Chapter 4

QUADRATIC EQUATIONS

QUADRATIC EQUATION

A Quadratic Equation in x is an equation that can be written in the form of $a x^2 + bx + c = 0$; where a, b, c are real numbers and $a \neq 0$, for example

$x^2 - 7x + 10 = 0$; $a = 1$, $b = -7$, $c = 10$

$x^2 = 4$; $a = 1$, $b = 0$, $c = -4$

Another name for a quadratic equation in x is 2^{nd} degree polynomial.

There are three basic techniques for solving a quadratic equation.

(i) By Factorization

(ii) By Completing Squares

(iii) By Applying the Quadratic Formula

EXERCISE 4.1

Solve the following equation by factorization:

Q.1 $\mathbf{3x^2 + 4x + 1 = 0}$

Solution:

$3x^2 + 4x + 1 = 0$

$\Rightarrow \quad 3x^2 + 3x + x + 1 = 0$

$\Rightarrow \quad 3x(x + 1) + 1(x + 1) = 0$

$\Rightarrow \quad (3x + 1)(x + 1) = 0$

Either $(3x + 1) = 0$ or $x + 1 = 0$

$3x = -1$ $\qquad$ $x = -1$

$x = -\frac{1}{3}$ $\qquad$ $x = -1$

Hence the solution set $= \left\{-\frac{1}{3}, -1\right\}$

Q.2 $x^2 + 7x + 12 = 0$ **(Lahore Board 2007)**

Solution:

$x^2 + 7x + 12 = 0$

$\Rightarrow \quad x^2 + 4x + 3x + 12 = 0$

$\Rightarrow \quad x(x + 4) + 3(x + 4) = 0$

$\Rightarrow \quad x(x + 4) + (x + 3) = 0$

Either $x + 4 = 0$ or $x + 3 = 0$

$x = -4$ $x = -3$

Hence the solution set $= \{-4, -3\}$

Q.3 $9x^2 - 12x - 5 = 0$

Solution:

$9x^2 - 12x - 5 = 0$

$\Rightarrow \quad 9x^2 - 15x + 3x - 5 = 0$

$\Rightarrow \quad 3x(3x - 5) + 1(3x - 5) = 0$

$\Rightarrow \quad (3x - 5)(3x + 1) = 0$

$\Rightarrow$ Either $3x - 5 = 0$ or $3x + 1 = 0$

$\Rightarrow \quad 3x = 5$ $3x = -1$

$\Rightarrow \quad x = \frac{5}{3}$ $x = -\frac{1}{3}$

Hence the solution set $= \left\{-\frac{1}{3}, \frac{5}{3}\right\}$

Q.4 $x^2 - x = 2$

Solution:

$x^2 - x = 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x - 2) + 1(x - 2) = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

$\Rightarrow$ Either $x - 2 = 0$ or $x + 1 = 0$

$\Rightarrow \quad x = 2,$ $x = -1$

Hence the solution set $= \{-1, 2\}$

Q.5 $x(x + 7) = (2x - 1)(x + 4)$ **(Lahore Board 2011)**

Solution:

$x(x + 7) = (2x - 1)(x + 4)$

$\Rightarrow \quad x^2 + 7x = 2x^2 + 8x - x - 4$

$\Rightarrow \quad 2x^2 + 8x - x - 4 - x^2 - 7x = 0$

$\Rightarrow \quad x^2 - 4 = 0$

$\Rightarrow \quad (x^2) - (2)^2 = 0$

$\Rightarrow \quad (x + 2)(x - 2) = 0$

$\Rightarrow$ Either $x + 2 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = -2,$ $x = 2$

Hence the solution set $= \{-2, 2\}$

Q.6 $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{5}{2}; \quad x \neq -1, 0$

Solution:

$$\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$$

$$\Rightarrow \quad \frac{x^2 + (x+1)^2}{x(x+1)} = \frac{5}{2}$$

$$\Rightarrow \quad \frac{x^2 + x^2 + 2x + 1}{x^2 + x} = \frac{5}{2}$$

$$\Rightarrow \quad \frac{2x^2 + 2x + 1}{x^2 + x} = \frac{5}{2}$$

$$\Rightarrow \quad 5(x^2 + x) = 2(2x^2 + 2x + 1)$$

$$\Rightarrow \quad 5x^2 + 5x = 4x^2 + 4x + 2$$

$$\Rightarrow \quad 5x^2 + 5x - 4x^2 - 4x - 2 = 0$$

$$\Rightarrow \quad x^2 + x - 2 = 0$$

$$\Rightarrow \quad x^2 + 2x - x - 2 = 0$$

$$\Rightarrow \quad x(x+2) - 1(x+2) = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad \text{Either} \quad x + 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\Rightarrow \quad x = -2 \quad \text{or} \quad x = 1$$

Hence the solution set $= \{-2, 1\}$

Q.7 $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{7}{x+5}; \quad x \neq -1, -2, -5$

Solution:

$$\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}$$

$$\Rightarrow \quad \frac{x + 2 + 2(x+1)}{(x+1)(x+2)} = \frac{7}{x+5}$$

$$\Rightarrow \quad \frac{x + 2 + 2x + 2}{x^2 + 2x + x + 2} = \frac{7}{x+5}$$

$$\Rightarrow \quad \frac{3x + 4}{x^2 + 3x + 2} = \frac{7}{x+5}$$

$$\Rightarrow \quad 7(x^2 + 3x + 2) = (x+5)(3x+4)$$

$\Rightarrow \quad 7x^2 + 21x + 14 = 3x^2 + 4x + 15x + 20$

$\Rightarrow \quad 7x^2 + 21x + 14 - 3x^2 - 4x - 15x - 20 = 0$

$\Rightarrow \quad 4x^2 + 2x - 6 = 0$

$\Rightarrow \quad 2(2x^2 + x - 3) = 0$

$\Rightarrow \quad 2x^2 + x - 3 = 0$

$\Rightarrow \quad 2x^2 + 3x - 2x - 3 = 0$

$\Rightarrow \quad x(2x + 3) - 1(2x + 3) = 0$

$\Rightarrow \quad (2x + 3)(x - 1) = 0$

$\Rightarrow \quad$ Either $2x + 3 = 0$ or $x - 1 = 0$

$\Rightarrow \quad 2x = -3 \qquad x = 1$

$\Rightarrow \quad x = -\frac{3}{2}$

Hence the solution set $= \left\{-\frac{3}{2}, 1\right\}$

Q.8 $\dfrac{\mathbf{a}}{\mathbf{ax-1}} + \dfrac{\mathbf{b}}{\mathbf{bx-1}} = \mathbf{a+b}\,; \quad \mathbf{x \neq \frac{1}{a}, \frac{1}{b}}$ **(Lahore Board 2008)**

Solution:

$$\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$$

$$\Rightarrow \quad \frac{a}{ax-1} - b + \frac{b}{bx-1} - a = 0$$

$$\Rightarrow \quad \frac{a - b(ax-1)}{ax-1} + \frac{b - a(bx-1)}{bx-1} = 0$$

$$\Rightarrow \quad \frac{a - abx + b}{ax-1} + \frac{b - abx + a}{bx-1} = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{1}{ax-1} + \frac{1}{bx-1}\right] = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{bx - 1 + ax - 1}{(ax-1)(bx-1)}\right] = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{ax + bx - 2}{(ax-1)(bx-1)}\right] = 0$$

$$\Rightarrow \quad \text{Either } a + b - abx = 0 \quad \text{or} \quad \frac{ax + bx - 2}{(ax-1)(bx-1)} = 0$$

$\Rightarrow$ $abx = a + b$ or $ax + bx - 2 = 0$

$\Rightarrow$ $x = \dfrac{a+b}{ab}$ or $x(a+b) - 2 = 0$

$x(a+b) = 2$

$x = \dfrac{2}{a+b}$

Hence the solution set $= \left\{ \dfrac{2}{a+b}, \dfrac{a+b}{ab} \right\}$

Solve the following equations by completing the square:

Q.9 $\mathbf{x^2 - 2x - 899 = 0}$ **(Lahore Board 2007)**

Solution:

