

Chapter 12

APPLICATION OF TRIGONOMETRY

EXERCISE 12.1

Q.1 Find the values of

(i) $\sin 53^\circ 40'$
 $= 0.8056$

(ii) $\cos 36^\circ 20'$
 $= 0.8056$

(iii) $\tan 19^\circ 30'$
 $= 0.3541$

(iv) $\cot 33^\circ 50'$
 $= \frac{1}{\tan 33^\circ 50'} = 1.4919$

(v) $\cos 42^\circ 38'$
 $= 0.7357$

(vi) $\tan 25^\circ 34'$
 $= 0.4784$

(vii) $\sin 18^\circ 31'$
 $= 0.3176$

(viii) $\cos 52^\circ 13'$
 $= 0.6127$

(ix) $\cot 89^\circ 9'$
 $= \frac{1}{\tan 89^\circ 9'} = 0.01483$

Q.2 Find θ , if

(i) $\sin \theta = 0.5791$

$\Rightarrow \theta = \sin^{-1}(0.5791) = 35^\circ 23'$

(ii) $\cos \theta = 0.9316$

$\theta = \cos^{-1}(0.9316) = 21^\circ 18'$

(iii) $\cos \theta = 0.5257$

$\theta = \cos^{-1}(0.5257) = 58^\circ 17'$

(iv) $\tan \theta = 1.705$

$\theta = \tan^{-1}(1.705) = 59^\circ 36'$

(v) $\tan \theta = 21.943$

$\theta = \tan^{-1}(21.943) = 87^\circ 23'$

(vi) $\sin \theta = 0.5186$

$\theta = \sin^{-1}(0.5186) = 31^\circ 14'$

EXERCISE 12.2

Q.1 Find the unknown angles and sides of the following triangles.

(i) $\alpha = 45^\circ$, $\beta = 90^\circ$, $\gamma = m \angle B$

(ii) $\alpha = 60^\circ$, $\beta = 90^\circ$, $\gamma = m \angle c$

(iii) $\alpha = 90^\circ$, $\beta = m \angle B$, $\gamma = m \angle c$

Solution:

(i) $\alpha = 45^\circ$, $\beta = 90^\circ$, $\gamma = m \angle B$

since α, β, γ are angles of triangle

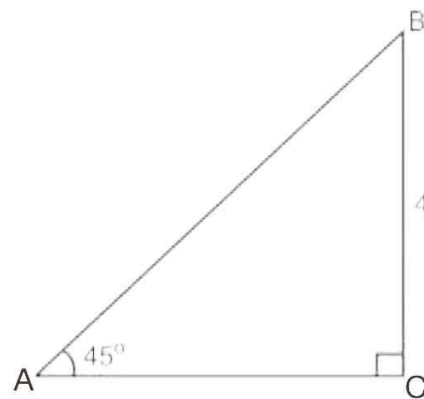
so $\alpha + \beta + \gamma = 180^\circ$

$45^\circ + 90^\circ + m \angle B = 180^\circ$

$m \angle B = 45^\circ$

$\sin 45^\circ = \frac{BC}{AB}$

$\sin 45^\circ = \frac{4}{AB}$



$$\frac{1}{\sqrt{2}} = \frac{4}{AB} \Rightarrow \boxed{AB = 4\sqrt{2}}$$

$$\cos 45^\circ = \frac{AC}{AB}$$

$$\frac{1}{\sqrt{2}} = \frac{AC}{4\sqrt{2}} \Rightarrow \boxed{AC = 4}$$

(ii) $\alpha = 60^\circ, \beta = 90^\circ, \gamma = m\angle c$

$$\text{since } \alpha + \beta + \gamma = 180^\circ$$

$$60^\circ + 90^\circ + \gamma = 180^\circ$$

$$\gamma = 30^\circ$$

$$\sin 60^\circ = \frac{BC}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{BC}{12} \Rightarrow BC = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$$

$$\cos 60^\circ = \frac{AB}{AC}$$

$$\frac{1}{2} = \frac{AB}{6\sqrt{3}} \Rightarrow AB = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$

(iii) $\alpha = 90^\circ, \beta = m\angle B, \gamma = m\angle c$

$$\cos \beta = \frac{AB}{BC}$$

$$\cos \beta = \frac{5}{10}$$

$$\beta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

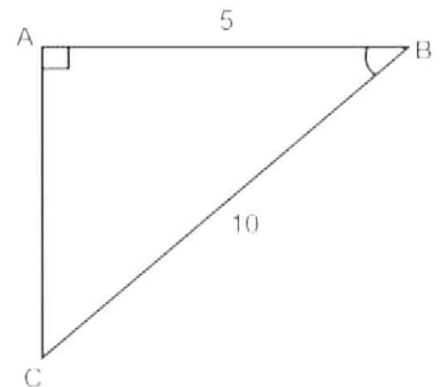
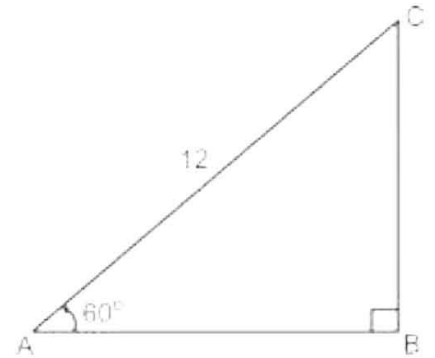
$$\sin 60^\circ = \frac{AC}{BC}$$

$$\sin 60^\circ = \frac{AC}{10}$$

$$\boxed{AC = 8.66}$$

$$\alpha + \beta + \gamma = 180^\circ$$

$$90^\circ + 60^\circ + \gamma = 180^\circ$$



$$\boxed{\gamma = 30^\circ}$$

(iv) $\alpha + \beta + \gamma = 180^\circ$
 $90^\circ + 40^\circ + \gamma = 180^\circ$

$$\boxed{\gamma = 50^\circ}$$

$$\sin 40^\circ = \frac{AC}{BC}$$

$$\sin 40^\circ = \frac{8}{BC}$$

$$\Rightarrow \boxed{BC = 12.45}$$

$$\tan 40^\circ = \frac{AC}{AB} = \frac{8}{AB}$$

$$\boxed{AB = 9.53}$$

(v) $\alpha + \beta + \gamma = 180^\circ$
 $90^\circ + \beta + 56^\circ = 180^\circ$

$$\boxed{\beta = 34^\circ}$$

$$\sin 56^\circ = \frac{AB}{BC}$$

$$\sin 56^\circ = \frac{AB}{15}$$

$$\Rightarrow \boxed{AB = 12.44}$$

$$\cos 56^\circ = \frac{AC}{BC}$$

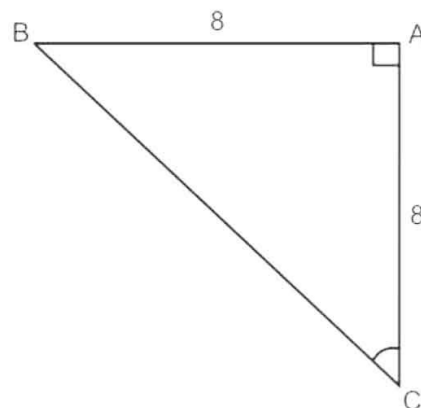
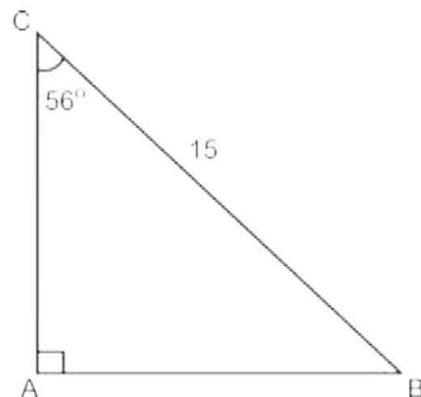
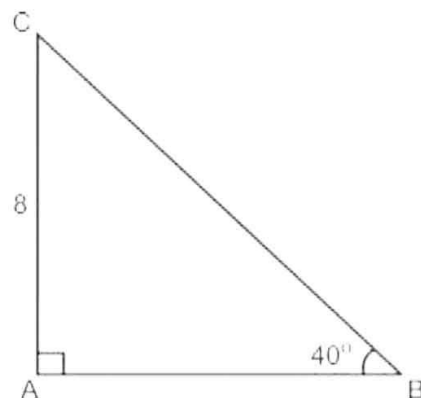
$$\cos 56^\circ = \frac{AC}{15}$$

$$\boxed{AC = 8.39}$$

$$\tan m \angle B = \frac{AC}{AB}$$

$$\tan m \angle B = \frac{8}{8}$$

$$m \angle B = \tan^{-1}(1)$$



$$\boxed{m \angle B = 45^\circ}$$

$$m \angle A + m \angle B + m \angle C = 180^\circ$$

$$45^\circ + 90^\circ + m \angle C = 180^\circ$$

$$\boxed{m \angle C = 45^\circ}$$

$$\sin 45^\circ = \frac{AC}{AB}$$

$$\frac{1}{\sqrt{2}} = \frac{8}{BC}$$

$$\boxed{BC = 8\sqrt{2}}$$

Q.2 Solve the right triangle ABC, in which $\gamma = 90^\circ$ $\alpha = 37^\circ 20'$, $a = 243$

Solution:

$$\text{Since } \alpha + \beta + \gamma = 180^\circ$$

$$37^\circ 20' + \beta + 90^\circ = 180^\circ$$

$$\boxed{\beta = 52^\circ 40'}$$

$$\sin 37^\circ 20' = \frac{BC}{AB}$$

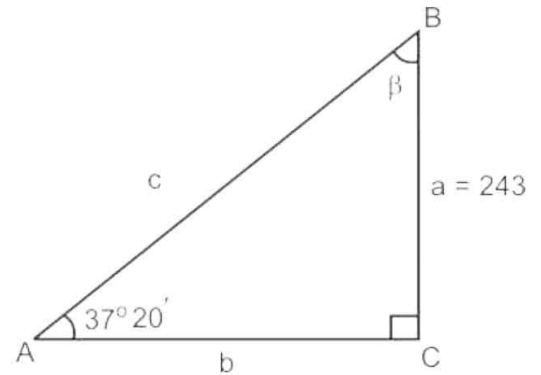
$$\sin 37^\circ 20' = \frac{243}{c}$$

$$\Rightarrow \boxed{c = 400}$$

$$\cos 37^\circ 20' = \frac{b}{c}$$

$$\cos 37^\circ 20' = \frac{b}{400}$$

$$\boxed{b = 318.04}$$



Q.3 $\gamma = 90^\circ$, $\alpha = 62^\circ 40'$, $b = 796$

Solution:

$$\text{Since } \alpha + \beta + \gamma = 180^\circ$$

$$62^\circ 40' + \beta + 90^\circ = 180^\circ$$

$$\boxed{\beta = 27^\circ 20'}$$

$$\tan 62^{\circ} 40' = \frac{a}{b}$$

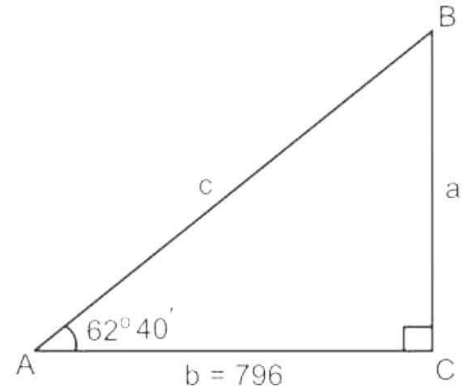
$$\tan 62^{\circ} 40' = \frac{a}{796}$$

$$\Rightarrow \boxed{a = 1540}$$

$$\sin 62^{\circ} 40' = \frac{a}{c}$$

$$\sin 62^{\circ} 40' = \frac{1540}{c}$$

$$\Rightarrow \boxed{c = 1733.55}$$



Q.4 $\gamma = 90^{\circ}$, $a = 3.28$, $b = 5.74$

Solution:

$$\tan \alpha = \frac{a}{b}$$

$$\tan \alpha = \frac{3.28}{5.74}$$

$$\alpha = \tan^{-1} \left(\frac{3.28}{5.74} \right)$$

$$\Rightarrow \boxed{\alpha = 29^{\circ} 44'}$$

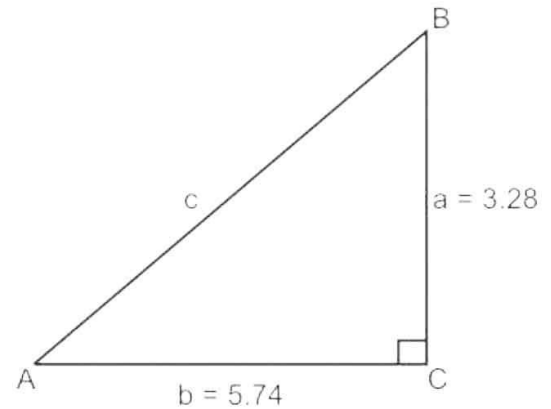
$$\text{Since } \alpha + \beta + \gamma = 180^{\circ}$$

$$29^{\circ} 44' + \beta + 90^{\circ} = 180^{\circ}$$

$$\boxed{\beta = 60^{\circ} 16'}$$

$$\sin 29^{\circ} 44' = \frac{3.28}{c}$$

$$\boxed{c = 6.61}$$



Q.5 $\gamma = 90^{\circ}$, $b = 68.4$, $c = 96.2$

Solution:

$$\cos \alpha = \frac{b}{c}$$

$$\cos \alpha = \frac{68.4}{96.2}$$

$$\alpha = \cos^{-1} \left(\frac{68.4}{96.2} \right)$$

$$\Rightarrow \boxed{\alpha = 44^{\circ} 40'}$$

$$\text{Since } \alpha + \beta + \gamma = 180^{\circ}$$

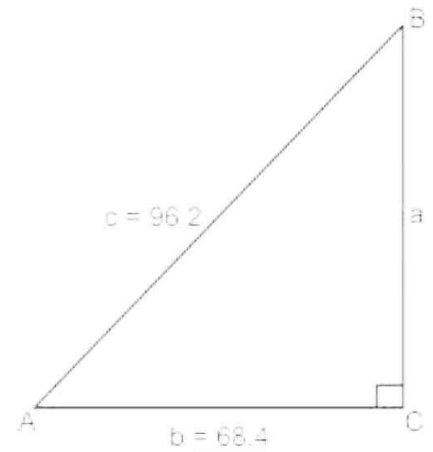
$$44^{\circ} 40' + \beta + 90^{\circ} = 180^{\circ}$$

$$\boxed{\beta = 45^{\circ} 20'}$$

$$\sin \alpha = \frac{a}{c}$$

$$\sin 44^{\circ} 40' = \frac{a}{96.2}$$

$$\boxed{a = 67.6}$$



Q.6 $\gamma = 90^{\circ}$, $a = 5429$, $c = 6294$

Solution:

$$\sin \alpha = \frac{a}{c}$$

$$\sin \alpha = \frac{5429}{6294}$$

$$\alpha = \sin^{-1} \left(\frac{5429}{6294} \right)$$

$$\Rightarrow \boxed{\alpha = 59^{\circ} 36'}$$

$$\text{Since } \alpha + \beta + \gamma = 180^{\circ}$$

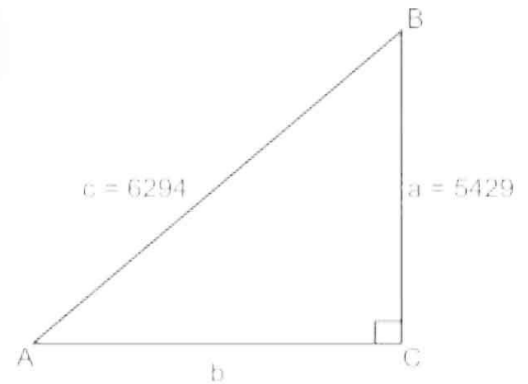
$$59^{\circ} 36' + \beta + 90^{\circ} = 180^{\circ}$$

$$\boxed{\beta = 30^{\circ} 24'}$$

$$\cos \alpha = \frac{b}{c}$$

$$\cos (59^{\circ} 36') = \frac{b}{6294}$$

$$\boxed{b = 3184}$$



Q.7 $\gamma = 90^\circ$, $\beta = 50^\circ 10'$, $c = 0.832$

Solution:

