 1}$

(d) $\frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$

[Gujranwala Board 2008]

(15) Partial fraction of $\frac{(x - a)(x - b)}{(x - c)(x - d)}$ will be of the form:

(a) $\frac{A}{x - c} + \frac{B}{x - d}$

(c) $\frac{A}{x - a} + \frac{B}{x - b}$

(b) $1 + \frac{A}{x - a} + \frac{B}{x - b}$

(d) $1 + \frac{B}{x - c} + \frac{A}{x - d}$

[Lahore Board 2010]

(16) Partial fraction of $\frac{x}{(x - 1)(x - 2)}$ will be of the form:

(a) $\frac{A}{x - 1} + \frac{B}{x - 2}$

(c) $\frac{1}{x + 2}$

(b) $\frac{1}{x - 1}$

(d) $1 + \frac{A}{x - 1} + \frac{B}{x + 2}$

[Gujranwala Board 2011]

(17) Conditional equation $3x - 1 = 0$ is true only if:

(a) $x = 3$

(c) $x = \frac{1}{3}$

(b) $x = -3$

(d) $x = -\frac{1}{3}$

[Lahore Board 2012]

(18) Partial fractions of $\frac{1}{(x+1)(x^2-1)}$ will be of the form:

- (a) $\frac{a}{x+1} + \frac{Bx+C}{x^2-1}$ (b) $\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+1}$
 (c) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$ (d) $\frac{A}{x-1} + \frac{Bx+C}{(x+1)^2}$

[Lahore Board 2012]

(19) The function of the form $\frac{p(x)}{q(x)}$, $q(x) \neq 0$ is called:

- (a) Identity (b) Equation
 (c) Fraction (d) Formula

[Lahore Board 2013]

(20) Partial fractions of $\frac{2}{x^2-1}$ are:

- (a) $\frac{1}{x-1} + \frac{1}{x+1}$ (b) $\frac{1}{x-1} - \frac{1}{x+1}$
 (c) $\frac{1}{2(x+1)} - \frac{1}{2(x-1)}$ (d) $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

[Lahore Board 2013]

(21) A conditional equation is also called:

- (a) Identity (b) Equation
 (c) Fraction (d) None of these

(22) An equation which holds good for all values of the variable is called:

- (a) Identity (b) Equation
 (c) Conditional equation (d) All of these

(23) Expressing a rational function as a sum of partial fractions is called:

- (a) Fraction (b) Partial fraction resolution
 (c) Equation (d) Identity

(24) $(x+3)(x+4) = x^2 + 7x + 12$ is a/an:

- (a) Fraction (b) Conditional equation
 (c) Identity (d) All of these

(25) $\frac{3x^2-1}{x-2} = \text{_____}$

- (a) $3x - 6 + \frac{A}{x-2}$ (b) $\frac{Ax+B}{x-2}$
 (c) $3x + 6 + \frac{Ax+B}{x^2-1}$ (d) $3x + 6 + \frac{11}{x-2}$

(26) Partial fraction of $\frac{x^2 - 10x + 13}{(x - 1)(x^2 - 5x + 6)}$ will be of the form:

(a) $\frac{A}{x - 1} + \frac{B}{x^2 + 5x + 6}$

(b) $1 + \frac{A}{x - 1} + \frac{Bx + C}{x^2 - 5x + 6}$

(c) $\frac{A}{x - 1} + \frac{Bx + C}{x^2 + 5x + 6}$

(d) $\frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$

(27) $\frac{4x^2 + 8x}{x^4 + 2x^2 + 9} = \text{_____}$

(a) $\frac{4x^2 + 8x}{(x^2 + 2x + 3)(x^2 + 2x + 3)}$

(b) $\frac{4x^2 + 8x}{(x^2 + 2x - 3)(x^2 + 2x + 3)}$

(c) $\frac{4x^2 + 8x}{(x^2 - 2x - 3)(x^2 + 2x + 3)}$

(d) $\frac{4x^2 + 8x}{(x^2 + 2x + 3)(x^2 - 2x + 3)}$

(28) Partial fraction of $\frac{x - b}{x^3 - b^3}$ where $x \neq b$ is of the form:

(a) $\frac{A}{x - b} + \frac{Bx + C}{x^2 + bx + b^2}$

(b) $\frac{A}{x - b} + \frac{B}{x^2 + bx + a^2}$

(c) $\frac{B}{x^2 + bx + b^2}$

(d) None of these

(29) Partial fraction of $\frac{3x - 4}{(x - 1)^2}$ is of the form:

(a) $\frac{A}{(x - 1)^2}$

(b) $\frac{A}{x - 1} + \frac{Bx + C}{(x - 1)^2}$

(c) $\frac{A}{x - 1} + \frac{Bx + C}{(x - 1)^2}$

(d) None of these

(30) $\cos^2\theta + \sin^2\theta = 1$ is a/an:

(a) Identity

(b) Conditional equation

(c) Fraction

(d) None of these

(31) An equation in which two algebraic expressions are equal for particular values of the variable is called:

(a) Identity

(b) Conditional equation

(c) Fraction

(d) None of these

(32) The quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$ with no common factors is called a/an:

- (a) Partial fraction resolution (b) Identity
(c) Rational fraction (d) None of these

(33) Partial fractions of $\frac{1}{x(x+1)}$ are of the form:

- (a) $\frac{1}{x-1} + \frac{1}{x+1}$ (b) $\frac{1}{x-1} - \frac{1}{x+1}$
(c) $\frac{1}{x} + \frac{1}{x+1}$ (d) $\frac{1}{x} - \frac{1}{x+1}$

[Lahore Board 2014]

Answers

(1)	c	(2)	b	(3)	a	(4)	c	(5)	b
(6)	b	(7)	c	(8)	b	(9)	c	(10)	b
(11)	b	(12)	d	(13)	b	(14)	c	(15)	d
(16)	a	(17)	c	(18)	c	(19)	c	(20)	b
(21)	b	(22)	a	(23)	b	(24)	c	(25)	d
(26)	d	(27)	d	(28)	d	(29)	d	(30)	a
(31)	b	(32)	c	(33)	d				

SOLUTION

Q.2 As

$$\frac{1}{x^2 - 1} = \frac{1}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

Q.3 As degree of numerator is greater than the degree of denominator so it is an improper fraction.

Q.4 As,

$$\frac{1}{x^3 + 1} = \frac{1}{(x + 1)(x^2 - x + 1)}$$

So let,

$$\frac{1}{(x + 1)(x^2 - x + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$$

Q.6
$$\frac{x^3 + 1}{(x - 1)(x + 2)} = \frac{x^3 + 1}{x^2 + x - 2}$$

As degree of the polynomial in numerator is greater than the degree of the polynomial in the denominator so it is improper fraction.

Q.7
$$\frac{7x^2 + 25}{(x + 3)(x + 4)} = \frac{7x^2 + 25}{x^2 + 7x + 12}$$

$$\begin{array}{r} 7 \\ x^2 + 7x + 12 \overline{) 7x^2 + 25} \\ \underline{- 7x^2 + 84 + 49x} \\ -49x - 59 \end{array}$$

$$\begin{aligned} \Rightarrow \frac{7x^2 + 25}{(x + 3)(x + 4)} &= 7 + \frac{-49x - 59}{(x + 3)(x + 4)} \\ &= 7 + \frac{A}{x + 3} + \frac{B}{x + 4} \end{aligned}$$

Q.9
$$\frac{1}{(x + 1)(x^2 - 1)} = \frac{1}{(x + 1)(x + 1)(x - 1)} = \frac{1}{(x - 1)(x + 1)^2}$$

We Let,

$$\frac{1}{(x - 1)(x + 1)^2} = \frac{A}{x - 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 1}$$

Q.10
$$\frac{x^2 + 1}{(x + 1)(x - 1)} = \frac{x^2 + 1}{x^2 - 1}$$

$$\frac{1}{x^2 - 1} \sqrt{x^2 + 1}$$

$$\frac{x^2 - 1}{-} +$$

$$2$$

$$\Rightarrow \frac{x^2 + 1}{x^2 - 1} = 1 + \frac{2}{x^2 - 1} = 1 + \frac{2}{(x + 1)(x - 1)}$$

Its P.F will be of the form.

$$1 + \frac{A}{x + 1} + \frac{B}{x - 1}$$

Q.14 As,

$$\frac{1}{x^3 - 1} = \frac{1}{(x - 1)(x^2 + x + 1)}$$

So, P.F of $\frac{1}{x^3 - 1}$ are of the form $\frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$

Q.24 As,

$$\frac{x^2 - 10x + 13}{(x - 1)(x^2 - 5x + 6)} = \frac{x^2 - 10x + 13}{(x - 1)(x - 2)(x - 3)}$$

$\Rightarrow$ P.F of $\frac{x^2 - 10x + 13}{(x - 1)(x - 2)(x - 3)}$ are of the form $\frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$

Q.25 As,

$$x^4 + 2x^2 + 9 = (x^2 + 2x + 3)(x^2 - 2x + 3)$$

$$\Rightarrow \frac{4x^2 + 8x}{(x^2 + 2x + 3)(x^2 - 2x + 3)}$$

Q.26 As,

$$\frac{x - b}{x^3 - b^3} = \frac{x - b}{(x - b)(x^2 + bx + b^2)} = \frac{1}{x^2 + bx + b^2}$$

Q.28 As it is true for all values of 'θ' so it is an identity.