$x^2 - 2x - 899 = 0$

$\Rightarrow$ $x^2 - 2x = 899$

add $\left(\dfrac{2}{2}\right)^2 = (1)^2$ on both sides

$\Rightarrow$ $x^2 - 2x + (1)^2 = 899 + (1)^2$

$\Rightarrow$ $(x-1)^2 = 900$

$\Rightarrow$ $(x-1)^2 = (30)^2$

$\Rightarrow$ $(x-1) = \pm 30$

$\Rightarrow$ Either $x - 1 = 30$ or $x - 1 = -30$

$\Rightarrow$ $x = 31$ or $x = -29$

Hence the solution set $= \{-29, 31\}$

Q.10 $\mathbf{x^2 + 4x - 1085 = 0}$

Solution:

$x^2 + 4x - 1085 = 0$

$\Rightarrow$ $x^2 + 4x = 1085$

add $\left(\dfrac{4}{2}\right)^2 = (2)^2$ to both sides

$\Rightarrow$ $x^2 + 4x + (2)^2 = 1085 + (2)^2$

$\Rightarrow$ $(x+2)^2 = 1089$

$\Rightarrow$ $(x+2)^2 = (33)^2$

$\Rightarrow \quad x + 2 = \pm 33$

$\Rightarrow$ Either $x + 2 = 33$ or $x + 2 = -33$

$x = 31$ or $x = -35$

Hence the solution set $= \{-35, 31\}$

Q.11 $\mathbf{x^2 + 6x - 567 = 0}$

Solution:

$x^2 + 6x - 567 = 0$

$\Rightarrow \quad x^2 + 6x = 567$

add $\left(\frac{6}{2}\right)^2 = (3)^2$ to both sides

$\Rightarrow \quad x^2 + 6x + (3)^2 = 567 + (3)^2$

$\Rightarrow \quad (x + 3)^2 = 567$

$\Rightarrow \quad (x + 3)^2 = (24)^2$

$\Rightarrow \quad x + 3 = \pm 24$

$\Rightarrow$ Either $x + 3 = 24$ or $x + 3 = -24$

$\Rightarrow \quad x = 21$ or $x = -27$

Hence the solution set $= \{-27, 21\}$

Q.12 $\mathbf{x^2 - 3x - 648 = 0}$

Solution:

$x^2 - 3x - 648 = 0$

$\Rightarrow \quad x^2 - 3x = 648$

add $\left(\frac{3}{2}\right)^2$ to both sides

$\Rightarrow \quad x^2 - 3x + \left(\frac{3}{2}\right)^2 = 648 + \left(\frac{3}{2}\right)^2$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = 648 + \frac{9}{4}$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = \frac{2601}{4}$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = \left(\frac{51}{2}\right)^2$

$\Rightarrow \quad x - \frac{3}{2} = \pm\frac{51}{2}$

$\Rightarrow \quad$ Either $x - \frac{3}{2} = \frac{51}{2}$ or $x - \frac{3}{2} = -\frac{51}{2}$

$\Rightarrow \quad x = \frac{51}{2} + \frac{3}{2}$ or $x = \frac{-51}{2} + \frac{3}{2}$

$\Rightarrow \quad x = \frac{54}{2} = 27$ or $x = \frac{-48}{2} = -24$

Hence the solution set $= \{-24, 27\}$

Q.13 $\mathbf{x^2 - x - 1806 = 0}$

Solution:

$x^2 - x - 1806 = 0$

$\Rightarrow \quad x^2 - x = 1806$

add $\left(\frac{1}{2}\right)^2$ to both sides

$\Rightarrow \quad x^2 - x + \left(\frac{1}{2}\right)^2 = 1806 + \left(\frac{1}{2}\right)^2$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = 1806 + \frac{1}{4}$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = \frac{7225}{4} = \left(\frac{85}{2}\right)^2$

$\Rightarrow \quad x - \frac{1}{2} = \pm\frac{85}{2}$

$\Rightarrow \quad$ Either $x - \frac{1}{2} = \frac{85}{2}$ or $x - \frac{1}{2} = -\frac{85}{2}$

$\Rightarrow \quad x = \frac{85}{2} + \frac{1}{2}$ or $x = \frac{-85}{2} + \frac{1}{2}$

$\Rightarrow \quad x = \frac{86}{2} = 43$ or $x = -\frac{84}{2} = -42$

Hence the solution set $= \{-42, 43\}$

Q.14 $\mathbf{2x^2 + 12x - 110 = 0}$

Solution:

$2x^2 + 12x - 110 = 0$

$\Rightarrow \quad 2(x^2 + 6x - 55) = 0$

$\Rightarrow \quad x^2 + 6x - 55 = 0$

$\Rightarrow \quad x^2 + 6x = 55$

add $\left(\frac{6}{2}\right)^2 = (3)^2$ to both sides

$\Rightarrow \quad x^2 + 6x + (3)^2 = 55 + (3)^2$

$\Rightarrow \quad (x + 3)^2 = 55 + 9$

$\Rightarrow \quad (x + 3)^2 = 64$

$\Rightarrow \quad (x + 3)^2 = (8)^2$

$\Rightarrow \quad (x + 3) = \pm 8$

$\Rightarrow \quad$ Either $x + 3 = 8$ or $x + 3 = -8$

$x = 5$ or $x = -11$

Hence the solution set $= \{-11, 5\}$

Find roots of the following equations by using quadratic formula.

Q.15 $5x^2 - 13x + 6 = 0$

Solution:

$5x^2 - 13x + 6 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 5, \quad b = -13, \quad c = 6$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(5)(6)}}{2(5)}$$

$$x = \frac{13 \pm \sqrt{169 - 120}}{10} = \frac{13 \pm \sqrt{49}}{10} = \frac{13 \pm 7}{10}$$

$\Rightarrow \quad$ Either $x = \frac{13 + 7}{10}$ or $x = \frac{13 - 7}{10}$

$\Rightarrow \quad x = \frac{20}{10}$ or $x = \frac{6}{10}$

$\Rightarrow \quad x = 2$ or $x = \frac{3}{5}$

Hence the solution set $= \left\{\frac{3}{5}, 2\right\}$

Q.16 $4x^2 + 7x - 1 = 0$

Solution:

$4x^2 + 7x - 1 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 4, \quad b = 7, \quad c = -1$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(4)(-1)}}{2(4)} = \frac{-7 \pm \sqrt{49 + 16}}{8} = \frac{-7 \pm \sqrt{65}}{8}$$

$\Rightarrow$ Either $x = \frac{-7 + \sqrt{65}}{8}$ or $x = \frac{-7 - \sqrt{65}}{8}$

Hence the solution set $= \left\{ \frac{-7 + \sqrt{65}}{8}, \frac{-7 - \sqrt{65}}{8} \right\}$

Q.17 $15x^2 + 2ax - a^2 = 0$

Solution:

$15x^2 + 2ax - a^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 15 \quad b = 2a \quad c = -a^2$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2a \pm \sqrt{(2a)^2 - 4(15)(-a^2)}}{2(15)}$$

$$= \frac{-2a \pm \sqrt{4a^2 + 60a^2}}{30} = \frac{-2a \pm \sqrt{64a^2}}{30} = \frac{-2a \pm 8a}{30}$$

$\Rightarrow$ Either $x = \frac{-2a + 8a}{30}$ or $\frac{-2a - 8a}{30}$

$\Rightarrow$ $x = \frac{6}{30}$ or $\frac{-10a}{30}$

$\Rightarrow$ $x = \frac{a}{5}$ or $-\frac{a}{3}$

Hence the solution set $= \left\{ -\frac{a}{3}, \frac{a}{5} \right\}$

Q.18 $16x^2 + 8x + 1 = 0$

Solution:

$16x^2 + 8x + 1 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 16, \quad b = 8, \quad c = 1$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{(8)^2 - 4(16)(1)}}{2(16)} = \frac{-8 \pm \sqrt{64 - 64}}{32} = \frac{-8 \pm 0}{32} = \frac{-8}{32}$$

$$\Rightarrow \quad x = -\frac{1}{4}$$

Hence the solution set $= \left\{-\frac{1}{4}\right\}$

Q.19 $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

Solution:

$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

$$\Rightarrow \quad x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ac = 0$$

$$\Rightarrow \quad 3x^2 - 2ax - 2bx - 2cx + ab + bc + ac = 0$$

$$\Rightarrow \quad 3x^2 - 2(a + b + c)x + ab + bc + ac = 0$$

Comparing with $ax^2 + bx + c = 0$, we have $a = 3$, $b = -2(a + b + c)$, $c = ab + bc + ac$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2(a + b + c) \pm \sqrt{4(a + b + c)^2 - 4(3)(ab + bc + ac)}}{2(3)}$$

$$= \frac{2(a + b + c) \pm \sqrt{4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ac) - 12(ab + bc + ac)}}{6}$$

$$= \frac{2(a + b + c) \pm 2\sqrt{4a^2 + 4b^2 + 4c^2 + 8ab + 8bc + 8ac - 12ab - 12bc - 12ac}}{6}$$

$$= \frac{2(a + b + c) \pm \sqrt{4a^2 + 4b^2 + 4c^2 - 4ab - 4bc - 4ac}}{6}$$

$$= \frac{2(a+b+c) \pm 2\sqrt{a^2+b^2+c^2-ab-bc-ac}}{6}$$

$$= \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ac}}{3}$$

Hence the solution set $= \left\{ \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ac}}{3} \right\}$

Q.20 $(a+b)x^2 + (a+2b+c)x + b + c = 0$

Solution:

$(a+b)x^2 + (a+2b+c)x + b + c = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = a+b$, $b = a+2b+c$, $c = b+c$

So using

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(a+2b+c) \pm \sqrt{(a+2b+c)^2 - 4(a+b)(b+c)}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ac-4ab-4ac-4b^2-4bc}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{a^2+c^2-2ac}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{(a-c)^2}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm (a-c)}{2(a+b)}$$

$\Rightarrow$ Either $x = \frac{-(a+2b+c)+(a-c)}{2(a+b)}$ or $x = \frac{-(a+2b+c)-(a-c)}{2(a+b)}$

$= \frac{-a-2b-c+a-c}{2(a+b)}$ or $= \frac{-a-2b-c-a+c}{2(a+b)}$

$= \frac{-2b-2c}{2(a+b)}$ or $= \frac{-2a-2b}{2(a+b)}$

$= \frac{-2(b+c)}{2(a+b)}$ or $= \frac{-2(a+b)}{2(a+b)}$

$= -\frac{b+c}{a+b}$ or $= -1$

Hence the solution set $= \left\{ -1, -\frac{b+c}{a+b} \right\}$

SOLUTION OF EQUATIONS REDUCIBLE TO THE QUADRATIC EQUATION

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic form.

Type I

The equations of the form $ax^{2n} + bx^n + c = 0;\ a \neq 0$. Put $x^n = y$ and get the given equation reduced to quadratic equation in y.

Type II

The equations of the form $(x + a)(x + b)(x + c)(x + d) = K$ where $a + b = c + d$.

Type III: EXPONENTIAL EQUATIONS

Equations in which variable occurs in exponent are called exponential equations. For example $2^{2x} - 3.2^{x+2} + 32 = 0$.

Type IV: RECIPROCAL EQUATIONS

An equation, which remains unchanged when x is replaced by $\frac{1}{x}$, is called a reciprocal equation. For example $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$.

EXERCISE 4.2

Solve the following equations:

Q.1 $\mathbf{x^4 - 6x^2 + 8 = 0.}$

Solution:

$x^4 - 6x^2 + 8 = 0$

Put $x^2 = y$ ..... (1)

$y^2 - 6y + 8 = 0$

$\Rightarrow \quad y^2 - 4y - 2y + 8 = 0$

$\Rightarrow \quad y(y - 4) - 2(y - 4) = 0$

$\Rightarrow \quad (y - 4)(y - 2) = 0$

$\Rightarrow \quad$ Either $y - 4 = 0$ or $y - 2 = 0$

$\Rightarrow \quad y = 4$ or $y = 2$

When $y = 2$ equation (1) becomes

$x^2 = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$

when $y = 4$ equation (1) becomes

$x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

Hence the solution set $= \{ \pm\sqrt{2}, \pm 2 \}$

Q.2 $x^{-2} - 10 = 3x^{-1}$. **(Gujranwala Board 2007, Lahore Board 2010)**

Solution:

$$x^{-2} - 10 = 3x^{-1}$$

$$\Rightarrow \quad x^{-2} - 3x^{-1} - 10 = 0$$

$$\Rightarrow \quad (x^{-1})^2 - 3x^{-1} - 10 = 0$$

Put $x^{-1} = y$ ..... (1)

$$\Rightarrow \quad y^2 - 3y - 10 = 0$$

$$\Rightarrow \quad y^2 - 5y + 2y - 10 = 0$$

$$\Rightarrow \quad y(y-5) + 2(y-5) = 0$$

$$\Rightarrow \quad (y-5)(y+2) = 0$$

$$\Rightarrow \quad \text{Either } y - 5 = 0 \quad \text{or} \quad y + 2 = 0$$

$$\Rightarrow \quad y = 5, \quad \text{or} \quad y = -2$$

when $y = 5$ equation (1) becomes

$$x^{-1} = 5$$

$$\Rightarrow \quad \frac{1}{x} = 5$$

$$\Rightarrow \quad x = \frac{1}{5}$$

when $y = -2$ equation (1) becomes

$$x^{-1} = -2 \quad \Rightarrow \quad \frac{1}{x} = -2 \Rightarrow x = -\frac{1}{2}$$

Hence the solution set $= \left\{-\frac{1}{2}, \frac{1}{5}\right\}$

Q.3 $x^6 - 9x^3 + 8 = 0$ **(Gujranwala Board 2007)**

Solution:

$$x^6 - 9x^3 + 8 = 0$$

$$\Rightarrow \quad (x^3)^2 - 9x^3 + 8 = 0$$

Put $x^3 = y$ ..... (1)

$$\Rightarrow \quad y^2 - 9y + 8 = 0$$

$$\Rightarrow \quad y^2 - 8y - y + 8 = 0$$

$$\Rightarrow \quad y(y-8) - 1(y-8) = 0$$

$\Rightarrow \quad (y-8)(y-1) = 0$

$\Rightarrow \quad$ Either $y-8 = 0 \quad$ or $\quad y-1 = 0$

$\Rightarrow \quad y = 8 \quad$ or $\quad y = 1$

when $y = 8$, equation (1) becomes

$x^3 = 8$

$\Rightarrow \quad x^3 - 8 = 0 \quad \Rightarrow \quad (x)^3 - (2)^3 = 0$

$\Rightarrow \quad (x-2)(x^2 + 2x + 4) = 0$

$\Rightarrow \quad$ Either $x-2 = 0 \quad$ or $\quad x^2 + 2x + 4 = 0$

$\Rightarrow \quad x = 2 \quad$ or $\quad x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= -1 \pm \sqrt{-3}$$

when $y = 1$ equation (1) becomes

$x^3 = 1 \quad \Rightarrow \quad (x)^3 - (1)^3 = 0$

$\Rightarrow \quad (x-1)(x^2 + x + 1) = 0$

$\Rightarrow \quad$ Either $x-1 = 0 \quad$ or $\quad x^2 + x + 1 = 0$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

Hence the solution set $= \left\{ 1, 2, \dfrac{-1 \pm \sqrt{-3}}{2}, -1 \pm \sqrt{-3} \right\}$

Q.4 $\mathbf{8x^6 - 19x^3 - 27 = 0}$

Solution:

$8x^6 - 19x^3 - 27 = 0$

$\Rightarrow \quad 8(x^3)^2 - 19x^3 - 27 = 0$

Put $x^3 = y \quad \ldots\ldots\ldots (1)$

$\Rightarrow \quad 8y^2 - 19y - 27 = 0$

$\Rightarrow \quad 8y^2 - 27y + 8y - 27 = 0$

$\Rightarrow \quad y(8y - 27) + 1(8y - 27) = 0$

$\Rightarrow \quad (8y - 27)(y + 1) = 0$

$\Rightarrow \quad$ Either $8y - 27 = 0$ or $y + 1 = 0$

$\Rightarrow \quad y = \frac{27}{8}$ or $y = -1$

when $y = \frac{27}{8}$ equation (1) becomes

$x^3 = \frac{27}{8} \Rightarrow 8x^3 = 27 \Rightarrow 8x^3 - 27 = 0$

$\Rightarrow \quad (2x)^3 - (3)^3 = 0$

$\Rightarrow \quad (2x - 3)(4x^2 + 6x + 9) = 0$

$\Rightarrow \quad$ Either $2x - 3 = 0$ or $4x^2 + 6x + 9 = 0$

$\Rightarrow \quad 2x = 3$ or $x = \frac{-6 \pm \sqrt{(6)^2 - 4(4)(9)}}{2(4)}$

$\Rightarrow \quad x = \frac{3}{2} = \frac{-6 \pm \sqrt{36 - 144}}{8}$

$= \frac{-6 \pm \sqrt{-108}}{8}$

$= \frac{-6 \pm 6\sqrt{-3}}{8}$

$= \frac{-3 \pm 3\sqrt{-3}}{4}$

when $y = -1$ equation (1) becomes

$x^3 = -1$

$\Rightarrow \quad x^3 + 1 = 0$

$\Rightarrow \quad (x + 1)(x^2 - x + 1) = 0$

$\Rightarrow \quad$ Either $x + 1 = 0$ or $x^2 - x + 1 = 0$

$\Rightarrow \quad x = -1$ or $x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$\Rightarrow \quad = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$

Hence the solution set $= \left\{-1, \frac{3}{2}, \frac{1 \pm \sqrt{-3}}{2}, \frac{-3 \pm 3\sqrt{-3}}{4}\right\}$

Q.5 $x^{2/5} + 8 = 6x^{1/5}$

Solution:

$x^{2/5} + 8 = 6x^{1/5}$

$x^{2/5} - 6x^{1/5} + 8 = 0$

$(x^{1/5})^2 - 6x^{1/5} + 8 = 0$

Put $x^{1/5} = y$ ..... (1)

$\Rightarrow$ $y^2 - 6y + 8 = 0$

$\Rightarrow$ $y^2 - 4y - 2y + 8 = 0$

$\Rightarrow$ $y(y-4) - 2(y-4) = 0$

$\Rightarrow$ $(y-4)(y-2) = 0$

$\Rightarrow$ Either $y - 4 = 0$ or $y - 2 = 0$

$\Rightarrow$ $y = 4$ or $y = 2$

when $y = 4$ equation (1) becomes

$x^{1/5} = 4 \Rightarrow x = (4)^5 \Rightarrow x = 1024$

when $y = 2$ equation (1) becomes

$x^{1/5} = 2 \Rightarrow x = 2^5 \Rightarrow x = 32$

Hence the solution set $= \{32, 1024\}$

Q.6 $(x + 1)(x + 2)(x + 3)(x + 4) = 24$

Solution:

$(x + 1)(x + 2)(x + 3)(x + 4) = 24$

As $1 + 4 = 2 + 3$

$\Rightarrow$ we combine

$[(x + 1)(x + 4)][(x + 2)(x + 3)] = 24$

$\Rightarrow$ $[x^2 + 4x + x + 4)][(x^2 + 3x + 2x + 6)] = 24$

$\Rightarrow$ $[x^2 + 5x + 4)][(x^2 + 5x + 6)] = 24$

Put $x^2 + 5x = y$ ..... (1)

$\Rightarrow \quad [y + 4]\,[y + 6] = 24$

$\Rightarrow \quad y^2 + 6y + 4y = 24 = 24$

$\Rightarrow \quad y^2 + 10y = 0$

$\Rightarrow \quad y\,(y + 10) = 0$

$\Rightarrow \quad$ Either $y = 0$ or $y + 10 = 0$

$\Rightarrow \quad$ or $y = -10$

Put $y = 0$ in equation (1)

$\Rightarrow \quad x^2 + 5x = 0$

$\Rightarrow \quad x\,(x + 5) = 0$

$\Rightarrow \quad$ Either $x = 0$ or $x + 5 = 0$

$\Rightarrow \quad$ or $x = -5$

Put $y = -10$ in equation (1)

$\Rightarrow \quad x^2 + 5x = -10$

$\Rightarrow \quad x^2 + 5x + 10 = 0$

$\Rightarrow \quad x = \dfrac{-5 \pm \sqrt{(5)^2 - 4\,(1)\,(10)}}{2\,(1)} = \dfrac{-5 \pm \sqrt{25 - 40}}{2} = \dfrac{-5 \pm \sqrt{-15}}{2}$

Hence the solution set $= \left\{ -5\,,\, 0\,,\, \dfrac{-5 \pm \sqrt{-15}}{2} \right\}$

Q.7 $(x - 1)\,(x + 5)\,(x + 8)\,(x + 2) - 880 = 0$

Solution:

$(x - 1)\,(x + 5)\,(x + 8)\,(x + 2) - 880 = 0$

As $-1 + 8 = 5 + 2$

So we combine

$[(x - 1)\,(x + 8)]\,[(x + 5)\,(x + 2)] - 880 = 0$

$\Rightarrow \quad [x^2 + 8x - x - 8]\,[x^2 + 5x + 2x + 10] - 880 = 0$

$\Rightarrow \quad [x^2 + 7x - 8]\,[x^2 + 7x + 10] - 880 = 0$

Put $x^2 + 7x = y$ ..... (1)

$\Rightarrow \quad [y - 8]\,[y + 10] - 880 = 0$

$\Rightarrow \quad y^2 + 10y - 8y - 80 - 880 = 0$

$\Rightarrow \quad y^2 + 2y - 960 = 0$

$\Rightarrow \quad y^2 + 32y - 30y - 960 = 0$

$\Rightarrow \quad y(y + 32) - 30(y + 32) = 0$

$\Rightarrow \quad (y + 32)(y - 30) = 0$

$\Rightarrow \quad$ Either $y + 32 = 0$ or $y - 30 = 0$

$\Rightarrow \quad y = -32$ or $y = 30$

Put $y = -32$ and $y = 30$ in equation (1)

$\Rightarrow \quad x^2 + 7x = -32$

$\Rightarrow \quad x^2 + 7x + 32 = 0$

$\Rightarrow \quad x = \dfrac{-7 \pm \sqrt{(7)^2 - 4(1)(32)}}{2(1)}$

$\Rightarrow \quad = \dfrac{-7 \pm \sqrt{49 - 128}}{2}$

$\Rightarrow \quad = \dfrac{-7 \pm \sqrt{-79}}{2}$

$\Rightarrow \quad x^2 + 7x = 30$

$\Rightarrow \quad x^2 + 7x - 30 = 0$

$\Rightarrow \quad x^2 + 10x - 3x - 30 = 0$

$\Rightarrow \quad x(x + 10) - 3(x + 10) = 0$

$\Rightarrow \quad (x + 10)(x - 3) = 0$

$\Rightarrow \quad$ Either $x + 10 = 0$ or $x - 3 = 0$

$\Rightarrow \quad x = -10$ or $x = 3$

Hence the solution set $= \left\{ -10, 3, \dfrac{-7 \pm \sqrt{-79}}{2} \right\}$

Q.8 $(x - 5)(x - 7)(x + 6)(x + 4) - 504 = 0$

Solution:

$(x - 5)(x - 7)(x + 6)(x + 4) - 504 = 0$

As $-5 + 4 = -7 + 6$

So we combine

$\Rightarrow \quad [(x - 5)(x + 4)][(x - 7)(x + 6)] - 504 = 0$

$\Rightarrow \quad [x^2 + 4x - 5x - 20][x^2 + 6x - 7x - 42] - 504 = 0$

$\Rightarrow \quad [x^2 - x - 20][x^2 - x - 42] - 504 = 0$

Put $x^2 - x = y$ ..... (1)

$\Rightarrow \quad [y - 20][y - 42] - 504 = 0$

$\Rightarrow \quad y^2 - 42y - 20y + 840 - 504 = 0$

$\Rightarrow \quad y^2 - 62y + 336 = 0$

$\Rightarrow \quad y^2 - 56y - 6y + 336 = 0$

$\Rightarrow \quad y(y - 56) - 6(y - 56) = 0$

$\Rightarrow \quad (y - 56)(y - 6) = 0$

$\Rightarrow \quad$ Either $y - 56 = 0$ or $y - 6 = 0$

$\Rightarrow \quad y = 56$ or $y = 6$

Put $y = 56$ and $y = 6$ in equation (1)