Since $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + 50^\circ 10' + 90^\circ = 180^\circ$$

$$\boxed{\alpha = 39^\circ 50'}$$

$$\sin \alpha = \frac{a}{c}$$

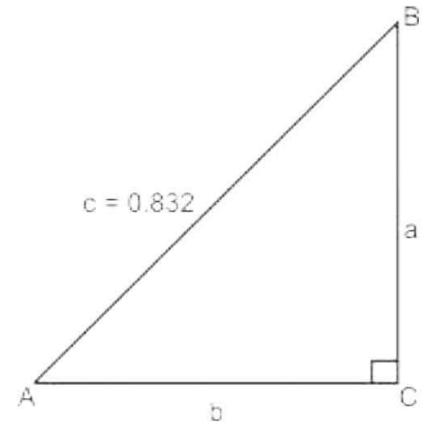
$$\sin 39^\circ 50' = \frac{a}{0.832}$$

$$\Rightarrow \boxed{a = 0.5329}$$

$$\cos \alpha = \frac{a}{b}$$

$$\cos 39^\circ 50' = \frac{0.5329}{b}$$

$$\boxed{c = 0.6939}$$



EXERCISE 12.3

Q.1 A vertical pole is 8m high and the length of its shadow is 6m. What is the angle of elevation of the sun at that moment?

Solution:

Let the required angle be $= \theta$

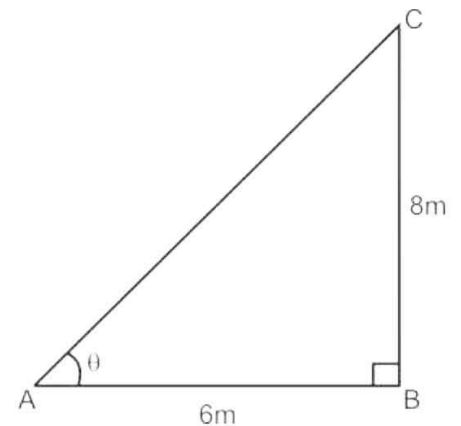
We know that

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan \theta = \frac{8}{6}$$

$$\theta = \tan^{-1} \left(\frac{8}{6} \right)$$

$$\theta = 53^\circ 7'$$



Q.2 A man 18dm tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is 32° . What is the height of the tree?

Solution:

Let the required height of the tree be $= x + 18$

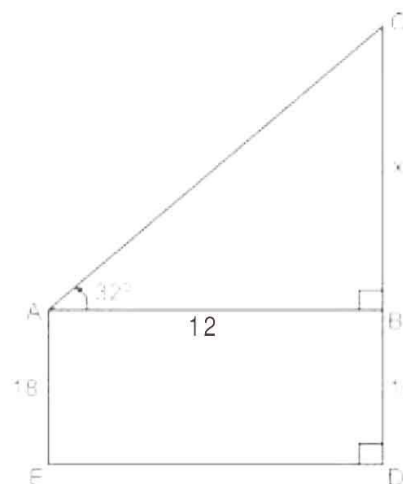
we know that

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 32^\circ = \frac{x}{12}$$

$$x = 7.50 \text{ m}$$

$$\begin{aligned} \text{Required height} &= x + 18 \\ &= 7.50 + 1.8 \text{ m} \quad (18\text{dm} = 1.8\text{m}) \\ &= 9.3\text{m} \end{aligned}$$



Q.3 At the top of a cliff 80m high, the angle of depression of a boat is 12° . How far is the boat from the cliff?

Solution:

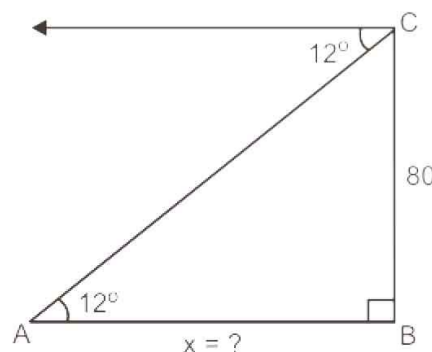
Let the boat from the cliff be $= x$

we know that

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 12^\circ = \frac{80}{x}$$

$$x = 376.37 \text{ m}$$



Q.4 A ladder leaning against a vertical wall makes an angle of 24° with the wall. Its foot is 5m from the wall. Find its length.

Solution:

Let length of ladder be $= l$

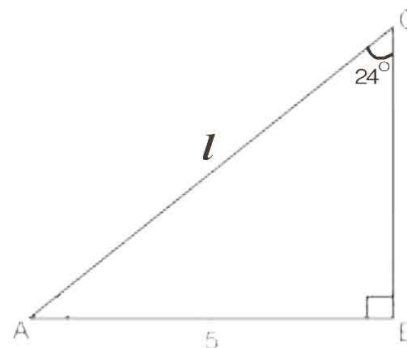
we know that

$$\sin \theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\sin 24^\circ = \frac{5}{l}$$

$$l = \frac{5}{\sin 24^\circ}$$

$$l = 12.30\text{m}$$



Q.5 A kite is flying at a height of 67.2m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.

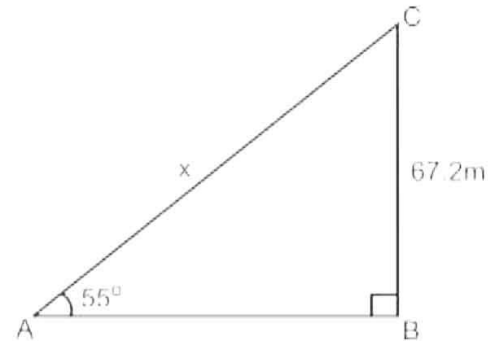
Solution:

Let the string of the kite be = x
we know that

$$\sin \theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\sin 55^\circ = \frac{67.2}{x}$$

$$x = 82 \text{ m}$$



Q.6 When the angle between the ground and the sun is 30° , flag pole casts a shadow of 40m long. Find the height of the top of flag. (Lahore Board 2008)

Solution:

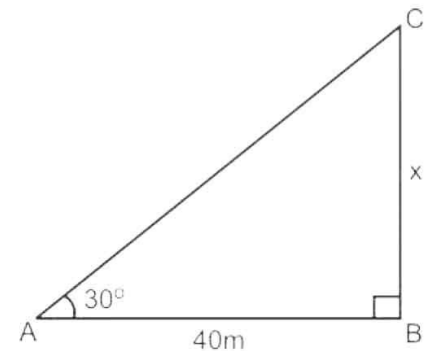
Let the height of the top of the flag be = x

we know that

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 30^\circ = \frac{x}{40}$$

$$x = 23.09 \text{ m}$$



Q.7 A plane flying directly above a post 6000m away from an anti-aircraft observes the gun at an angle of depression of 27° . Find the height of the plane.

Solution:

Let the height of plane be = x

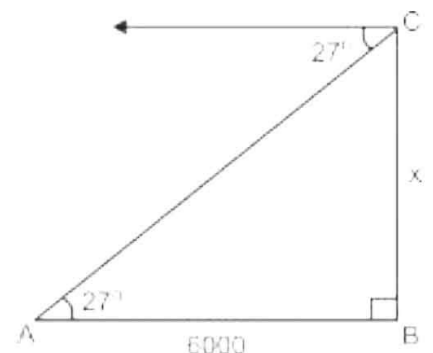
we know that

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan \theta = \frac{x}{6000}$$

$$\tan 27^\circ = \frac{x}{6000}$$

$$x = 3057.15 \text{ m}$$



Q.8 A man on the top of a 100m high light house tower is in line with two ships on the same side of it, whose angle of depression from the man are 17° and 19° respectively. Find the distance between the ships.

Solution:

Let the distance between the ships be $= x$

First consider $\triangle DAC$

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 17^\circ = \frac{100}{b}$$

$$\Rightarrow b = 327.08 \text{ m}$$

now we consider $\triangle DBC$

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 19^\circ = \frac{100}{b - x}$$

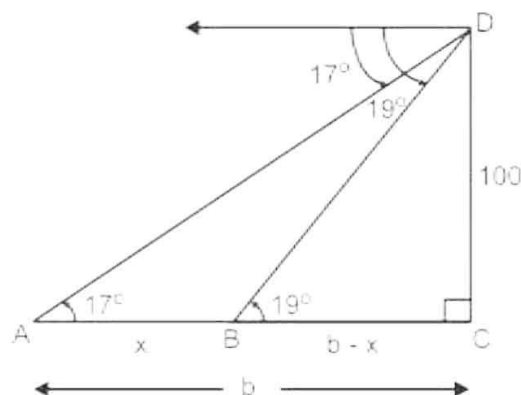
$$0.344 = \frac{100}{327 - x}$$

$$112.59 - 0.344x = 100$$

$$12.59 = 0.344x$$

$$\frac{12.59}{0.344} = x$$

$$x = 36.58 \text{ m}$$



Q.9 P and Q are two points in line with a tree. If the distance between P and Q be 30m and the angle of elevation of the top of the tree at P and Q be 12° and 15° respectively. Find the height of tree.

Solution:

Let the height of the tree be $= h$

First consider $\triangle SQR$

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

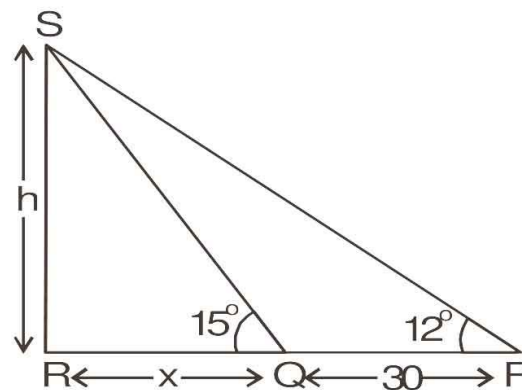
$$\tan 15^\circ = \frac{h}{x}$$

$$\Rightarrow \boxed{h = x \tan 15^\circ} \quad \dots\dots\dots (1)$$

Now we consider $\triangle SPR$

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 12^\circ = \frac{h}{x + 30}$$



$$\Rightarrow \boxed{h = (x + 30) \tan 12^\circ} \dots\dots\dots (2)$$

Using eq. (1) and (2).

$$x \tan 15^\circ = x \tan 12^\circ + 30 \tan 12^\circ$$

$$x (\tan 15^\circ - \tan 12^\circ) = 30 \tan 12^\circ$$

$$0.055x = 6.38$$

$$\boxed{x = 115.94} \text{ put in eq. (1)}$$

$$h = 115.94 \tan 15^\circ$$

$$h = 31.066 \text{ m}$$

Q.10 Two men are on the opposite sides of a 100m high tower. If the measure of the angles of the angles of elevation of the top of the tower are 18° & 22° respectively. Find the distance between them.

Solution:

Let the distance between the two men = $x + y$

Consider $\triangle CAD$

$$\tan 18^\circ = \frac{100}{x}$$

$$x = \frac{100}{\tan 18^\circ}$$

$$x = 307.77$$

Consider $\triangle CBD$

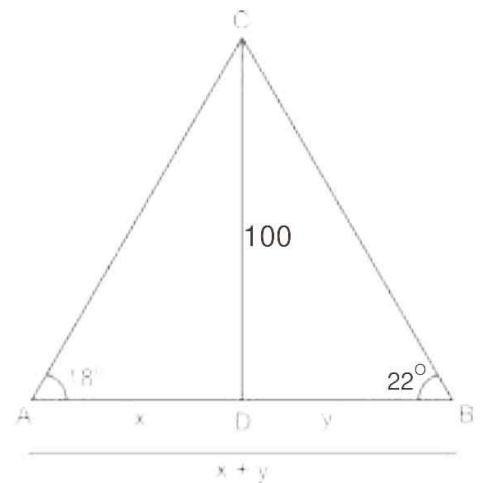
$$\tan 22^\circ = \frac{100}{y}$$

$$y = \frac{100}{\tan 22^\circ}$$

$$y = 247.51$$

Required distance between them = $x + y$

$$= 307.77 + 247.51 = 555.28 \text{ m}$$



Q.11 A man standing 60m away from a tower notices that the angle of elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find the length of the flag staff.

Solution:

Let the length of the flag staff be = y

First consider $\triangle BMA$

$$\tan 62^\circ = \frac{x}{60}$$

$$x = 112.84$$

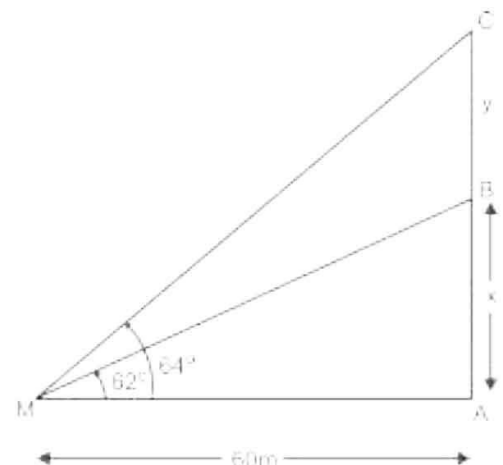
Now consider $\triangle CMA$

$$\tan 64^\circ = \frac{x + y}{60}$$

$$60^\circ \times \tan 64^\circ = 112.84 + y$$

$$123 = x + 112.84$$

$$x = 10.2 \text{ m}$$



Q.12 The angle of elevation of the top of 60m high tower from a point A, on the same level as the foot of the tower is 25° . Find the angle of elevation of the top of the tower from a point B, 20m nearer to A from the foot of the tower.