$\Rightarrow \quad x^2 - x = 56$ $\qquad$ $\Rightarrow \quad x^2 - x = 6$

$\Rightarrow \quad x^2 - x - 56 = 0$ $\qquad$ $\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad x^2 - 8x + 7x - 56 = 0$ $\qquad$ $\Rightarrow \quad x^2 - 3x + 2x - 6 = 0$

$\Rightarrow \quad x(x - 8) + 7(x - 8) = 0$ $\qquad$ $\Rightarrow \quad x(x - 3) + 2(x - 3) = 0$

$\Rightarrow \quad (x - 8)(x + 7) = 0$ $\qquad$ $\Rightarrow \quad (x - 3)(x + 2) = 0$

$\Rightarrow \quad$ Either $x - 8 = 0$ or $x + 7 = 0$ $\qquad$ $\Rightarrow \quad$ Either $x - 3 = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = 8$ or $x = -7$ $\qquad$ $\Rightarrow \quad x = 3$ or $x = -2$

Hence the solution set $= \{-7, -2, 3, 8\}$

Q.9 $(x - 1)(x - 2)(x - 8)(x + 5) + 360 = 0$

Solution:

$(x - 1)(x - 2)(x - 8)(x + 5) + 360 = 0$

So we combine

$[(x - 1)(x - 2)][(x - 8)(x + 5)] + 360 = 0$

$\Rightarrow \quad [x^2 - 2x - x + 2][x^2 + 5x - 8x - 40] + 360 = 0$

$\Rightarrow \quad [x^2 - 3x + 2][x^2 - 3x - 40] + 360 = 0$

Put $x^2 - 3x = y$ $\qquad$ ..... (1)

$\Rightarrow \quad [y + 2][y - 40] + 360 = 0$

$\Rightarrow \quad y^2 - 40y + 2y - 80 + 360 = 0$

$\Rightarrow \quad y^2 - 38y + 280 = 0$

$\Rightarrow \quad y^2 - 28y - 10y + 280 = 0$

$\Rightarrow \quad y(y - 28) - 10(y - 28) = 0$

$\Rightarrow \quad$ Either $y - 28 = 0$ or $y - 10 = 0$

$\Rightarrow \quad y = 28$ or $y = 10$

Put $y = 28$ and $y = 10$ in equation (1)

$\Rightarrow \quad x^2 - 3x = 28$ $\Rightarrow \quad x^2 - 3x = 10$

$\Rightarrow \quad x^2 - 3x - 28 = 0$ $\Rightarrow \quad x^2 - 3x - 10 = 0$

$\Rightarrow \quad x^2 - 7x + 4x - 28 = 0$ $\Rightarrow \quad x^2 - 5x + 2x - 10 = 0$

$\Rightarrow \quad x(x-7) + 4(x-7) = 0$ $\Rightarrow \quad x(x-5) + 2(x-5) = 0$

$\Rightarrow \quad (x-7)(x+4) = 0$ $\Rightarrow \quad (x-5)(x+2) = 0$

$\Rightarrow \quad$ Either $x - 7 = 0$ or $x + 4 = 0$ $\Rightarrow \quad$ Either $x - 5 = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = 7$ or $x = -4$ $\Rightarrow \quad x = 5$ or $x = -2$

Hence the solution set $= \{-4, -2, 5, 7\}$

Q.10 $(x + 1)(2x + 3)(2x + 5)(x + 3) = 945$ **(Lahore Board 2007)**

Solution:

$(x + 1)(2x + 3)(2x + 5)(x + 3) = 945$

Here we combine

$[(x + 1)(x + 3)][(2x + 3)(2x + 5)] = 945$

$\Rightarrow \quad [x^2 + 3x + x + 3][4x^2 + 10x + 6x + 15] = 945$

$\Rightarrow \quad [x^2 + 4x + 3][4x^2 + 16x + 15] = 945$

$\Rightarrow \quad [x^2 + 4x + 3][4(x^2 + 4x) + 15] = 945$

Put $x^2 + 4x = y$ ..... (1)

$\Rightarrow \quad [y + 3](4y + 15] = 945$

$\Rightarrow \quad 4y^2 + 15y + 12y + 45 = 945$

$\Rightarrow \quad 4y^2 + 27y + 45 - 945 = 0$

$\Rightarrow \quad 4y^2 + 27y - 900 = 0$

$\Rightarrow \quad 4y^2 + 75y - 48y - 900 = 0$

$\Rightarrow \quad y(4y + 75) - (4y + 75) = 0$

$\Rightarrow \quad (4y + 75)(y - 12) = 0$

$\Rightarrow \quad$ Either $4y + 75 = 0$ or $y - 12 = 0$

$\Rightarrow \quad y = \frac{-75}{4}$ or $y = 12$

Put $y = \frac{-75}{4}$ and $y = 12$ in equation (1)

$\Rightarrow \quad x^2 + 4x = \frac{-75}{4}$ $\Rightarrow \quad x^2 + 4x = 12$

$\Rightarrow \quad 4x^2 + 16x = -75$

$\Rightarrow \quad 4x^2 + 16x + 75 = 0$

$\Rightarrow \quad x = \dfrac{-16 \pm \sqrt{(16)^2 - 4\,(4)\,(75)}}{2\,(4)}$

$\Rightarrow \quad = \dfrac{-16 \pm \sqrt{-944}}{8}$

$\Rightarrow \quad = \dfrac{-16 \pm 4\sqrt{-59}}{8}$

$\Rightarrow \quad = \dfrac{-4 \pm \sqrt{-59}}{2}$

Hence the solution set $= \left\{ -6\,,\, 2\,,\, \dfrac{-4 \pm \sqrt{-59}}{2} \right\}$

$\Rightarrow \quad x^2 + 4x - 12 = 0$

$\Rightarrow \quad x^2 + 6x - 2x - 12 = 0$

$\Rightarrow \quad x\,(x + 6) - 2\,(x + 6) = 0$

$\Rightarrow \quad$ Either $x + 6 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = -6$ or $x = 2$

Q.11 $(2x - 7)\,(x^2 - 9)\,(2x + 5) - 91 = 0$

Solution:

$(2x - 7)\,(x^2 - 9)\,(2x + 5) - 91 = 0$

$\Rightarrow \quad (2x - 7)\,(x + 3)\,(x - 3)\,(2x + 5) - 91 = 0$

Here we combine

$[(2x - 7)\,(x + 3)]\,[(x - 3)\,(2x + 5)] - 91 = 0$

$\Rightarrow \quad [2x^2 + 6x - 7x - 21]\,[2x^2 + 5x - 6x - 15] - 91 = 0$

$\Rightarrow \quad [2x^2 - x - 21]\,[2x^2 - x - 15] - 91 = 0$

Put $2x^2 - x = y \qquad \ldots\ldots\ldots (1)$

$\Rightarrow \quad [y - 21]\,[y - 15] - 91 = 0$

$\Rightarrow \quad y^2 - 15y - 21y + 315 - 91 = 0$

$\Rightarrow \quad y^2 - 36y + 224 = 0$

$\Rightarrow \quad y^2 - 28y - 8y + 224 = 0$

$\Rightarrow \quad y\,(y - 28) - 8\,(y - 28) = 0$

$\Rightarrow \quad (y - 28)\,(y - 8) = 0$

$\Rightarrow \quad$ Either $y - 28 = 0$ or $y - 8 = 0$

$\Rightarrow \quad y = 28$ or $y = 8$

Put $y = 28$ and $y = 8$ in equation (1)

$\Rightarrow \quad 2x^2 - x = 28 \qquad \Rightarrow \quad 2x^2 - x = 8$

$\Rightarrow \quad 2x^2 - x - 28 = 0 \qquad \Rightarrow \quad 2x^2 - x - 8 = 0$

$\Rightarrow \quad 2x^2 - 8x + 7x - 28 = 0 \qquad \Rightarrow \quad x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-8)}}{2(2)}$

$\Rightarrow \quad 2x(x-4) + 7(x-4) = 0 \qquad \Rightarrow \quad = \dfrac{1 \pm \sqrt{1+64}}{4}$