Solution:

Let the required angle of elevation be $= \theta$

First consider $\triangle DAC$

$$\tan 25^\circ = \frac{60}{x + 20} \Rightarrow x + 20 = \frac{60}{\tan 25^\circ}$$

$$x = 128.67 - 20$$

$$x = 108.67$$

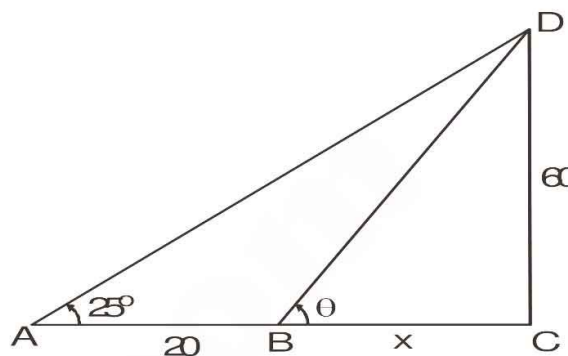
Now we consider $\triangle DBC$

$$\tan \theta = \frac{60}{x}$$

$$\tan \theta = \frac{60}{108.67}$$

$$\tan \theta = \frac{60}{108.67}$$

$$\theta = \tan^{-1} \left(\frac{60}{108.67} \right) = 28^\circ 54'$$



Q.13 Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20° . the angle of elevation from the base of the building B to the top of the building A is 50° . Find the height of the building B.

Solution:

Let the required height of building B be $= x + y$

We know that $\tan \theta = \frac{\text{perpendicular}}{\text{base}}$

$$\tan 20^\circ = \frac{y}{100}$$

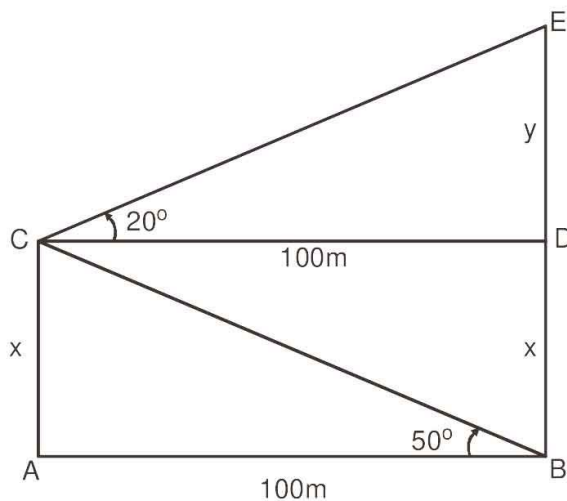
$$y = 36.39$$

Now consider $\triangle ABC$

$$\tan 50^\circ = \frac{x}{100}$$

$$x = 119.2$$

$$\begin{aligned} \text{required height of building B} &= x + y \\ &= 119.2 + 36.39 \\ &= 115.6 \end{aligned}$$



Q.14 A window washer is working in a hotel building at a distance of 20m from the building finds angle of elevation of the worker to be of 30° . The worker climbs up 12m and the observer moves 4m further away from the building. Find the new angle of elevation of the worker.

Solution:

Let the required angle of elevation = θ

First consider $\triangle ABC$

$$\tan 30^\circ = \frac{x}{20}$$

$$\Rightarrow x = 11.55\text{m}$$

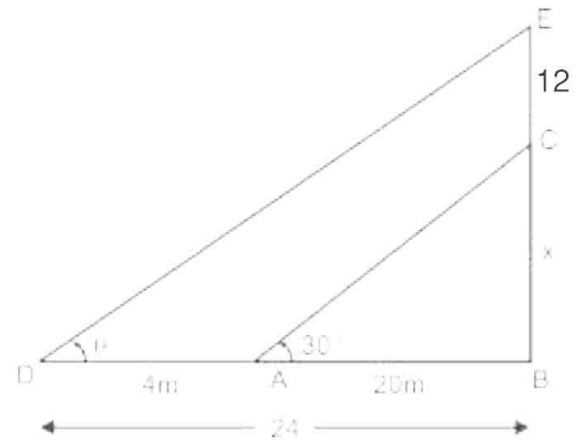
Now we consider $\triangle DBE$

$$\tan \theta = \frac{12 + x}{24}$$

$$\tan \theta = \frac{12 + 11.55}{24}$$

$$\theta = \tan^{-1}(0.98125)$$

$$\theta = 44^\circ 27'$$



Q.15 A man standing on a bank of canal observes that the measure of the angle of elevation of a tree on the other side of the canal is 60° . On retreating 40m from the bank, he finds the measure of elevation of tree is 30° . find the height of the tree and width of the canal.

Solution:

Let height of tree = h & width of canal = x

First consider $\triangle BAC$

$$\tan 30^\circ = \frac{h}{x + 40}$$

$$h = (x + 40) \tan 30^\circ \quad \dots\dots\dots (1)$$

Now we consider $\triangle BDC$

$$\tan 60^\circ = \frac{h}{x} \Rightarrow h = x \tan 60^\circ \quad \dots\dots\dots (2)$$

From eq. (1) and (2).

$$(x + 40) \tan 30^\circ = x \tan 60^\circ$$

$$x \tan 30^\circ + 40 \tan 30^\circ = x \tan 60^\circ$$

$$x \tan 60^\circ - x \tan 30^\circ = 40 \tan 30^\circ$$

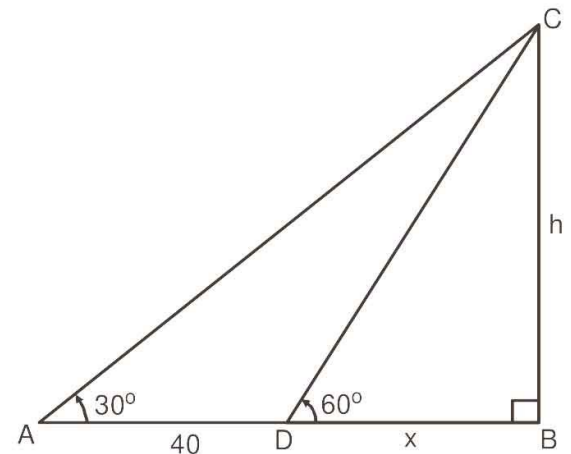
$$x(\tan 60^\circ - \tan 30^\circ) = 40 \tan 30^\circ$$

$$1.15x = 23.09$$

$$\boxed{x = 20} \text{ put in (2).}$$

$$h = 20 \tan 60^\circ$$

$$\boxed{h = 34.64}$$



OBLIQUE TRIANGLE:

A triangle, which is not right, is called an oblique triangle.

For the solution of oblique triangles we can use law of sines, law of cosines, law of tangents.

1. The Law of Cosine

In any triangle ABC, with usual notations

$$(i) \quad a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$(ii) \quad b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$(iii) \quad c^2 = a^2 + b^2 - 2ab \cos \gamma$$

2. Law of Sines

In any triangle ABC, with usual notations

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

3. The Law of Tangents

In any triangle ABC, with usual notations

$$(i) \quad \frac{a-b}{a+b} = \frac{\tan \frac{\alpha-\beta}{2}}{\tan \frac{\alpha+\beta}{2}} \qquad (ii) \quad \frac{b-c}{b+c} = \frac{\tan \frac{\beta-\gamma}{2}}{\tan \frac{\beta+\gamma}{2}}$$

$$(iii) \quad \frac{c-a}{c+a} = \frac{\tan \frac{\gamma-\alpha}{2}}{\tan \frac{\gamma+\alpha}{2}}$$

4. Half angle Formulas

In any triangle ABC, with usual notation

$$(i) \quad \sin \frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$$

$$(ii) \quad \sin \frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{ac}}$$

$$(iii) \quad \sin \frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$(i) \quad \cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$(ii) \quad \cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$(iii) \quad \cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$(i) \quad \tan \frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

$$(ii) \quad \tan \frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$$

$$(iii) \quad \tan \frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

EXERCISE 12.4

Q.1 Solve triangle ABC if

$$\beta = 60^\circ, \quad \gamma = 15^\circ, \quad b = \sqrt{6}$$

(Gujranwala Board 2007)

Solution:

$$\text{Since } \alpha + \beta + \gamma = 180^\circ$$

$$\alpha + 60^\circ + 15^\circ = 180^\circ$$

$$\alpha = 180^\circ - 60^\circ - 15^\circ$$

$$\boxed{\alpha = 105}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\frac{a}{\sin 105^\circ} = \frac{\sqrt{6}}{\sin 60^\circ} \Rightarrow a = \frac{\sqrt{6}}{\sin 60^\circ} \times \sin 105^\circ$$

$$\boxed{a = 2.732}$$

again

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\frac{c}{\sin 15^\circ} = \frac{\sqrt{6}}{\sin 60^\circ} \Rightarrow c = \frac{\sqrt{6}}{\sin 60^\circ} \times \sin 15^\circ$$

$$\boxed{c = 0.7320}$$

Q.2 $\beta = 52^\circ$, $\gamma = 89^\circ 35'$, $a = 89.35$

Solution:

$$\text{Since } \alpha + \beta + \gamma = 180^\circ$$

$$\alpha + 52^\circ + 89^\circ 35' = 180^\circ$$

$$\alpha = 180^\circ - 52^\circ - 89^\circ 35'$$

$$\boxed{\alpha = 38^\circ 25'}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\Rightarrow \frac{89.35}{\sin 38^{\circ} 25'} = \frac{b}{\sin 52^{\circ}}$$

$$b = \frac{89.35}{\sin 38^{\circ} 25'} \times \sin 52^{\circ}$$

$$\boxed{b = 113.18}$$

again

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha}$$

$$\frac{c}{\sin 89^{\circ} 35'} = \frac{89.35}{\sin 38^{\circ} 25'}$$

$$c = \frac{89.35}{\sin 38^{\circ} 25'} \times \sin 89^{\circ} 35'$$

$$\boxed{c = 143.79}$$

Q.3 $b = 125$, $\gamma = 53^{\circ}$, $\alpha = 47^{\circ}$

Solution:

$$\text{Since } \alpha + \beta + \gamma = 180^{\circ}$$

$$47^{\circ} + \beta + 35^{\circ} = 180^{\circ}$$

$$\beta = 180^{\circ} - 53^{\circ} - 47^{\circ}$$

$$\boxed{\beta = 80^{\circ}}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\frac{a}{\sin 47^{\circ}} = \frac{125}{\sin 80^{\circ}}$$

$$a = \frac{125}{\sin 80^{\circ}} \times \sin 47^{\circ}$$

$$\boxed{a = 92.8}$$

and

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\frac{c}{\sin 53^{\circ}} = \frac{125}{\sin 80^{\circ}}$$

$$c = \frac{125}{\sin 80^{\circ}} \times \sin 53^{\circ}$$

$$\boxed{c = 101}$$

Q.4 $c = 16.1$, $\alpha = 53^\circ 45'$, $\gamma = 74^\circ 32'$

Solution:

Since $\alpha + \beta + \gamma = 180^\circ$

$$42^\circ 45' + \beta + 74^\circ 32' = 180^\circ$$

$$\beta = 180^\circ - 42^\circ 45' - 74^\circ 32'$$

$$\boxed{\beta = 62^\circ 43'}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$\frac{a}{\sin 42^\circ 45'} = \frac{16.1}{\sin 74^\circ 32'}$$

$$a = \frac{16.1}{\sin 74^\circ 32'} \times \sin 42^\circ 45'$$

$$\boxed{a = 11.3}$$

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\frac{b}{\sin 62^\circ 43'} = \frac{16.1}{\sin 74^\circ 32'}$$

$$b = \frac{16.1}{\sin 74^\circ 32'} \times \sin 62^\circ 43'$$

$$\boxed{b = 14.8}$$

Q.5 $a = 53$, $\beta = 88^\circ 36'$, $\gamma = 31^\circ 54'$

(Gujranwala Board 2006, Lahore Board 2007)

Solution:

Since $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + 88^\circ 36' + 31^\circ 54' = 180^\circ$$

$$\alpha = 180^\circ - 88^\circ 36' - 31^\circ 54'$$

$$\boxed{\alpha = 59^\circ 30'}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\frac{53}{\sin 59^\circ 30'} = \frac{b}{\sin 88^\circ 36'}$$

$$b = \frac{53}{\sin 59^\circ 30'} \times \sin 88^\circ 36'$$

$$\boxed{b = 61.49}$$

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\frac{61.49}{\sin 88^\circ 36'} = \frac{c}{\sin 31^\circ 54'}$$

$$c = \frac{61.49}{\sin 88^\circ 36'} \times \sin 31^\circ 54'$$

$$\boxed{c = 32.5}$$

EXERCISE 12.5

Solve the triangle ABC, in which

Q.1 $b = 59$, $c = 34$, and $\alpha = 52^\circ$ (Gujranwala Board 2007)

Solution:

Using law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\begin{aligned} a^2 &= (95)^2 + (34)^2 - 2(95)(34) \cos 52^\circ \\ &= 9025 + 1156 - 3977 \end{aligned}$$

$$a^2 = 6204$$

$$\boxed{a = 78.76}$$

$$\therefore \cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Now } \beta = \cos^{-1} \left[\frac{a^2 + c^2 - b^2}{2ac} \right]$$

$$\beta = \cos^{-1} \left[\frac{(78.76)^2 + (34)^2 - (95)^2}{2(78.76)(34)} \right]$$

$$\boxed{\beta = 71^\circ 53'}$$

$$\begin{aligned} \text{so } \gamma &= 180^\circ - \beta - \alpha \\ &= 180^\circ - 71^\circ 53' - 52^\circ \end{aligned}$$

$$\boxed{\gamma = 56^\circ 7'}$$

$$\begin{aligned} \therefore b^2 &= a^2 + c^2 - 2ac \cos \beta \\ 2ac \cos \beta &= a^2 + c^2 - b^2 \\ \cos \beta &= \frac{a^2 + c^2 - b^2}{2ac} \end{aligned}$$

Q.2 $b = 12.5$, $c = 23$, & $\alpha = 38^\circ 20'$

Solution:

By law of cosines

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos \alpha \\&= (12.5)^2 + (23)^2 - 2 (12.5) (23) \cos 38^\circ 20'\end{aligned}$$

$$a^2 = 234.21$$

$$\boxed{a = 15.3}$$

$$\therefore \cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Now } \beta = \cos^{-1} \left[\frac{a^2 + c^2 - b^2}{2ac} \right]$$

$$\cos^{-1} \left[\frac{(15.3)^2 + (23)^2 - (12.5)^2}{2 (15.3) (23)} \right]$$

$$\boxed{\beta = 30^\circ 26'}$$

$$\begin{aligned}\text{so } \gamma &= 180^\circ - \alpha - \beta \\&= 180^\circ - 38^\circ 20' - 30^\circ 26'\end{aligned}$$

$$\boxed{\gamma = 111^\circ 14'}$$

Q.3 $a = \sqrt{3} - 1$, $b = \sqrt{3} + 1$, $\gamma = 60^\circ$

Solution:

$$a = \sqrt{3} - 1 = 0.7320$$

$$b = \sqrt{3} + 1 = 2.7320$$

By law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$c^2 = (0.7320)^2 + (2.7320)^2 - 2 (0.7320) (2.7320) \cos 60^\circ$$

$$c^2 = 6$$

$$\boxed{c = \sqrt{6}}$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\begin{aligned}\alpha &= \cos^{-1} \left[\frac{b^2 + c^2 - a^2}{2bc} \right] \\&= \cos^{-1} \left[\frac{(2.7320)^2 + (6)^2 - (0.7320)^2}{2 (2.7320) (\sqrt{6})} \right]\end{aligned}$$

$$\boxed{\alpha = 15^\circ}$$

$$\begin{aligned}\text{since } \alpha + \beta + \gamma &= 180^\circ \\ \beta &= 180^\circ - 15^\circ - 60^\circ\end{aligned}$$

$$\boxed{\beta = 105^\circ}$$

$$\begin{aligned}\therefore a^2 &= b^2 + c^2 - 2ac \cos \alpha \\ 2ac \cos \alpha &= b^2 + c^2 - a^2 \\ \cos \alpha &= \frac{b^2 + c^2 - a^2}{2ac}\end{aligned}$$

Q.4 $\alpha = 3^\circ$, $c = 6$, $\beta = 36^\circ 20'$

Solution:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos \beta \\ &= (3)^2 + (6)^2 - 2(3)(6) \cos 36^\circ 20' \\ b^2 &= 16 \end{aligned}$$

$$\boxed{b = 4}$$

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\alpha = \cos^{-1} \left[\frac{b^2 + c^2 - a^2}{2bc} \right] \Rightarrow \cos^{-1} \left[\frac{16 + 36 - 9}{2(4)(6)} \right]$$

$$\boxed{\alpha = 26^\circ 23'}$$

$$\begin{aligned} \gamma &= 180^\circ - \alpha - \beta \\ &= 180^\circ - 26^\circ 23' - 36^\circ 20' \end{aligned}$$

$$\boxed{\gamma = 117^\circ 17'}$$

Q.5 $a = 7$, $b = 3$ and $\gamma = 38^\circ 13'$

Solution:

By law of cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos \gamma \\ c^2 &= (7)^2 + (3)^2 - 2(7)(3) \cos 38^\circ 13' \\ c^2 &= 25 \Rightarrow \boxed{c = 5} \end{aligned}$$

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Now $\alpha = \cos^{-1} \left[\frac{b^2 + c^2 - a^2}{2bc} \right]$

$$\alpha = \cos^{-1} \left[\frac{9 + 25 - 49}{2(3)(5)} \right]$$

$$\alpha = \cos^{-1} \left(\frac{-1}{2} \right)$$

$$\boxed{\alpha = 120^\circ}$$

$$\begin{aligned} \beta &= 180^\circ - \alpha - \gamma \\ &= 180^\circ - 120^\circ - 38^\circ 13' \end{aligned}$$

$$\boxed{\beta = 21^\circ 47'}$$

Solve the following triangles, using first law of tangents, and then law of sines.

Q.6 **a = 36.21** , **b = 42.09** , **$\gamma = 44^\circ 29'$**

Solution:

We know that $\alpha + \beta + \gamma = 180^\circ$

$$\begin{aligned}\beta + \alpha &= 180^\circ - \gamma \\ &= 180^\circ - 44^\circ 29'\end{aligned}$$

$$\beta + \alpha = 135^\circ 31' \quad \dots\dots\dots (1)$$

$$\frac{b - a}{b + a} = \frac{\tan\left(\frac{\beta - \alpha}{2}\right)}{\tan\left(\frac{\beta + \alpha}{2}\right)}$$

$$\frac{42.09 - 36.21}{42.09 + 36.21} = \frac{\tan\left(\frac{\beta - \alpha}{2}\right)}{\tan 67^\circ 45'}$$

$$\frac{5.88}{78.3} = \frac{\tan\left(\frac{\beta - \alpha}{2}\right)}{2.4443}$$

$$(0.0750)(2.4443) = \tan\left(\frac{\beta - \alpha}{2}\right)$$

$$\tan^{-1}(0.18355) = \frac{\beta - \alpha}{2}$$

$$2(10^\circ 24') = \beta - \alpha$$

$$\beta - \alpha = 20^\circ 48' \quad \dots\dots\dots (2)$$

Adding (1) to (2).

$$\beta + \alpha = 135^\circ 31'$$

$$\beta - \alpha = 20^\circ 48'$$

adding
$$2\beta = 156^\circ 19'$$

$$\boxed{\beta = 78^\circ 10'}$$

Put in (2)

$$78^\circ 10' - \alpha = 20^\circ 48'$$

$$78^\circ 10' - 20^\circ 48' = \alpha$$

$$\boxed{\alpha = 57^\circ 22'}$$

We find 'c' using law of sines

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$c = \frac{42.09}{\sin 78^\circ 10'} \times \sin 44^\circ 29'$$

$$\boxed{c = 30.13}$$

Q.7 $a = 93$, $c = 101$, $\beta = 80^\circ$

Solution:

We know that $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + \gamma = 180^\circ - \beta$$

$$\alpha + \gamma = 180^\circ - 80^\circ$$

$$\alpha + \gamma = 100^\circ$$

By law of tangents.

$$\frac{c - a}{c + a} = \frac{\tan\left(\frac{\gamma - \alpha}{2}\right)}{\tan\left(\frac{\gamma + \alpha}{2}\right)}$$

$$\frac{101 - 93}{101 + 93} = \frac{\tan\left(\frac{\gamma - \alpha}{2}\right)}{\tan\left(\frac{\gamma + \alpha}{2}\right)}$$

$$\frac{8}{194} = \frac{\tan\left(\frac{\gamma - \alpha}{2}\right)}{\tan\left(\frac{100^\circ}{2}\right)}$$

$$0.04124 = \frac{\tan\left(\frac{\gamma - \alpha}{2}\right)}{1.1918}$$

$$\tan\left(\frac{\gamma - \alpha}{2}\right) = (0.04124)(1.1918)$$

$$\frac{\gamma - \alpha}{2} = \tan^{-1}(0.04915)$$

$$\gamma - \alpha = 2(2^\circ 48')$$

$$\gamma - \alpha = 5^\circ 37'$$

$$\gamma - \alpha = 5^\circ 37'$$

By adding.

$$\alpha + \gamma = 100^\circ$$

$$-\alpha + \gamma = 5^\circ 37'$$

$$2\gamma = 105^\circ 37'$$

$$\boxed{\gamma = 52^\circ 49'} \quad \text{Put in (2)}$$

$$52^\circ 49' - \alpha = 5^\circ 37'$$

$$52^\circ 49' - 5^\circ 37' = \alpha$$

$$\boxed{\alpha = 47^\circ 11'}$$

To find 'b' using law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$b = \frac{93}{\sin 47^\circ 11'} \times \sin 80^\circ$$

$$\boxed{b = 125}$$

Q.8 $b = 14.8$, $c = 16.1$, $\alpha = 42^\circ 45'$

Solution:

$$\therefore \quad \alpha + \beta + \gamma = 180^\circ$$

$$42^\circ 45' + \beta + \gamma = 180^\circ$$

$$\beta + \gamma = 180^\circ - 42^\circ 45'$$

$$\beta + \gamma = 137^\circ 15' \quad \dots\dots\dots (1)$$

By law of tangents.

$$\frac{c-b}{c+b} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{\tan\left(\frac{\gamma+\beta}{2}\right)}$$

$$\frac{16.1-14.8}{16.1+14.8} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{\tan\left(\frac{137^\circ 15'}{2}\right)}$$

$$\frac{1.3}{30.9} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{2.555}$$

$$\tan\left(\frac{\gamma-\beta}{2}\right) = (0.0420)(2.555)$$

$$\frac{\gamma-\beta}{2} = \tan^{-1}(0.1075)$$

$$\gamma - \beta = 2(6^\circ 8')$$

$$\gamma - \beta = 12^\circ 16' \quad \dots\dots\dots (2)$$

Adding (1) to (2).

$$\beta + \gamma = 137^\circ 15'$$

$$\gamma - \beta = 12^\circ 16'$$

$$2\gamma = 149^\circ 31'$$

$$\boxed{\gamma = 74^\circ 45'}$$

$$\boxed{\beta = 62^\circ 29'}$$

To find 'α' using law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$a = \frac{b}{\sin \beta} \times \sin \alpha$$

$$= \frac{14.8}{\sin 62^\circ 29'} \times \sin 42^\circ 45'$$

$$\boxed{a = 11.33}$$

Q.9 **a = 319 , b = 168 , γ = 110° 22'**

Solution:

Since $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + \beta = 180^\circ - \gamma$$

$$\alpha + \beta = 180^\circ - 110^\circ 22'$$

$$\alpha + \beta = 69^\circ 38' \quad \dots\dots\dots (1)$$

By law of tangent.

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{319-168}{319+168} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{69^\circ 38'}{2}\right)}$$

$$\frac{151}{487} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{0.695}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3101) (0.695)$$

$$\frac{\alpha-\beta}{2} = \tan^{-1} (0.2155)$$

$$\alpha - \beta = 2(12^\circ 9')$$

$$\alpha - \beta = 24^\circ 20' \quad \dots\dots\dots (2)$$

Adding (1) and (2).

$$\alpha + \beta = 69^\circ 38'$$

$$\alpha - \beta = 24^\circ 20'$$

$$\hline 2\alpha = 93^\circ 58'$$

$$\boxed{\alpha = 46^\circ 58'} \quad \text{Put in (ii)}$$

$$\boxed{\beta = 22^\circ 39'}$$

to find 'c' using law of sines

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$c = \frac{319}{\sin 46^\circ 60'} \times \sin 110^\circ 22'$$

$$\boxed{c = 408.9}$$

Q.10 $b = 61$, $c = 32$, $\alpha = 59^\circ 30'$

Solution:

since $\alpha + \beta + \gamma = 180^\circ$

$$\beta + \gamma = 180^\circ - 59^\circ 30'$$

$$\beta + \gamma = 120^\circ 30' \quad \dots\dots\dots (1)$$

By law of tangent.

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{61-32}{61+32} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{120^\circ 30'}{2}\right)}$$

$$\frac{29}{93} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan(60^\circ 15')}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3118) (1.7496)$$

$$\frac{\beta-\gamma}{2} = \tan^{-1} (0.5455)$$

$$\beta - \gamma = 2(28^\circ 36')$$

$$\beta - \gamma = 57^\circ 14' \quad \dots\dots\dots (2)$$

Adding eq. (1) and eq. (2).

$$\beta + \gamma = 120^\circ 30'$$

Adding
$$\frac{\beta - \gamma = 57^\circ 14'}{2\beta = 177^\circ 44'}$$

$$\boxed{\beta = 88^\circ 51'} \quad \text{Put in (3)}$$

$$\boxed{\gamma = 31^\circ 38'}$$

To find 'a' we will use law of sines

$$\frac{a}{\sin \alpha} = \frac{B}{\sin \beta}$$

$$a = \frac{b}{\sin \beta} \times \sin \alpha$$

$$a = \frac{61}{\sin 88^\circ 52'} \times \sin 59^\circ 30'$$

$$\boxed{a = 53}$$

Q.11 Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure 57°. Find remaining two angles.

Solution:

Let a and b the two sides of a triangle are in the ratio 3 : 2 and include angle is $\gamma = 57^\circ$.

Then $a = 3$, $b = 2$, and $\gamma = 57^\circ$

since $\alpha + \beta + \gamma = 180^\circ$

$$\beta + \gamma = 180^\circ - \alpha$$

$$\alpha + \beta = 180^\circ - \gamma$$

$$\alpha + \beta = 123^\circ \quad \dots\dots\dots (1)$$

By law of tangent.

$$\frac{a - b}{a + b} = \frac{\tan\left(\frac{\alpha - \beta}{2}\right)}{\tan\left(\frac{\alpha + \beta}{2}\right)}$$

$$\frac{3 - 2}{3 + 2} = \frac{\tan\left(\frac{\alpha - \beta}{2}\right)}{\tan\left(\frac{123^\circ}{2}\right)}$$

$$\frac{1}{5} = \frac{\tan\left(\frac{\alpha - \beta}{2}\right)}{\tan(61^\circ 30')}$$

$$\tan\left(\frac{\alpha - \beta}{2}\right) = (0.2) (1.8418)$$

$$\frac{\alpha - \beta}{2} = \tan^{-1}(0.3684)$$

$$\alpha - \beta = 2(20^\circ 13')$$

$$\alpha - \beta = 24^\circ 27' \quad \dots\dots\dots (1)$$

Adding eq. (1) and (2).

$$\alpha + \beta = 123^\circ$$

adding
$$\frac{\alpha - \beta = 24^\circ 27'}{2\alpha = 163^\circ 27'}$$

$$\boxed{\alpha = 81^\circ 44'} \quad \text{Put in (2)}$$

$$\boxed{\beta = 41^\circ 16'}$$

Q.12 Two forces of 40N and 30N are represented by $\vec{AB}$ and $\vec{BC}$ which are inclined at an angle of $147^\circ 25'$. Find $\vec{AC}$, the resultant of $\vec{AB}$ & $\vec{BC}$.