$\Rightarrow \quad (x-4) + (2x+7) = 0 \qquad \Rightarrow \quad = \dfrac{1 \pm \sqrt{65}}{4}$

$\Rightarrow$ Either $x - 4 = 0$ or $2x + 7 = 0$

$\Rightarrow \quad x = 4$ or $x = \dfrac{-7}{2}$

Hence the solution set $= \left\{ -\dfrac{7}{2}, 4, \dfrac{1 \pm \sqrt{65}}{4} \right\}$

Q.12 $(x^2 + 6x + 8)(x^2 + 14x + 48) = 105$

Solution:

$(x^2 + 6x + 8)(x^2 + 14x + 48) = 105$

$\Rightarrow \quad (x^2 + 4x + 2x + 8)(x^2 + 8x + 6x + 48) = 105$

$\Rightarrow \quad [x(x+4) + 2(x+4)][(x(x+8) + 6(x+8)] = 105$

$\Rightarrow \quad [(x+4)(x+2)][(x+8)(x+6)] = 105$

$\Rightarrow \quad (x+4)(x+2)(x+8)(x+6) = 105$

As $\quad 2 + 8 = 4 + 6$

So we combine

$[(x+2)(x+8)][(x+4)(x+6)] = 105$

$\Rightarrow \quad [x^2 + 8x + 2x + 16][x^2 + 6x + 4x + 24] = 105$

$\Rightarrow \quad [x^2 + 10x + 16][x^2 + 10x + 24] = 105$

Put $x^2 + 10x = y \qquad \ldots\ldots\ldots (1)$

$\Rightarrow \quad [y + 16][y + 24] = 105$

$\Rightarrow \quad y^2 + 24y + 16y + 384 = 105$

$\Rightarrow \quad y^2 + 40y + 384 - 105 = 0$

$\Rightarrow \quad y^2 + 40y + 279 = 0$

$\Rightarrow \quad y^2 + 31y + 9y + 279 = 0$

$\Rightarrow \quad y(y+31)+9(y+31) = 0$

$\Rightarrow \quad (y+9)(y+31) = 0$

$\Rightarrow \quad$ Either $y+9 = 0 \quad$ or $\quad y+31 = 0$

$\Rightarrow \quad y = -9 \quad$ or $\quad y = -31$

Put $y = -9$ and $y = -31$ in equation (1)

$\Rightarrow \quad x^2+10x = -9$	$\Rightarrow \quad x^2+10x = -31$
$\Rightarrow \quad x^2+10x+9 = 0$	$\Rightarrow \quad x^2+10x+31 = 0$
	$\Rightarrow \quad x = \dfrac{-10 \pm \sqrt{(10)^2-4(1)(31)}}{2(1)}$
$\Rightarrow \quad x^2+9x+x+9 = 0$	$\Rightarrow \quad = \dfrac{-10 \pm \sqrt{100-124}}{2}$
$\Rightarrow \quad x(x+9)+1(x+9) = 0$	$\Rightarrow \quad = \dfrac{-10 \pm \sqrt{-24}}{2}$
$\Rightarrow \quad (x+1)(x+9) = 0$	$\Rightarrow \quad = \dfrac{-10 \pm 2\sqrt{-6}}{2}$
$\Rightarrow \quad$ Either $x+1 = 0$ or $x+9 = 0$	$\Rightarrow \quad = -5 \pm \sqrt{-6}$
$\Rightarrow \quad x = -1$ or $x = -9$	

Hence the solution set $= \{-1, -9, -5 \pm \sqrt{-6}\}$

Q.13 $(x^2+6x-27)(x^2-2x-35) = 385$

Solution:

$(x^2+6x-27)(x^2-2x-35) = 385$

$\Rightarrow \quad (x^2+9x-3x-27)(x^2-7x+5x-35) = 385$

$\Rightarrow \quad [x(x+9)-3(x+9)][x(x-7)+5(x-7)] = 385$

$\Rightarrow \quad [(x+9)(x-3)][(x-7)(x+5)] = 385$

$\Rightarrow \quad (x+9)(x-3)(x-7)(x+5) = 385$

As $9-7 = 5-3$

So we combine

$[(x+9)(x-7)][(x+5)(x-3)] = 385$

$\Rightarrow \quad [x^2-7x+9x-63][x^2-3x+5x-15] = 385$

$\Rightarrow \quad [x^2+2x-63][x^2+2x-15] = 385$

Put $x^2 + 2x = y$ ..... (1)

$\Rightarrow$ $[y - 63][y - 15] = 385$

$\Rightarrow$ $y^2 - 15y - 63y + 945 = 385$

$\Rightarrow$ $y^2 - 78y + 945 - 385 = 0$

$\Rightarrow$ $y^2 - 78y + 560 = 0$

$\Rightarrow$ $y^2 - 70y - 8y + 560 = 0$

$\Rightarrow$ $y(y - 70) - 8(y - 70) = 0$

$\Rightarrow$ $(y - 70)(y - 8) = 0$

$\Rightarrow$ Either $y - 70 = 0$ or $y - 8 = 0$

$\Rightarrow$ $y = 70$ or $y = 8$

Put $y = 70$ and $y = 8$ in equation (1)

$\Rightarrow$ $x^2 + 2x = 70$	$\Rightarrow$ $x^2 + 2x = 8$
$\Rightarrow$ $x^2 + 2x - 70 = 0$	$\Rightarrow$ $x^2 + 2x - 8 = 0$
$\Rightarrow$ $x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-70)}}{2(1)}$	$\Rightarrow$ $x^2 + 4x - 2x - 8 = 0$
$\Rightarrow$ $= \frac{-2 \pm \sqrt{4 + 280}}{2}$	$\Rightarrow$ $x(x + 4) - 2(x + 4) = 0$
$\Rightarrow$ $= \frac{-2 \pm \sqrt{284}}{2}$	$\Rightarrow$ $(x + 4)(x - 2) = 0$
$\Rightarrow$ $= \frac{-2 \pm 2\sqrt{71}}{2}$	$\Rightarrow$ Either $x + 4 = 0$ or $x - 2 = 0$
$\Rightarrow$ $= -1 \pm \sqrt{71}$	$\Rightarrow$ $x = -4$ or $x = 2$

Hence the solution set $= \{-4, 2, -1 \pm \sqrt{71}\}$

Q.14 $\mathbf{4.2^{2x+1} - 9.2^x + 1 = 0}$ **(Gujranwala Board 2005)**

Solution:

$4.2^{2x+1} - 9.2^x + 1 = 0$

$4.2^{2x} \cdot 2 - 9.2^x + 1 = 0$

$8(2^x)^2 - 9.2^x + 1 = 0$

Put $2^x = y$ ..... (1)

$\Rightarrow$ $8y^2 - 9y + 1 = 0$

$\Rightarrow \quad 8y^2 - 8y - y + 1 = 0$

$\Rightarrow \quad 8y(y-1) - 1(y-1) = 0$

$\Rightarrow \quad (y-1)(8y-1) = 0$

$\Rightarrow \quad$ Either $y - 1 = 0 \quad$ or $\quad 8y - 1 = 0$

$\Rightarrow \quad y = 1 \quad$ or $\quad y = \frac{1}{8}$

Put $y = 1$ and $y = \frac{1}{8}$ in equation (1)

$\Rightarrow \quad 2^x = 1$	$\Rightarrow \quad 2^x = \frac{1}{8}$
$\Rightarrow \quad 2^x = 2^0$	$\Rightarrow \quad 2^x = \frac{1}{2^3}$
$\Rightarrow \quad x = 0$	$\Rightarrow \quad 2^x = 2^{-3}$
	$\Rightarrow \quad x = -3$

Hence the solution set $= \{-3, 0\}$

Q.15 $\mathbf{2^x + 2^{-x+6} - 20 = 0}$

Solution:

$2^x + 2^{-x+6} - 20 = 0$

$2^x + 2^{-x} \cdot 2^6 - 20 = 0$

$2^x + 64\, x^{-x} - 20 = 0$

Put $2^x = y \quad \ldots\ldots\ldots (1)$

$\Rightarrow \quad 2^{-x} = y^{-1}$

$\Rightarrow \quad y + 64\, y^{-1} - 20 = 0$

$\Rightarrow \quad y + \frac{64}{y} - 20 = 0$

Multiplying by 'y'