Solution:

$$\text{Given } \vec{AB} = 40 \text{ N} \quad \therefore c = 40$$

$$\vec{BC} = 30 \text{ N} \quad \therefore a = 30$$

$$\text{Now } m \angle ABC = 147^\circ 25'$$

$$\beta = 147^\circ 25'$$

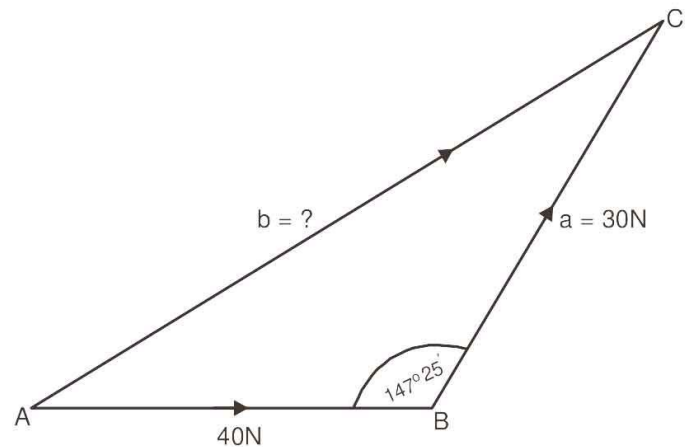
By law of cosines

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos \beta \\ &= (30)^2 + (40)^2 - 2(30)(40) \cos 147^\circ 25' \end{aligned}$$

$$b^2 = 4522.262$$

$$b = 67.25$$

$$\vec{AC} = 67.25 \text{ N}$$



EXERCISE 12.6

Q.1 Solve the following triangles, in which

$$a = 7, \quad b = 7, \quad c = 9$$

Solution:

$$S = \frac{a+b+c}{2} = \frac{7+7+9}{2} = \frac{23}{2} = 11.5$$

$$S - a = 11.5 - 7 = 4.5$$

$$S - b = 11.5 - 7 = 4.5$$

$$S - c = 11.5 - 9 = 2.5$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\cos \frac{\alpha}{2} = \sqrt{\frac{11.5(4.5)}{7 \times 9}}$$

$$\alpha = 2 \times \cos^{-1}(0.9063)$$

$$\boxed{\alpha = 50^\circ}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{11.5(4.5)}{7 \times 9}}$$

$$\beta = 2 \times \cos^{-1}(0.9063)$$

$$\boxed{\beta = 50^\circ}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{11.5 \times 2.5}{49}}$$

$$\gamma = 2 \times \cos^{-1}(0.7659)$$

$$\boxed{\gamma = 80^\circ}$$

Q.2 $a = 32, \quad b = 40, \quad c = 66$ (Lahore Board 2008)

Solution:

$$S = \frac{a+b+c}{2} = \frac{32+40+66}{2} = 69$$

$$S - a = 69 - 32 = 37$$

$$S - b = 69 - 40 = 29$$

$$S - c = 69 - 66 = 3$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{69 \times 37}{40 \times 66}}$$

$$\boxed{\alpha = 20^\circ 56'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{69 \times 29}{32 \times 66}}$$

$$\boxed{\beta = 26^\circ 30'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{69 \times 3}{32 \times 40}}$$

$$\boxed{\gamma = 132^\circ 34'}$$

Q.3 $a = 28.3$, $b = 31.7$, $c = 42.8$

Solution:

$$S = \frac{a+b+c}{2} = \frac{28.3+31.7+42.8}{2} = 51.4$$

$$S - a = 51.4 - 28.3 = 23.1$$

$$S - b = 51.4 - 31.7 = 19.7$$

$$S - c = 51.4 - 42.8 = 8.6$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{51.4 \times 23.1}{31.7 \times 42.8}}$$

$$\boxed{\alpha = 41^\circ 23'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{51.4 \times 19.7}{28.3 \times 42.8}}$$

$$\boxed{\beta = 47^\circ 46'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{51.4 \times 8.6}{28.3 \times 31.7}}$$

$$\boxed{\gamma = 90^\circ 51'}$$

Q.4 $a = 31.9$, $b = 56.31$, $c = 40.27$

Solution:

$$S = \frac{a+b+c}{2} = \frac{31.9+56.31+40.27}{2} = 64.24$$

$$S - a = 64.24 - 31.9 = 32.34$$

$$S - b = 64.24 - 56.31 = 7.93$$

$$S - c = 64.24 - 40.27 = 23.97$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{64.24 \times 32.34}{56.31 \times 40.27}}$$

$$\boxed{\alpha = 33^\circ 39'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{64.24 \times 7.93}{31.9 \times 56.31}}$$

$$\boxed{\beta = 115^\circ 38'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{64.24 \times 23.97}{31.9 \times 56.31}}$$

$$\boxed{\gamma = 30^\circ 43'}$$

Q.5 $a = 4584$, $b = 5140$, $c = 3624$

Solution:

$$S = \frac{a+b+c}{2} = \frac{4584+5140+3624}{2} = 6674$$

$$S - a = 6674 - 4584 = 2090$$

$$S - b = 6674 - 5140 = 1534$$

$$S - c = 6674 - 3624 = 3050$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{6674 \times 2090}{18627360}}$$

$$\boxed{\alpha = 60^{\circ} 9'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{6674 \times 1534}{16612416}}$$

$$\boxed{\beta = 76^{\circ} 34'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{6674 \times 3050}{23561760}}$$

$$\boxed{\gamma = 43^{\circ} 17'}$$

**Q.6 Find the smallest angle of the triangle ABC when
a = 37.34 , b = 3.24 , c = 35.06**

Solution:

By law of cosines

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$2ac \cos \beta = a^2 + c^2 - b^2$$

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\beta = \cos^{-1} \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$\beta = \cos^{-1} \left(\frac{(37.34)^2 + (35.06)^2 - (3.24)^2}{2(37.34)(35.06)} \right)$$

$$\boxed{\beta = 3^{\circ} 39'}$$

Q.7 find the measure of the greatest angle, if sides of triangle are 16, 20, 23.

(Lahore Board 2010)

Solution:

$$a = 16 , b = 20 , c = 33 , \gamma = ?$$

Now by law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$2ab \cos \gamma = a^2 + b^2 - c^2$$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\gamma = \cos^{-1} \left(\frac{a^2 + b^2 - c^2}{2ab} \right)$$

$$\gamma = \cos^{-1} \left(\frac{(16)^2 + (20)^2 - (33)^2}{2 \times 16 \times 20} \right)$$

$$\gamma = 132^{\circ} 35'$$

Q.8 The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Prove that the greatest angle of the triangle is 120° . (Gujranwala Board 2006)

Solution:

$$a = x^2 + x + 1, \quad b = 2x + 1, \quad c = x^2 - 1, \quad \alpha = ?$$

$$S = \frac{a+b+c}{2} = \frac{x^2 + x + 1 + 2x + 1 + x^2 - 1}{2} = \frac{2x^2 + 3x + 1}{2}$$

$$S - a = \frac{2x^2 + 3x + 1}{2} - (x^2 + x + 1)$$

$$S - a = \frac{2x^2 + 3x + 1 - 2x^2 - 2x - 2}{2} = \frac{x - 1}{2}$$

By half angle formula

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{\frac{2x^2 + 3x + 1}{2} \left(\frac{x - 1}{2} \right)}{(2x + 1)(x^2 - 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{(2x^2 + 2x + x + 1)(x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{[2x(x + 1) + (x + 1)](x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{(2x + 1)(x + 1)(x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$\alpha = 2 \cos^{-1} \left(\frac{1}{2} \right) = 2 \times 60^\circ = 120^\circ$$

Hence proved.

Q.9 The measures of the sides of a triangular plot are 413, 214 and 375 meters.
Find the measure of the corner angles of the plot.

Solution:

$$a = 413, \quad b = 214, \quad c = 375$$

$$S = \frac{a + b + c}{2} = \frac{413 + 214 + 375}{2} = 501$$

$$S - a = 501 - 413 = 88$$

$$S - b = 501 - 214 = 287$$

$$S - c = 501 - 375 = 126$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{501 \times 88}{214 \times 375}}$$

$$\boxed{\alpha = 84^\circ 20'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{501 \times 287}{413 \times 375}}$$

$$\boxed{\beta = 31^\circ 2'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{501 \times 126}{413 \times 214}}$$

$$\boxed{\gamma = 64^\circ 38'}$$

Q.10 Three villages A, B, C are connected by straight roads 6km, 9km, 13km.
What angles these roads makes with each other.

Solution:

Let, $a = 6, \quad b = 9, \quad c = 13$

$$S = \frac{a + b + c}{2} = \frac{6 + 9 + 13}{2} = 14$$

$$S - a = 14 - 6 = 8$$

$$S - b = 14 - 9 = 5$$

$$S - c = 14 - 13 = 1$$

Now by half angle formulas

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{14 \times 8}{117 \times 375}}$$

$$\boxed{\alpha = 23^\circ 51'}$$

$$\cos \frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2 \cos^{-1} \sqrt{\frac{14 \times 5}{78}}$$

$$\boxed{\beta = 37^\circ 21'}$$

$$\cos \frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2 \cos^{-1} \sqrt{\frac{14 \times 1}{54}}$$

$$\boxed{\gamma = 118^\circ 46'}$$

Area of Triangle

Case 1: When two sides & included angle is given

$$\Delta = \frac{1}{2} bc \sin \alpha = \frac{1}{2} ab \sin \gamma = \frac{1}{2} ac \sin \beta$$

Case II: When one side & two angles are given

$$\text{Area of triangle} = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha} = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin \beta} = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}$$

Case III: When three sides are given

Area of triangle by Hero's formula is

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

where a, b, c sides of triangle ABC

and $S = \frac{a+b+c}{2}$

Exercise 12.7

Q.1 Find the area of triangle ABC, given two sides and their included angle

(i) $a = 200, b = 120, \gamma = 150^\circ$ (Lahore Board 2006, Gujranwala Board 2007)

(ii) $b = 37, C = 45^\circ, \alpha = 30^\circ 50'$

(Gujranwala Board 2005, Lahore Board 2008,2011)

(iii) $a = 4.33, b = 9.25, \gamma = 56^\circ 44'$

Solution:

(i) $a = 200, b = 120, \gamma = 150^\circ$

$$\begin{aligned} \Delta &= \frac{1}{2} ab \sin \gamma \\ &= \frac{1}{2} (200) \times 120 \sin 150^\circ \\ &= 12000 \times 0.50 \\ &= 6000 \text{ sq. units.} \end{aligned}$$

(ii) $b = 37, C = 45^\circ, \alpha = 30^\circ 50'$

$$\begin{aligned} \Delta &= \frac{1}{2} bc \sin \alpha \\ &= \frac{1}{2} (37) \times 45 \sin 30^\circ 50' \\ &= 426.692 \text{ sq. units.} \end{aligned}$$

(iii) $a = 4.33, b = 9.25, \gamma = 56^\circ 44'$

$$\Delta = \frac{1}{2} ab \sin \gamma$$

$$= \frac{1}{2} (4.33) \times 9.25 \sin 56^\circ 44'$$

$$= 16.745 \text{ sq. units.}$$

Q.2 Find area of triangle ABC, given one side and two angles.

(i) $b = 25.4, \gamma = 36^\circ 41', \alpha = 45^\circ 17'$

(ii) $c = 32, \alpha = 47^\circ 24', \beta = 70^\circ 16'$

(iii) $a = 4.8, \gamma = 83^\circ 42', \gamma = 37^\circ 12'$

(Lahore Board 2010)

Solution:

(i) $b = 25.4, \gamma = 36^\circ 41', \alpha = 45^\circ 17'$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - 36^\circ 41' - 45^\circ 17'$$

$$\beta = 98^\circ 2'$$

We know that area of triangle

$$\Delta = \frac{1}{2} b^2 \frac{\sin \alpha \times \sin \gamma}{\sin \beta}$$

$$= \frac{1}{2} (25.4)^2 \frac{\sin 45^\circ 17' \sin 36^\circ 41'}{\sin 98^\circ 2'}$$

$$\Delta = 138.29 \text{ sq. units.}$$

(ii) $c = 32, \alpha = 47^\circ 24', \beta = 70^\circ 16'$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta$$

$$\gamma = 180^\circ - 47^\circ 24' - 70^\circ 16'$$

$$\gamma = 62^\circ 20'$$

$$\text{Area of triangle} = \frac{1}{2} c^2 \frac{\sin \alpha \times \sin \beta}{\sin \gamma}$$

$$\Delta = \frac{1}{2} (32)^2 \frac{\sin 47^\circ 24' \sin 70^\circ 16'}{\sin 62^\circ 20'}$$

$$\Delta = 400.57 \text{ sq. units.}$$

(iii) $a = 4.8, \gamma = 83^\circ 42', \gamma = 37^\circ 12'$

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 83^\circ 42' - 37^\circ 12'$$

$$\beta = 59^\circ 6'$$

$$\text{Area of triangle} = \frac{1}{2} a^2 \frac{\sin \beta \times \sin \gamma}{\sin \alpha}$$

$$\Delta = \frac{1}{2} (4.8)^2 \frac{\sin 59^\circ 6' \sin 37^\circ 12'}{\sin 83^\circ 42'}$$

$$\Delta = 6.02 \text{ sq. units.}$$

Q.3 Find the area of the triangle ABC, given three sides

(i) $a = 18, b = 24, c = 30$

(ii) $a = 524, b = 276, c = 315$

(iii) $a = 32.65, b = 42.81, c = 64.92$

Solution:

(i) $a = 18, b = 24, c = 30$

$$S = \frac{a + b + c}{2} = \frac{18 + 24 + 30}{2} = 36$$

$$S - a = 36 - 18 = 18$$

$$S - b = 36 - 24 = 12$$

$$S - c = 36 - 30 = 6$$

$$\begin{aligned} \text{Required Area} &= \sqrt{S(S-a)(S-b)(S-c)} \\ &= \sqrt{36 \times 18 \times 12 \times 6} = 216 \text{ sq. units.} \end{aligned}$$

(ii) $a = 524, b = 276, c = 315$

$$S = \frac{a + b + c}{2} = \frac{524 + 276 + 315}{2} = 557.5$$

$$S - a = 557.5 - 524 = 33.5$$

$$S - b = 557.5 - 276 = 281.5$$

$$S - c = 557.5 - 315 = 242.5$$

$$\text{Required Area} = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{557.5 \times 33.5 \times 281.5 \times 242.5} = 35705.89 \text{ sq. units.}$$

(iii) **a = 32.65, b = 42.81, c = 64.92**

$$S = \frac{a + b + c}{2} = \frac{32.65 + 42.81 + 64.92}{2} = 70.19$$

$$S - a = 70.19 - 32.65 = 37.54$$

$$S - b = 70.19 - 42.81 = 27.38$$

$$S - c = 70.19 - 64.92 = 5.27$$

$$\begin{aligned} \text{Required Area} &= \sqrt{S(S-a)(S-b)(S-c)} \\ &= \sqrt{70.19 \times 37.54 \times 27.38 \times 5.27} = 616.60 \text{ sq. units.} \end{aligned}$$

Q.4 The area of triangle is 2437. If a = 79, c = 97. Find angle β .

(Lahore Board 2005).

Solution:

$$\Delta = 2437, \quad a = 79, \quad c = 97, \quad \beta = ?$$

$$\text{area of triangle} = \frac{1}{2} ac \sin \beta$$

$$2437 = \frac{1}{2} \times 79 \times 97 \sin \beta$$

$$\beta = \sin^{-1} \frac{2437}{3831.5}$$

$$\beta = 39^\circ 30'$$

Q.5 The area of triangle is 121.34. If $\alpha = 32^\circ 15'$, $\beta = 65^\circ 37'$ find c and γ .

(Lahore Board 2004)

Solution:

$$\therefore \alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 32^\circ 15' - 65^\circ 37'$$

$$\gamma = 82^\circ 8'$$

$$\text{Area of triangle} = \frac{1}{2} c^2 \frac{\sin \alpha \times \sin \beta}{\sin \gamma}$$

$$121.34 = \frac{1}{2} c^2 \frac{\sin 32^\circ 15' \sin 65^\circ 37'}{\sin 82^\circ 8'}$$

$$c^2 = \frac{240.3962}{0.4860}$$

$$c^2 = 494.64$$

$$c = 22.24 \text{ sq. units.}$$

Q.6 One side of a triangular garden is 30m. If its two corner angles are $22^{\circ}\frac{1}{2}$ and $112^{\circ}\frac{1}{2}$. Find the cost of planting the grass at the rate of Rs. 5 per square meter.

Solution:

$$a = 30$$

$$\beta = 22^{\circ}\frac{1}{2} = 22.5^{\circ} = 22^{\circ} 30'$$

$$\gamma = 112^{\circ}\frac{1}{2} = 112^{\circ} 30'$$

$$\alpha = ?$$

$$\because \alpha + \beta + \gamma = 180^{\circ}$$

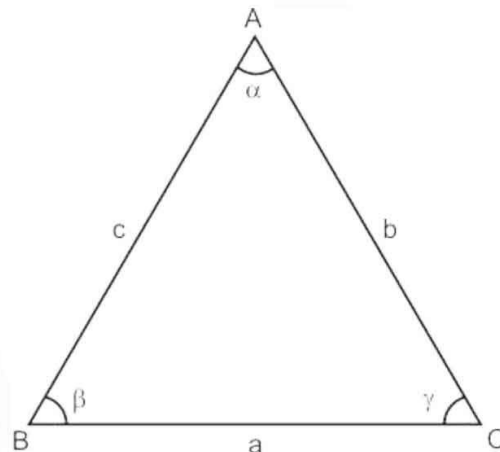
$$\begin{aligned}\alpha &= 180^{\circ} - \beta - \gamma \\ &= 180^{\circ} - 22^{\circ} 30' - 112^{\circ} 30'\end{aligned}$$

$$\alpha = 45^{\circ}$$

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2} a^2 \frac{\sin \beta \sin \gamma}{\sin \alpha} \\ &= \frac{1}{2} (30)^2 \frac{\sin 22^{\circ} 30' \sin 112^{\circ} 30'}{\sin 45^{\circ}}\end{aligned}$$

$$\Delta = 225 \text{ sq. m}$$

$$\begin{aligned}\text{Grass planting @ Rs. 5/sq. m} &= 225 \times 5 \\ &= \text{Rs. 1125} \quad \text{Ans.}\end{aligned}$$



EXERCISE 12.8

Q.1 Show that

$$(i) \quad r = 4 R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$(ii) \quad S = 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution:

$$(i) \quad r = 4 R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$\begin{aligned}\text{R.H.S.} &= 4 R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \\ &= 4 \frac{abc}{4 \Delta} \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{(S-a)(S-b)}{ab}}\end{aligned}$$

$$\begin{aligned}
&= \frac{a b c}{\Delta} \sqrt{\frac{(S-b)^2 (S-c)^2 (S-a)^2}{a^2 b^2 c^2}} \\
&= \frac{a b c}{\Delta} \frac{(S-a)(S-b)(S-c)}{a b c} \\
&= \frac{1}{\Delta} \frac{S(S-a)(S-b)(S-c)}{S} \quad (\text{multiply and dividing by } S) \\
&= \frac{1}{\Delta} \times \frac{\Delta^2}{S} \\
&= \frac{\Delta}{S} = r \\
&= \text{L.H.S. Hence proved.}
\end{aligned}$$

(ii) $S = 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$

$$\begin{aligned}
\text{R.H.S.} &= 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\
&= 4 \frac{a b c}{4 \Delta} \sqrt{\frac{S(S-a)}{bc}} \sqrt{\frac{S(S-b)}{ac}} \sqrt{\frac{S(S-c)}{ab}} \\
&= \frac{a b c}{\Delta} \sqrt{\frac{S^3 (S-a)(S-b)(S-c)}{a^2 b^2 c^2}} \\
&= \frac{S a b c}{\Delta a b c} \sqrt{S(S-a)(S-b)(S-c)} \\
&= \frac{S}{\Delta} \times \Delta \\
&= S = \text{L.H.S.}
\end{aligned}$$

Hence proved.

Q.2 Show that $r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} = c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$

Solution:

$$\begin{aligned}
\text{R.H.S.} &= a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} \\
&= a \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{(S-a)(S-b)}{ab}} \sqrt{\frac{bc}{S(S-a)}} \\
&= a \sqrt{\frac{(S-a)(S-c)(S-a)(S-b)bc}{a^2 bc S(S-a)}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^2}{a} \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}} \\
&= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}
\end{aligned}$$

Next R.H.S. = $b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2}$

$$\begin{aligned}
&= b \sqrt{\frac{(S-a)(S-b)}{ab}} \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{ac}{S(S-b)}} \\
&= b \sqrt{\frac{(S-a)(S-b)(S-b)(S-c)ac}{a b^2 c S^2 (S-b)}} \\
&= \frac{b}{b} \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}} \\
&= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}
\end{aligned}$$

Next R.H.S. = $c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$

$$\begin{aligned}
&= c \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{ab}{S(S-c)}} \\
&= c \sqrt{\frac{(S-b)(S-c)(S-c)(S-a)ab}{ba c^2 S (S-c)}} \\
&= \frac{c}{c} \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}} \\
&= \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S} \\
&= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}
\end{aligned}$$

Q.3 Prove that

(i) $r_1 = 4 R \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$

(ii) $r_2 = 4 R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$

(iii) $r_3 = 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$

Solution:

$$(i) \quad \mathbf{r_1} = 4 \mathbf{R} \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$\text{R.H.S.} = 4 \mathbf{R} \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$\begin{aligned}
&= 4 \frac{a b c}{4 \Delta} \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{S(S-b)}{ac}} \sqrt{\frac{S(S-c)}{ab}} \\
&= \frac{a b c}{\Delta} \sqrt{\frac{(S-b)^2 (S-c)^2 S^2}{a^2 b^2 c^2}} \\
&= \frac{abc}{\Delta abc} \frac{S(S-b)(S-c)(S-a)}{S-a} \\
&= \frac{\Delta^2}{\Delta (S-a)} \\
&= \frac{\Delta}{S-a} = r_1 \quad \text{L.H.S.} \quad \text{Hence proved}
\end{aligned}$$

(ii) $r_2 = 4 R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$

$$\begin{aligned}
\text{R.H.S.} &= 4 R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2} \\
&= 4 \left(\frac{abc}{4 \Delta} \right) \sqrt{\frac{S(S-a)}{bc}} \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{S(S-c)}{ab}} \\
&= \frac{a b c}{\Delta} \sqrt{\frac{S^2 (S-a)^2 (S-c)^2}{a^2 b^2 c^2}} \\
&= \frac{a b c}{\Delta} \times \frac{S(S-a)(S-c)}{a b c} \\
&= \frac{S(S-a)(S-b)(S-c)}{(S-b) \Delta} \\
&= \frac{\Delta^2}{\Delta (S-b)} \\
&= \frac{\Delta}{S-b} = r_2 = \text{L.H.S.} \quad \text{Hence proved}
\end{aligned}$$

Hence proved.

(iii) $r_3 = 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$

$$\text{R.H.S.} = 4 R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$\begin{aligned}
&= 4 \left(\frac{abc}{4\Delta} \right) \sqrt{\frac{S(S-a)}{bc}} \sqrt{\frac{S(S-b)}{ab}} \sqrt{\frac{(S-a)(S-b)}{ab}} \\
&= \frac{abc}{\Delta} \sqrt{\frac{S^2(S-a)^2(S-b)^2}{a^2b^2c^2}} \\
&= \frac{abc}{\Delta} \frac{S(S-a)(S-b)(S-c)}{abc(S-c)} \\
&= \frac{\Delta^2}{\Delta(S-c)} \\
&= \frac{\Delta}{S-c} = r_3 = \text{L.H.S. Hence proved.}
\end{aligned}$$

Q.4 Show that

(i) $r_1 = S \tan \frac{\alpha}{2}$

(ii) $r_2 = S \tan \frac{\beta}{2}$ (Lahore Board 2007)

(iii) $r_3 = S \tan \frac{\gamma}{2}$ (Gujranwala Board 2005)

Solution:

(i) $r_1 = S \tan \frac{\alpha}{2}$

$$\begin{aligned}
\text{R.H.S.} &= S \tan \frac{\alpha}{2} \\
&= S \sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \\
&= S \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-a)^2}} \\
&= S \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-a)} \\
&= \frac{\Delta}{S-a} = r_1 = \text{L.H.S. Hence proved}
\end{aligned}$$

$$(ii) \quad r_2 = S \tan \frac{\beta}{2}$$

$$\begin{aligned} \text{R.H.S.} &= S \tan \frac{\alpha}{2} \\ &= S \sqrt{\frac{(S-a)(S-c)}{S(S-b)}} \\ &= S \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-b)^2}} \\ &= S \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-b)} \\ &= \frac{\Delta}{S-b} = r_2 = \text{L.H.S. Hence proved} \end{aligned}$$

$$(iii) \quad r_3 = S \tan \frac{\gamma}{2}$$

$$\begin{aligned} \text{R.H.S.} &= S \tan \frac{\gamma}{2} \\ &= S \sqrt{\frac{(S-a)(S-b)}{S(S-c)}} \\ &= S \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}} \\ &= S \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-c)} \\ &= S \times \frac{\Delta}{S(S-c)} = r_3 = \text{L.H.S. Hence proved} \end{aligned}$$

Q.5 Prove that

$$(i) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2$$

(Lahore Board 2009)

$$(ii) \quad r r_1 r_2 r_3 = \Delta^2$$

$$(iii) \quad r_1 + r_2 + r_3 - r = 4R$$

$$(iv) \quad r_1 r_2 r_3 = rS^2$$

Solution:

$$(i) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2$$

$$\text{L.H.S.} = r_1 r_2 + r_2 r_3 + r_3 r_1$$

$$= \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} + \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} + \frac{\Delta}{S-c} \times \frac{\Delta}{S-a}$$

$$\begin{aligned}
&= \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-b)(S-c)} + \frac{\Delta^2}{(S-c)(S-a)} \\
&= \Delta^2 \left[\frac{S-c + S-a + S-b}{(S-a)(S-b)(S-c)} \right] \\
&= \Delta^2 \left[\frac{3S - (a+b+c)}{(S-a)(S-b)(S-c)} \right] \\
&= \Delta^2 \left[\frac{3S - 2S}{(S-a)(S-b)(S-c)} \right] \\
&= \frac{\Delta^2 S \cdot S}{S(S-a)(S-b)(S-c)} \\
&= \frac{\Delta^2 S^2}{\Delta^2} = S^2 = \text{R.H.S. Hence proved}
\end{aligned}$$

(ii) $r_1 r_2 r_3 = \Delta^2$

$$\begin{aligned}
\text{L.H.S.} &= r_1 r_2 r_3 \\
&= \frac{\Delta}{S} \times \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} \\
&= \frac{\Delta^4}{S(S-a)(S-b)(S-c)} \\
&= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S. Hence proved}
\end{aligned}$$

(iii) $r_1 + r_2 + r_3 - r = 4R$

$$\begin{aligned}
\text{L.H.S.} &= r_1 + r_2 + r_3 - r \\
&= \frac{\Delta}{S-a} + \frac{\Delta}{S-b} + \frac{\Delta}{S-c} - \frac{\Delta}{S} = \Delta \left[\frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c} - \frac{1}{S} \right] \\
&= \Delta \left[\frac{S-b + S-a}{(S-a)(S-b)} + \frac{S-(S-c)}{S(S-c)} \right] \\
&= \Delta \left[\frac{2S-a-b}{(S-a)(S-b)} + \frac{S-S+c}{S(S-c)} \right] \\
&= \Delta \left[\frac{2S-a-b-c+c}{(S-a)(S-b)} + \frac{c}{S(S-c)} \right] \\
&= \Delta \left[\frac{2S-(a+b+c)+c}{(S-a)(S-b)} + \frac{c}{S(S-c)} \right] \\
&= \Delta \left[\frac{2S-2S+c}{(S-a)(S-b)} + \frac{c}{S(S-c)} \right] \\
&= \Delta \cdot C \left[\frac{1}{(S-a)(S-b)} + \frac{1}{S(S-c)} \right] \\
&= C\Delta \left[\frac{S(S-c) + (S-a)(S-b)}{S(S-a)(S-b)(S-c)} \right]
\end{aligned}$$

$$\begin{aligned}
&= C\Delta \left[\frac{S^2 - CS + S^2 - bS - aS + ab}{\Delta^2} \right] \\
&= \frac{C}{\Delta} [2S^2 - S(a+b+C) + ab] \\
&= \frac{C}{\Delta} [2S^2 - S(2S) + ab] \\
&= \frac{C}{\Delta} [2S^2 - 2S^2 + ab] \\
&= \frac{abc}{\Delta} = \frac{4abc}{4\Delta} = 4R \\
&= \text{R.H.S. Hence proved}
\end{aligned}$$

$$(iv) \quad r_1 r_2 r_3 = r S^2$$

$$L.H.S. = r_1 r_2 r_3$$

$$\begin{aligned} \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} &= \frac{\Delta^3}{(S-a)(S-b)(S-c)} \\ &= \frac{S \Delta^3}{S(S-a)(S-b)(S-c)} \\ &= \frac{S \Delta^3}{\Delta^2} \\ &= S \Delta \\ &= S \Delta \times \frac{S}{S} \\ &= S^2 r \\ &= R.H.S. \end{aligned}$$

Hence proved.

Q.6 Find R, r, r_1, r_2, r_3 if measures of the sides of triangle $\triangle ABC$ are (Gujranwala Board 2005, 2007) (Lahore Board, 2004, 2005, 2009)

(i) $a = 13, b = 14, c = 15$

(ii) $a = 34, b = 20, c = 42$

Solution:

(i) $a = 13, b = 14, c = 15$

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$\begin{aligned} \Delta &= \sqrt{S(S-a)(S-b)(S-c)} \\ &= \sqrt{21(21-13)(21-14)(21-15)} \\ &= \sqrt{21 \times 8 \times 7 \times 6} = 84 \end{aligned}$$

$$R = \frac{abc}{4\Delta} = \frac{13 \times 14 \times 15}{4(84)} = \frac{2730}{336} = 8.125$$

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{S-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{S-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{6} = 14$$

(ii) $a = 34, b = 20, c = 42$

$$S = \frac{a+b+c}{2} = \frac{34+20+42}{2} = \frac{96}{2} = 48$$

$$\begin{aligned}\Delta &= \sqrt{S(S-a)(S-b)(S-c)} \\ &= \sqrt{48(48-34)(48-20)(48-42)} \\ &= \sqrt{48(14)(28)(6)} = 336\end{aligned}$$

$$R = \frac{abc}{4\Delta} = \frac{34 \times 20 \times 42}{4 \times 336} = \frac{28560}{1344} = 21.25$$

$$r = \frac{\Delta}{S} = \frac{336}{48} = 7$$

$$r_1 = \frac{\Delta}{S-a} = \frac{336}{14} = 24$$

$$r_2 = \frac{\Delta}{S-b} = \frac{336}{28} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{336}{6} = 56$$

Q.7 Prove that in an equilateral triangle

(i) $r : R : r_1 = 1 : 2 : 3$

(ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

Solution:

(i) $r : R : r_1 = 1 : 2 : 3$

We know that in an equilateral triangle

$$a = b = c$$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S-a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S-b = S-a = \frac{a}{2}$$

$$S-c = S-a = \frac{a}{2}$$

$$\Delta = \sqrt{S(S-a)(S-a)(S-a)} = \sqrt{\frac{3a}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3} a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\frac{\sqrt{3} a^2}{4}}{\frac{3a}{2}} = \frac{\sqrt{3} a^2}{4} \div \frac{3a}{2}$$

$$r = \frac{\sqrt{3} a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3} a}{6}$$

$$R = \frac{abc}{4\Delta} = \frac{a^3}{4 \times \frac{\sqrt{3} a^2}{4}} = \frac{a^3}{\sqrt{3} a^2} = \frac{a}{\sqrt{3}}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\frac{\sqrt{3} a^2}{4}}{\frac{a}{2}} = \frac{\sqrt{3} a^2}{4} \div \frac{a}{2}$$

$$r_1 = \frac{\sqrt{3} a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3} a}{2}$$

$$\text{L.H.S.} = r : R : r_1$$

$$\frac{\sqrt{3} a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3} a}{2}$$

$$\text{Dividing throughout by } \frac{\sqrt{3} a}{6}$$

$$\frac{\sqrt{3} a}{6} \times \frac{6}{\sqrt{3} a} : \frac{a}{\sqrt{3}} \times \frac{6}{\sqrt{3} a} : \frac{\sqrt{3} a}{2} \times \frac{6}{\sqrt{3} a}$$

$$= 1 : \frac{6}{3} : 3$$

$$= 1 : 2 : 3$$

$$= \text{R.H.S.}$$

Hence proved.

(ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

We know that in an equilateral triangle

$$a = b = c$$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S-a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S-b = \frac{3a}{2} - a = \frac{a}{2}$$

$$S-c = \frac{3a}{2} - a = \frac{a}{2}$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{\frac{3a}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3} a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3} a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3} a}{6}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\sqrt{3} a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3} a}{2}$$

$$R = \frac{a b c}{4\Delta} = \frac{a^3}{4 \times \frac{\sqrt{3} a^2}{4}} = \frac{a}{\sqrt{3}}$$

$$r_2 = \frac{\Delta}{S-b} = \frac{\sqrt{3} a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3} a}{2}$$

$$r_3 = \frac{\Delta}{S-c} = \frac{\sqrt{3} a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3} a}{2}$$

$$\text{L.H.S.} = r : R : r_1 : r_2 : r_3$$

$$= \frac{\sqrt{3} a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3} a}{2} : \frac{\sqrt{3} a}{2} : \frac{\sqrt{3} a}{2}$$

$$\text{Multiplying throughout by } \frac{6}{\sqrt{3} a}$$

$$= \frac{\sqrt{3} a}{6} \times \frac{6}{\sqrt{3} a} : \frac{a}{\sqrt{3}} \times \frac{6}{\sqrt{3} a} : \frac{\sqrt{3} a}{2} \times \frac{6}{\sqrt{3} a} : \frac{\sqrt{3} a}{2} \times \frac{6}{\sqrt{3} a} : \frac{\sqrt{3} a}{2} \times \frac{6}{\sqrt{3} a}$$

$$= 1 : 2 : 3 : 3 : 3$$

$$= \text{R.H.S. Hence proved.}$$

Q.8 Prove that

$$(i) \quad \Delta = r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

$$(ii) \quad r = S \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \quad (\text{Lahore Board 2010})$$

$$(iii) \quad \Delta = 4 R r \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution:

$$(i) \quad \Delta = r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

$$\text{R.H.S.} = r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S} \right)^2 \sqrt{\frac{S(S-a)}{(S-b)(S-c)}} \sqrt{\frac{S(S-b)}{(S-a)(S-c)}} \sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$$

$$\begin{aligned}
&= \frac{\Delta^2}{S^2} \sqrt{\frac{S^3 (S-a) (S-b) (S-c)}{(S-b)^2 (S-c)^2 (S-a)^2}} \\
&= \frac{\Delta^2}{S^2} S \sqrt{\frac{S (S-a) (S-b) (S-c)}{(S-a)^2 (S-b)^2 (S-c)^2}} \\
&= \frac{\Delta^2}{S} \sqrt{\frac{S \cdot S}{S (S-a) (S-b) (S-c)}} \\
&= \frac{\Delta^2}{S} \frac{S}{\sqrt{S (S-a) (S-b) (S-c)}} = \frac{\Delta^2}{\Delta} = \Delta = \text{L.H.S. Hence proved}
\end{aligned}$$

(ii) $r = S \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$

$$\begin{aligned}
\text{R.H.S.} &= S \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \\
&= S \sqrt{\frac{(S-b) (S-c)}{S(S-a)}} \sqrt{\frac{(S-a) (S-c)}{S (S-b)}} \sqrt{\frac{(S-a) (S-b)}{S(S-c)}} \\
&= S \sqrt{\frac{(S-b) (S-c) (S-a) (S-c) (S-a) (S-b)}{S^3 (S-a) (S-b) (S-c)}} \\
&= \frac{S}{S} \sqrt{\frac{(S-a)^2 (S-b)^2 (S-c)^2}{S (S-a) (S-b) (S-c)}} \\
&= \sqrt{\frac{S (S-a) (S-b) (S-c)}{S^2}} \\
&= \frac{\sqrt{S (S-a) (S-b) (S-c)}}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}
\end{aligned}$$

Hence proved.

(iii) $\Delta = 4 R r \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$

$$\begin{aligned}
\text{R.H.S.} &= 4 R r \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\
&= 4 \left(\frac{abc}{4\Delta} \right) \left(\frac{\Delta}{S} \right) \sqrt{\frac{S (S-a)}{bc}} \sqrt{\frac{S (S-b)}{ac}} \sqrt{\frac{S (S-c)}{ab}} \\
&= \frac{abc}{S} \sqrt{\frac{S^3 (S-a) (S-b) (S-c)}{a^2 b^2 c^2}} \\
&= \frac{abc S}{S} \sqrt{\frac{S (S-a) (S-b) (S-c)}{a^2 b^2 c^2}} \\
&= \frac{abc}{abc} \sqrt{S (S-a) (S-b) (S-c)} = \Delta = \text{L.H.S. Hence proved.}
\end{aligned}$$

Q.9 Show that

$$(i) \quad \frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

$$(ii) \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Solution:

$$(i) \quad \frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

$$\begin{aligned} \text{L.H.S.} &= \frac{1}{2rR} \\ &= \frac{1}{2 \left(\frac{\Delta}{S} \right) \left(\frac{abc}{4\Delta} \right)} = \frac{2S}{abc} = \frac{a+b+c}{abc} \\ &= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} = \text{R.H.S.} \text{ Hence proved} \end{aligned}$$

$$(ii) \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$\begin{aligned} \text{R.H.S.} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\ &= \frac{S-a}{\Delta} + \frac{S-b}{\Delta} + \frac{S-c}{\Delta} \\ &= \frac{S-a+S-b+S-c}{\Delta} = \frac{3S-(a+b+c)}{\Delta} \\ &= \frac{3S-2S}{\Delta} = \frac{S}{\Delta} = \frac{1}{r} = \text{L.H.S.} \text{ Hence proved} \end{aligned}$$

Q.10 Prove that

$$r = \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} = \frac{b \sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}{\cos \frac{\beta}{2}} = \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}}$$

Solution:

$$\begin{aligned} r &= \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} \\ \text{R.H.S.} &= \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{a \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-a)}{bc}}} \\
&= a \sqrt{\frac{\frac{(S-a)^2 (S-c)(S-b)}{a^2 bc}}{\frac{S(S-a)}{bc}}} \\
&= a \sqrt{\frac{(S-a)^2 (S-c)(S-b)}{a^2 bc}} \times \frac{abc}{S(S-a)} \\
&= a \sqrt{\frac{(S-a)(S-b)(S-c)S}{S^2 a^2}} \\
&= \frac{a}{a} \times \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}
\end{aligned}$$

$$r = \frac{b \sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}{\cos \frac{\beta}{2}}$$

$$\text{R.H.S.} = \frac{b \sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}{\cos \frac{\beta}{2}}$$

$$\begin{aligned}
&= b \frac{\sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-b)}{ac}}} \\
&= b \sqrt{\frac{(S-b)^2 (S-c)(S-a)}{a b^2 c}} \times \frac{ac}{S(S-b)} \\
&= b \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2 b^2}} \\
&= \frac{b}{b} \times \frac{\Delta}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}
\end{aligned}$$

$$r = \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}}$$

$$\begin{aligned}
\text{R.H.S.} &= \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}} \\
&= \frac{c \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{(S-a)(S-c)}{ac}}}{\sqrt{\frac{S(S-c)}{ab}}} \\
&= c \sqrt{\frac{(S-b)(S-c)^2(S-a)}{a b c^2}} \times \frac{ba}{S(S-c)} \\
&= c \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2 c^2}} \\
&= \frac{\Delta}{S} = r = \text{L.H.S.} \text{ Hence proved}
\end{aligned}$$

Q.11 Prove that $abc (\sin \alpha + \sin \beta + \sin \gamma) = 4 \Delta S$ (Lahore Board 2007, 2011)

Solution:

$$\begin{aligned}
\text{L.H.S.} &= abc (\sin \alpha + \sin \beta + \sin \gamma) \\
&= abc \left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \right) \quad \because R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{\sin\gamma} \\
&= abc \left(\frac{a+b+c}{2R} \right) = abc \left(\frac{2S}{2\left(\frac{abc}{4\Delta}\right)} \right) \\
&= abc \left(\frac{4\Delta S}{abc} \right) = 4\Delta S = \text{R.H.S.} \text{ Hence proved}
\end{aligned}$$

Q.12 Prove that

(i) $(r_1 + r_2) \tan \frac{\gamma}{2} = C$ (Lahore Board 2006) (Gujranwala Board 2007)

(ii) $(r^3 - r) \cot \frac{\gamma}{2} = C$

Solution:

(i) $(r_1 + r_2) \tan \frac{\gamma}{2} = C$

$$\begin{aligned}
\text{L.H.S.} &= (r_1 + r_2) \tan \frac{\gamma}{2} \\
&= \left(\frac{\Delta}{S-a} + \frac{\Delta}{S-b} \right) \sqrt{\frac{(S-a)(S-b)}{S(S-c)}} \\
&= \Delta \left(\frac{S-b+S-a}{(S-a)(S-b)} \right) \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}}
\end{aligned}$$

$$\begin{aligned}
&= \Delta \left(\frac{2S - b - a}{(S - a)(S - b)} \right) \frac{\Delta}{S(S - c)} \\
&= \frac{\Delta^2 (a + b + c - b - a)}{S(S - a)(S - b)(S - c)} \quad (\because 2S = a + b + c) \\
&= \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S. Hence proved.}
\end{aligned}$$

(ii) $(r^3 - r) \cot \frac{\gamma}{2} = C$

$$\begin{aligned}
\text{L.H.S.} &= (r^3 - r) \cot \frac{\gamma}{2} \\
&= \left(\frac{\Delta}{S - c} - \frac{\Delta}{S} \right) \sqrt{\frac{S(S - c)}{(S - a)(S - b)}} \\
&= \Delta \left(\frac{1}{S - c} - \frac{1}{S} \right) \sqrt{\frac{S(S - a)(S - b)(S - c)}{(S - a)^2 (S - b)^2}} \\
&= \Delta \left(\frac{S - S + c}{S(S - c)} \right) \frac{\sqrt{S(S - a)(S - b)(S - c)}}{(S - a)(S - b)} \\
&= \frac{\Delta C}{S(S - c)} \times \frac{\Delta}{(S - a)(S - b)} \\
&= \frac{\Delta^2 C}{S(S - a)(S - b)(S - c)} = \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S.}
\end{aligned}$$

Hence proved.

APPLICATION OF TRIGONOMETRY

MULTIPLE CHOICE QUESTIONS

(1) The value of $R =$

(a) $\frac{abc}{S - a}$

(b) $\frac{abc}{4\Delta}$

(c) $\frac{4abc}{\Delta}$

(d) $\frac{abc}{\Delta}$

[Lahore Board 2005]

(2) In the law of tangents $\frac{\tan \frac{\alpha - \beta}{2}}{\tan \frac{\alpha + \beta}{2}} =$

(a) $\frac{a - b}{a + b}$

(b) $\frac{a - \beta}{a + \beta}$

(c) $\frac{a + \beta}{a - \beta}$

(d) $\frac{a + b}{a - b}$

[Lahore Board 2005]

(3) Radius of the inscribed circle is:

(a) $r = \frac{\Delta}{S}$

(b) $r = \frac{abc}{4\Delta}$

(c) $r = \frac{S}{\Delta}$

(d) $r = \frac{S - a}{\Delta}$

[Lahore Board 2005]

(4) Hero's formula is used to calculate:

- (a) Area of Δ
- (b) Sides of Δ
- (c) Angles of Δ
- (d) None of these

[Lahore Board 2005]

(5) If three sides of a triangle are given then area of triangle =

- (a) $S(S - a)(S - b)(S - c)$
- (b) $(S - a)(S - b)(S - c)$
- (c) $\sqrt{S(S - a)(S - b)(S - c)}$
- (d) $\sqrt{S(S - a)(S - b)(S - c)}$

[Lahore Board 2005]

(6) $\frac{c + a}{c - a} =$

- (a) $\frac{\tan \frac{\gamma + \alpha}{2}}{\tan \frac{\gamma - \alpha}{2}}$
- (b) $\frac{\tan \frac{\alpha + \gamma}{2}}{\tan \frac{\alpha - \gamma}{2}}$
- (c) $\frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}}$
- (d) None of these

[Lahore Board 2005]

(7) For any triangle ABC $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$ is called _____

- (a) Cosine law
- (b) Sine law
- (c) Law of tangent
- (d) Hero's formula

[Gujranwala Board 2005]

(8) By Hero's formula $\Delta =$

- (a) $S(S - a)(S - b)(S - c)$
- (b) $(S - a)(S - b)(S - c)$
- (c) $\sqrt{S(S - a)(S - b)(S - c)}$
- (d) None of these

[Gujranwala Board 2005]

(9) $\cos \frac{\gamma}{2} =$

- (a) $\sqrt{\frac{S(S - c)}{ab}}$
- (b) $\sqrt{\frac{S(S - b)}{ca}}$
- (c) $\sqrt{\frac{S(S - a)}{ab}}$
- (d) None of these

[Gujranwala Board 2005]

(10) The in-radius r is equal to:

- | | |
|------------------------|-------------------------|
| (a) $S\Delta$ | (b) $\frac{1}{S\Delta}$ |
| (c) $\frac{S}{\Delta}$ | (d) $\frac{\Delta}{S}$ |

[Gujranwala Board 2005]

[Lahore Board 2008-10]

(11) Circle down inside a triangle touching its three sides is called ————

- | | |
|-------------------|-------------------|
| (a) Circum-circle | (b) In-circle |
| (c) Ex-circle | (d) None of these |

[Gujranwala Board 2005]

(12) $\Delta =$

- | | |
|----------------------|------------------|
| (a) $\frac{abc}{4R}$ | (b) rS |
| (c) $r_2(S - b)$ | (d) All of these |

[Gujranwala Board 2005]

(13) A circle passing through the vertices of a triangle is called:

- | | |
|---------------------|-------------------|
| (a) Circum-circle | (b) In-circle |
| (c) Escribed circle | (d) None of these |

[Gujranwala Board 2006-08]

(14) With usual notation, the value of $a - b + c$ is:

- | | |
|--------------|----------------|
| (a) $S + b$ | (b) $S - b$ |
| (c) $2S - b$ | (d) $2(S - b)$ |

[Gujranwala Board 2006]

(15) Area of circumcircle is:

- | | |
|----------------------|--------------------------------|
| (a) $\frac{abc}{4R}$ | (b) abC/R |
| (c) rS | (d) $\frac{1}{2}bc \sin\alpha$ |

[Gujranwala Board 2006]

(16) Radius of escribed circle opposite to the vertex A is:

(a) $\frac{\Delta}{a}$

(b) $\frac{\Delta}{S}$

(c) $\frac{S - a}{\Delta}$

(d) $\frac{\Delta}{S - a}$

[Lahore Board 2006, Lahore Board 2015]

[Gujranwala Board 2010]

(17) $S(S - a)(S - b)(S - c) =$

(a) Δ

(b) Δ^2

(c) Δ^3

(d) None of these

[Lahore Board 2006]

(18) In case of escribed circle, $\frac{\Delta}{S - C}$ is equal to:

(a) r_1

(b) r_2

(c) r_3

(d) None of these

[Lahore Board 2006]

(19) The radius of the circle passes through three vertices of a triangle is called:

(a) Circum-radius

(b) In-radius

(c) Ex-radius

(d) None of these

[Lahore Board 2006]

(20) $\frac{S - a}{\Delta} =$

(a) r

(b) r_1

(c) $\frac{1}{r_1}$

(d) None of these

[Lahore Board 2006]

(21) The circum radius R is given by:

(a) $\frac{a + b + c}{4\Delta}$

(b) $\frac{a + b + c}{\Delta}$

(c) $\frac{abc}{\Delta}$

(d) $\frac{abc}{4\Delta}$

[Lahore Board 2007]

[Gujranwala Board 2007]

(22) If in a right angled triangle base = 30.8 , hypoteneuse = 37.2 then $\cos\alpha$ will be:

- | | |
|------------|------------|
| (a) 0.7032 | (b) 0.8280 |
| (c) 0.7513 | (d) 0.8655 |

[Lahore Board 2007]

(23) The greatest angle is opposite to:

- | | |
|-------------------|-------------------|
| (a) Smallest side | (b) Greatest side |
| (c) Same side | (d) Right side |

[Lahore Board 2007]

(24) In a right triangle no angle is greater than:

- | | |
|----------------|----------------|
| (a) 90° | (b) 80° |
| (c) 60° | (d) 45° |

[Lahore Board 2008]

(25) If a, b, c have their usual meanings then $\frac{c^2 + a^2 - b^2}{2ac}$ is equal to:

- | | |
|------------------|-----------------|
| (a) $\cos\alpha$ | (b) $\cos\beta$ |
| (c) $\cos\gamma$ | (d) $\sin\beta$ |

[Gujranwala Board 2009]

(26) If $\triangle ABC$ is right angled then the law of cosines reduces to:

- | | |
|---------------------|------------------------|
| (a) Law of sines | (b) Law of cosines |
| (c) Law of tangents | (d) Pythagoras theorem |

[Lahore Board 2009]

(27) With usual notations for triangle R equals:

- | | |
|-----------------------------|----------------------------|
| (a) $\frac{b}{2\sin\alpha}$ | (b) $\frac{a}{2\sin\beta}$ |
| (c) $\frac{c}{2\sin\gamma}$ | (d) $\frac{\Delta}{5}$ |

[Lahore Board 2009]

[Lahore Board 2010]

(28) The value of R = _____

- | | |
|---------------------------|-----------------------------|
| (a) $\frac{\Delta}{S}$ | (b) $\frac{a}{\sin\alpha}$ |
| (c) $\frac{b}{\sin\beta}$ | (d) $\frac{c}{2\sin\gamma}$ |

[Lahore Board 2011]

(29) In any triangle ABC with usual notation $\frac{a^2 + b^2 - c^2}{2ab} =$

- | | |
|------------------|------------------|
| (a) $\cos\alpha$ | (b) $\cos\beta$ |
| (c) $\cos\gamma$ | (d) $\sin\gamma$ |

[Lahore Board 2012]

(30) In any triangle ABC, $a = 4$, $b = 10$, $\gamma = 30^\circ$ then area of triangle is:

- | | |
|--------|--------|
| (a) 5 | (b) 40 |
| (c) 10 | (d) 20 |

[Lahore Board 2012]

(31) Circle passing through three vertices of a triangle is called:

- | | |
|------------------|----------------|
| (a) Circumcircle | (b) In-circle |
| (c) E-circle | (d) Semicircle |

[Lahore Board 2014, Lahore Board 2013]

(32) Radius of E-circle, r_1 is equal to:

- | | |
|----------------------------|-----------------------------------|
| (a) $\frac{\Delta}{S - a}$ | (b) $\frac{\Delta}{\sqrt{S - a}}$ |
| (c) $\frac{\Delta}{S - C}$ | (d) $\frac{\Delta}{S}$ |

[Lahore Board 2013]

(33) $r_1 =$

- | | |
|----------------------------|----------------------------|
| (a) $\frac{\Delta}{S - b}$ | (b) $\frac{\Delta}{S - a}$ |
| (c) $\frac{\Delta}{S - C}$ | (d) $\frac{S - a}{\Delta}$ |

[Lahore Board 2013, Lahore Board 2015]

(34) If a, b, c are sides of a triangle then $\frac{C^2 + a^2 - b^2}{2aC} =$

- | | |
|------------------|-----------------|
| (a) $\cos\alpha$ | (b) $\cos\beta$ |
| (c) $\cos\gamma$ | (d) $\sin\beta$ |

[Lahore Board 2013]

(35) The angle made by an object at a lower position is called:

- | | |
|------------------------|-------------------------|
| (a) Angle of elevation | (b) Angle of depression |
| (c) Right angle | (d) None of these |

[Lahore Board 2014]

- (36) By law of cosines $\cos \alpha =$ _____
- (a) $\frac{b^2 - c^2 - a^2}{2bc}$ (b) $\frac{a^2 + b^2 - c^2}{2bc}$
- (c) $\frac{c^2 + a^2 - b^2}{2ca}$ (d) $\frac{b^2 + c^2 - a^2}{2bc}$
- (37) A triangle can be constructed if:
- (a) One side is given (b) 2 sides are given
- (c) Three sides are given (d) Three angle are given
- (38) If $\alpha = 35^\circ 17'$, $\beta = 45^\circ 13'$ then $\gamma = ?$
- (a) 99° (b) $99^\circ 30'$
- (c) $99^\circ 35'$ (d) $99^\circ 40'$
- (39) If $b = 3$, $c = 5$ and $\alpha = 120^\circ$ then $a =$
- (a) 5 (b) 6
- (c) 7 (d) 8
- (40) In escribed circles $r_3 =$
- (a) $\frac{\Delta}{S}$ (b) $\frac{\Delta}{S - a}$
- (c) $\frac{\Delta}{S - b}$ (d) $\frac{\Delta}{S - c}$
- (41) If the measures of the sides of a triangle ABC are 17, 10, 21 then $r_2 =$
- (a) 12 (b) 6
- (c) 84 (d) 8
- (42) In a triangle ABC, with usual notations $2R =$
- (a) $\frac{b}{\sin \beta}$ (b) $\frac{c}{\sin \gamma}$
- (c) Both 'a' and 'b' (d) None of these
- (43) The point of intersection of the right bisectors of the sides of a triangle is called:
- (a) Circum-centre (b) In-centre
- (c) Ex-centre (d) None of these
- (44) The point of intersection of the bisectors of angles of the triangle is called:
- (a) Circum-centre (b) In-centre
- (c) Ex-centre (d) None of these
- (45) In $\triangle ABC$, with usual notations $(S - a) \tan \frac{\alpha}{2} =$
- (a) r (b) r_1
- (c) r_2 (d) r_3

- (46) In an equilateral triangle $r_1 : R : r =$
- (a) $1 : 2 : 3$ (b) $1 : 2 : 2$
(c) $1 : 1 : 2$ (d) $3 : 2 : 1$
- (47) In a triangle ABC, with usual notations if $a = 7$, $b = 3$, $c = 5$, $\alpha =$
- (a) 90° (b) 120°
(c) 150° (d) 180°
- (48) In a triangle ABC, with usual notations if $\alpha = 57^\circ 21'$, $a = 36.21$, $\beta = 78^\circ 10'$, $b =$
- (a) 40 (b) 41.9
(c) 42.09 (d) 43.5
- (49) If $\sin \frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$ then $\frac{\alpha}{2}$ is:
- (a) An acute angle (b) An obtuse angle
(c) A right angle (d) None of these
- (50) $2 \sin^2 \frac{\alpha}{2} =$
- (a) $1 + \cos \alpha$ (b) $\cos^2 \alpha - \sin^2 \alpha$
(c) $1 - \cos \alpha$ (d) $\sin^2 \alpha - \cos^2 \theta$
- (51) A triangle which is not right angle is called:
- (a) Equilateral triangle (b) Oblique triangle
(c) Isosceles triangle (d) Acute triangle
- (52) Sum of opposite angles of a cyclic quadrilateral =
- (a) 90° (b) 180°
(c) 270° (d) 360°
- (53) In-radius of a circle is given by:
- (a) $R = \frac{a}{2 \sin \alpha}$ (b) $r = \frac{\Delta}{S}$
(c) $r_1 = \frac{\Delta}{S - a}$ (d) None of these

Answers

(1)	b	(2)	b	(3)	a	(4)	a	(5)	d
(6)	a	(7)	b	(8)	c	(9)	a	(10)	d
(11)	b	(12)	d	(13)	a	(14)	d	(15)	a
(16)	d	(17)	b	(18)	c	(19)	a	(20)	c
(21)	d	(22)	b	(23)	b	(24)	a	(25)	b
(26)	d	(27)	c	(28)	d	(29)	c	(30)	c
(31)	a	(32)	a	(33)	b	(34)	b	(35)	b
(36)	d	(37)	d	(38)	b	(39)	c	(40)	d
(41)	b	(42)	c	(43)	a	(44)	b	(45)	a
(46)	d	(47)	b	(48)	c	(49)	a	(50)	c
(51)	b	(52)	b	(53)	b				

Q.14 $a - b + c = a + b + c - b - b$
 $= a + b + c - 2b$
 $= 2S - 2b$
 $= 2(S - b)$

Q.15 As radius of circum-circle $= R = \frac{abc}{4\Delta}$

$$\Rightarrow \Delta = \frac{abc}{4R}$$

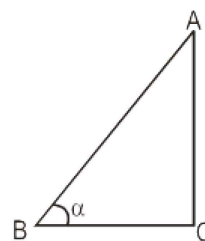
Q.20 As,

$$r_1 = \frac{\Delta}{S - a}$$

$$\Rightarrow \frac{1}{r_1} = \frac{S - a}{\Delta}$$

Q.22 As,

$$\begin{aligned} \cos \alpha &= \frac{\text{base}}{\text{hypoteneuse}} \\ &= \frac{30.8}{37.2} \\ &= 0.8279 \end{aligned}$$



Q.32 As

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha \\ 2bc \cos \alpha &= b^2 + c^2 - a^2 \\ \cos \alpha &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

Q.35 As,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha \\ &= (3)^2 + (5)^2 - 2(3)(5) \cos(120^\circ) \\ &= 9 + 25 - 30 \left(-\frac{1}{2} \right) \\ &= 9 + 25 + 5 \\ &= 49 \\ a &= 7 \end{aligned}$$

Q.37 As, $r_2 = \frac{\Delta}{S - b}$

And, $S = \frac{a + b + c}{2} = \frac{17 + 10 + 21}{2} = \frac{48}{2} = 24$

$$\Delta = \sqrt{S(S - a)(S - b)(S - c)}$$

$$= \sqrt{24(7)(14)(3)} = 84$$

So, $r_2 = \frac{\Delta}{S - b} = \frac{84}{24 - 10} = \frac{84}{14} = 6$

Q.41 $(S - a) \tan \frac{\alpha}{2} = (S - a) \sqrt{\frac{(S - b)(S - c)}{S(S - a)}}$

$$= \sqrt{\frac{(S - a)(S - b)(S - c)}{S}}$$

$$= \sqrt{\frac{S(S - a)(S - b)(S - c)}{S^2}}$$

$$= \frac{\Delta}{S} = r$$

Q.42 As triangle is equilateral.

$$\Rightarrow a = b = c$$

$$\Rightarrow S = \frac{a + a + a}{2} = \frac{3a}{2}$$

$$\Delta = \sqrt{\frac{3a}{2} \left(\frac{3a}{2} - a \right) \left(\frac{3a}{2} - a \right) \left(\frac{3a}{2} - a \right)}$$

$$= \frac{\sqrt{3} a^2}{4}$$

$$r_1 = \frac{\Delta}{S - a} = \frac{\frac{\sqrt{3} a^2}{4}}{\frac{3a}{2} - a} = \frac{\sqrt{3} a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}}{2} a$$

$$R = \frac{a \cdot a \cdot a}{4\Delta} = \frac{a^3}{4 \left(\frac{\sqrt{3} a^2}{4} \right)} = \frac{a}{\sqrt{3}}$$

$$r = \frac{\Delta}{S} = \frac{\frac{\sqrt{3} a^2}{4}}{3a/2} = \frac{a}{2\sqrt{3}}$$

$$\begin{aligned}
 r_1 : R : r &= \frac{\sqrt{3} a}{2} & : & \frac{a}{\sqrt{3}} & : & \frac{a}{2\sqrt{3}} \\
 &= \frac{\sqrt{3} \cdot \sqrt{3} a}{2\sqrt{3}} & : & \frac{a}{\sqrt{3}} & : & \frac{a}{2\sqrt{3}} \\
 &= \frac{3}{2} & : & 1 & : & \frac{1}{2}
 \end{aligned}$$

Multiplying by 2.

$$= 3 : 2 : 1$$

Q.43 As,

$$\begin{aligned}
 a^2 &= b^2 + c^2 - 2bc \cos \alpha \\
 \cos \alpha &= \frac{b^2 + c^2 - a^2}{2bc} \\
 &= \frac{(3)^2 + (5)^2 - (7)^2}{2(3)(5)} \\
 &= \frac{9 + 25 - 49}{30} \\
 &= \frac{-15}{30} = -\frac{1}{2} \\
 \alpha &= \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ
 \end{aligned}$$

Q.44 As,

$$\begin{aligned}
 \frac{a}{\sin \alpha} &= \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \\
 \Rightarrow b &= \frac{a \sin \beta}{\sin \alpha} = \frac{(36.21) \sin 78^\circ 10'}{\sin 57^\circ 21'} = \frac{(36.21)(0.9788)}{0.8420} \\
 &= 42.09
 \end{aligned}$$