$y^2 + 64 - 20y = 20$

$\Rightarrow \quad y^2 - 20y + 64 = 0$

$\Rightarrow \quad y^2 - 16y - 4y + 64 = 0$

$\Rightarrow \quad y(y-16) - 4(y-16) = 0$

$\Rightarrow \quad (y-16)(y-4) = 0$

$\Rightarrow$ Either $y - 16 = 0$ or $y - 4 = 0$

$y = 16$ or $y = 4$

Put $y = 16$ and $y = 4$ in equation (1)

$\Rightarrow \quad 2^x = 16$	$\Rightarrow \quad 2^x = 4$
$\Rightarrow \quad 2^x = 2^4$	$\Rightarrow \quad 2^x = 2^2$
$\Rightarrow \quad x = 4$	$\Rightarrow \quad x = 2$

Hence the solution set $= \{2, 4\}$

Q.16 $\mathbf{4^x - 3.2^{x+3} + 128 = 0}$

Solution:

$4^x - 3.2^{x+3} + 128 = 0$

$\Rightarrow \quad (2^2)^x - 3.2^x \cdot 2^3 + 128 = 0$

$\Rightarrow \quad 2 - 24.2^x + 128 = 0$

Put $2^x = y$ ..... (1)

$\Rightarrow \quad y^2 - 24y + 128 = 0$

$\Rightarrow \quad y^2 - 16y - 8y + 128 = 0$

$\Rightarrow \quad y(y - 16) - 8(y - 16) = 0$

$\Rightarrow \quad (y - 16)(y - 8) = 0$

$\Rightarrow$ Either $y - 6 = 0$ or $y - 8 = 0$

$\Rightarrow \quad y = 16$ or $y = 8$

Put $y = 16$ and $y = 8$ in equation (1)

$\Rightarrow \quad 2^x = 6$	$\Rightarrow \quad 2^x = 8$
$\Rightarrow \quad 2^x = 2^4$	$\Rightarrow \quad 2^x = 2^3$
$\Rightarrow \quad x = 4$	$\Rightarrow \quad x = 3$

Hence the solution set $= \{3, 4\}$

Q.17 $\mathbf{3^{2x-1} - 12.3^x + 81 = 0}$

Solution:

$3^{2x-1} - 12.3^x + 81 = 0$

$\Rightarrow \quad 3^{2x} \cdot 3^{-1} - 12 \cdot 3^x + 81 = 0$

$\Rightarrow \quad \frac{3^{2x}}{3} - 12.3^x + 81 = 0$

Multiplying by '3'

$\Rightarrow \quad 3^{2x} - 36 \cdot 3^{x} + 243 = 0$

$\Rightarrow \quad (3^{x})^{2} - 36 \cdot 3^{x} + 243 = 0$

Put $3^{x} = y \quad \ldots\ldots\ldots (1)$

$\Rightarrow \quad y^{2} - 36y + 243 = 0$

$\Rightarrow \quad y^{2} - 27y - 9y + 243 = 0$

$\Rightarrow \quad y(y - 27) - 9(y - 27) = 0$

$\Rightarrow \quad (y - 27) - (y - 9) = 0$

$\Rightarrow \quad$ Either $y - 27 = 0 \quad$ or $\quad y - 9 = 0$

$\Rightarrow \quad y = 27 \quad$ or $\quad y = 9$

Put $y = 27$ and $y = 9$ in equation (1)

$\Rightarrow \quad 3^{x} = 27$	$\Rightarrow \quad 3^{x} = 9$
$\Rightarrow \quad 3^{x} = 3^{3}$	$\Rightarrow \quad 3^{x} = 3^{2}$
$\Rightarrow \quad x = 3$	$\Rightarrow \quad x = 2$

Hence the solution set $= \{2, 3\}$

Q.18 $\left(x + \frac{1}{x}\right)^{2} - 3\left(x + \frac{1}{x}\right) - 4 = 0$

Solution:

$\left(x + \frac{1}{x}\right)^{2} - 3\left(x + \frac{1}{x}\right) - 4 = 0$

Put $x + \frac{1}{x} = y \quad \ldots\ldots\ldots (1)$

$y^{2} - 3y - 4 = 0$

$\Rightarrow \quad y^{2} - 4y + y - 4 = 0$

$\Rightarrow \quad y(y - 4) + 1(y - 4) = 0$

$\Rightarrow \quad (y - 4) + 1(y - 4) = 0 \Rightarrow (y - 4)(y + 1) = 0$

$\Rightarrow \quad$ Either $y - 4 = 0 \quad$ or $\quad y + 1 = 0$

$\Rightarrow \quad y = 4 \quad$ or $\quad y = -1$

Put $y = 4$ and $y = -1$ in equation (1)

$\Rightarrow \quad x + \frac{1}{x} = +4$	$\Rightarrow \quad x + \frac{1}{x} = -1$

$\Rightarrow \quad x + \frac{1}{x} - 4 = 0$

$\Rightarrow \quad x^2 + 1 - 4x = 0$

$\Rightarrow \quad x^2 - 4x + 1 = 0$

$\Rightarrow \quad x = \frac{4 \pm \sqrt{(-4)^2 - 4\,(1)\,(1)}}{2\,(1)}$

$\Rightarrow \quad x = \frac{4 \pm \sqrt{16 - 4}}{2}$

$\Rightarrow \quad x = \frac{4 \pm \sqrt{12}}{2}$

$\Rightarrow \quad x = \frac{4 \pm 2\sqrt{3}}{2}$

$\Rightarrow \quad x = 2 \pm \sqrt{3}$

$\Rightarrow \quad x + \frac{1}{x} + 1 = 0$

$\Rightarrow \quad x^2 + 1 + x = 0$

$\Rightarrow \quad x^2 + x + 1 = 0$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{(1)^2 - 4\,(1)\,(1)}}{2\,(1)}$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1 - 4}}{2}$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-3}}{2}$

Hence the solution set $= \left\{ 2 \pm \sqrt{3}\,,\ \frac{-1 \pm \sqrt{-3}}{2} \right\}$

Q.19 $\mathbf{x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0}$

Solution:

$x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$

$\left(x^2 + \frac{1}{x^2} \right) + \left(x + \frac{1}{x} \right) - 4 = 0$

Put $x + \frac{1}{x} = y$ ..... (1)

Then $\left(x + \frac{1}{x} \right)^2 = y^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2 = y^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = y^2 - 2$

$\Rightarrow$ given equation becomes

$\Rightarrow \quad y^2 - 2 + y - 4 = 0$

$\Rightarrow \quad y^2 + y - 6 = 0$

$\Rightarrow \quad y^2 + 3y - 2y - 6 = 0$

$\Rightarrow \quad y(y+3) - 2(y+3) = 0$

$\Rightarrow \quad (y+3)(y-2) = 0$

$\Rightarrow \quad$ Either $y + 3 = 0$ or $y - 2 = 0$

$\Rightarrow \quad y = -3$ or $y = 2$

Put $y = -3$ and $y = 2$ in equation (1)

$\Rightarrow \quad x + \frac{1}{x} = -3$	$\Rightarrow \quad x + \frac{1}{x} = 2$
Multiplying by 'x'	Multiplying by 'x'
$\Rightarrow \quad x^2 + 1 = -3x$	$\Rightarrow \quad x^2 + 1 = 2x$
$\Rightarrow \quad x^2 + 3x + 1 = 0$	$\Rightarrow \quad x^2 - 2x + 1 = 0$
$\Rightarrow \quad x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$	$\Rightarrow \quad (x-1)^2 = 0$
$\Rightarrow \quad x = \frac{-3 \pm \sqrt{9-4}}{2}$	$\Rightarrow \quad x - 1 = 0$
$\Rightarrow \quad x = \frac{-3 \pm \sqrt{5}}{2}$	$\Rightarrow \quad x = 1$

Hence the solution set $= \left\{ 1, \frac{-3 \pm \sqrt{5}}{2} \right\}$

Q.20 $\left(\mathbf{x - \frac{1}{x}} \right)^2 + 3\left(\mathbf{x + \frac{1}{x}} \right) = \mathbf{0}$

Solution:

$$\left(x - \frac{1}{x} \right)^2 + 3\left(x + \frac{1}{x} \right) = 0$$

or $\quad x^2 + \frac{1}{x^2} - 2 + 3\left(x + \frac{1}{x} \right) = 0$

or $\quad x^2 + \frac{1}{x^2} + 3\left(x + \frac{1}{x} \right) - 2 = 0 \quad \ldots\ldots\ldots (A)$

Put $x + \frac{1}{x} = y \quad \ldots\ldots\ldots (1)$

then $\left(x + \frac{1}{x}\right)^2 = y^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2 = y^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = y^2 - 2$

$\Rightarrow$ Above equation (A) becomes

$y^2 - 2 + 3y - 2 = 0$

$\Rightarrow \quad y^2 + 3y - 4 = 0$

$\Rightarrow \quad y^2 + 4y - y - 4 = 0$

$\Rightarrow \quad y(y + 4) - 1(y + 4) = 0$

$\Rightarrow \quad (y + 4)(y - 1) = 0$

$\Rightarrow$ Either $y + 4 = 0$ or $y - 1 = 0$

$\Rightarrow \quad y = -4$ or $y = 1$

Put $y = -4$ and $y = 1$ in equation (1)

$\Rightarrow \quad x + \frac{1}{x} = -4$

$\Rightarrow \quad x^2 + 1 = -4x$

$\Rightarrow \quad x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2(1)}$

$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16 - 4}}{2}$

$\Rightarrow \quad x = \frac{-4 \pm \sqrt{12}}{2}$

$\Rightarrow \quad x = \frac{-4 \pm 2\sqrt{3}}{2}$

$\Rightarrow \quad x = -2 \pm \sqrt{3}$

$\Rightarrow \quad x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 + 1 = x$

$\Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 - 4}}{2}$

$x = \frac{1 \pm \sqrt{-3}}{2}$

Hence the solution set $= \left\{-2 \pm \sqrt{3}, \frac{1 \pm \sqrt{-3}}{2}\right\}$

Q.21 $2x^4 - x^3 + x^2 - 3x + 2 = 0$

Solution:

$2x^4 - x^3 + x^2 - 3x + 2 = 0$

Dividing each by x^2

$$\Rightarrow \quad \frac{2x^4}{x^2} - \frac{3x^3}{x^2} - \frac{x^2}{x^2} - \frac{3x}{x^2} + \frac{2}{x^2} = \frac{0}{x^2}$$

$$\Rightarrow \quad 2x^2 - 3x - 1 - \frac{3}{x} + \frac{2}{x^2} = 0$$

$$\Rightarrow \quad 2x^2 + \frac{2}{x^2} - 3x - \frac{3}{x} - 1 = 0$$

$$\Rightarrow \quad 2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 1 = 0 \qquad \ldots\ldots\ldots (A)$$

Put $x + \frac{1}{x} = y$ $\qquad \ldots\ldots\ldots (1)$

Then $\left(x + \frac{1}{x}\right)^2 = y^2$

$$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2 = y^2 \qquad \Rightarrow \quad x^2 + \frac{1}{x^2} = y^2 - 2$$

Above equation (A) becomes

$$\Rightarrow \quad 2(y^2 - 2) - 3y - 1 = 0$$

$$\Rightarrow \quad 2y^2 - 3y - 5 = 0$$

$$\Rightarrow \quad 2y^2 - 5y + 2y - 5 = 0$$

$$\Rightarrow \quad y(2y - 5) + 1(2y - 5) = 0$$

$$\Rightarrow \quad (2y - 5)(y + 1) = 0$$

$$\Rightarrow \quad \text{Either} \quad 2y - 5 = 0 \quad \text{or} \quad y + 1 = 0$$

$$\Rightarrow \quad y = \frac{5}{2} \quad \text{or} \quad y = -1$$

Put $y = \frac{5}{2}$ and $y = -1$ in equation (1)

$$\Rightarrow \quad x + \frac{1}{x} = \frac{5}{2} \qquad \Rightarrow \quad x + \frac{1}{x} = -1$$

$$\Rightarrow \quad x^2 + 1 = \frac{5}{2}x \qquad \Rightarrow \quad x^2 + 1 = -x$$

$\Rightarrow \quad 2x^2 + 2 = 5x$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad 2x^2 - 4x - x + 2 = 0$

$\Rightarrow \quad 2x(x-2) - 1(x-2) = 0$

$\Rightarrow \quad (2x-1)(x-2) = 0$

$\Rightarrow \quad x^2 + x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow$ Either $2x - 1 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = \dfrac{1}{2}$ or $x = 2$

Hence the solution set $= \left\{ \dfrac{1}{2}, 2, \dfrac{-1 \pm \sqrt{-3}}{2} \right\}$

Q.22 $\mathbf{2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0}$

Solution:

$2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Dividing by x^2

$\Rightarrow \quad \dfrac{2x^4}{x^2} + \dfrac{3x^3}{x^2} - \dfrac{4x^2}{x^2} - \dfrac{3x}{x^2} + \dfrac{2}{x^2} = \dfrac{0}{x^2}$

$\Rightarrow \quad 2x^2 + 3x - 4 - \dfrac{3}{x} + \dfrac{2}{x^2} = 0$

$\Rightarrow \quad 2x^2 + \dfrac{2}{x^2} + 3x - \dfrac{3}{x} - 4 = 0$

$\Rightarrow \quad 2\left(x^2 + \dfrac{1}{x^2}\right) + 3\left(x - \dfrac{1}{x}\right) - 4 = 0 \qquad \ldots\ldots\ldots (A)$

Put $x - \dfrac{1}{x} = y \qquad \ldots\ldots\ldots (1)$

Then $\left(x - \dfrac{1}{x}\right)^2 \Rightarrow x^2 + \dfrac{1}{x^2} - 2 = y^2 \Rightarrow x^2 + \dfrac{1}{x^2} = y^2 + 2$

$\Rightarrow$ Above equation (A) becomes

$2(y^2 + 2) + 3y - 4 = 0$

$\Rightarrow \quad 2y^2 + 4 + 3y - 4 = 0$

$\Rightarrow \quad 2y^2 + 3y = 0$

$\Rightarrow \quad y(2y + 3) = 0$

$\Rightarrow \quad$ Either $y = 0$ or $2y + 3 = 0$

$\Rightarrow \quad$ or $y = \frac{-3}{2}$

Put $y = 0$ and $y = \frac{-3}{2}$ in equation (1)

$\Rightarrow \quad x - \frac{1}{x} = 0$

$\Rightarrow \quad x^2 - 1 = 0$

$\Rightarrow \quad (x + 1)(x - 1) = 0$

$\Rightarrow \quad$ Either $x + 1 = 0$ or $x - 1 = 0$

$\Rightarrow \quad x = -1$ or $x = 1$

$\Rightarrow \quad x - \frac{1}{x} = -\frac{3}{2}$

$\Rightarrow \quad x^2 - 1 = \frac{-3}{2}x$

$\Rightarrow \quad 2x^2 - 2 = -3x$

$\Rightarrow \quad 2x^2 + 3x - 2 = 0$

$\Rightarrow \quad 2x^2 + 4x - x - 2 = 0$

$\Rightarrow \quad 2x(x + 2) - 1(x + 2) = 0$

$\Rightarrow \quad (x + 2)(2x - 1) = 0$

$\Rightarrow \quad$ Either $x + 2 = 0$ or $2x - 1 = 0$

$\Rightarrow \quad x = -2$ or $x = \frac{1}{2}$

Hence the solution set $= \left\{-2, -1, \frac{1}{2}, 1\right\}$

Q.23 $\mathbf{6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0}$

Solution:

$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

Dividing by x^2

$\Rightarrow \quad \frac{6x^4}{x^2} - \frac{35x^3}{x^2} + \frac{62x^2}{x^2} - \frac{35x}{x^2} + \frac{6}{x^2} = \frac{0}{x^2}$

$\Rightarrow \quad 6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$

$\Rightarrow \quad 6x^2 + \frac{6}{x^2} - 35x - \frac{35}{x} + 62 = 0$

$\Rightarrow \quad 6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62 = 0 \quad \ldots\ldots\ldots (A)$

Put $x+\frac{1}{x} = y \quad \ldots\ldots\ldots (1)$

Then $\left(x+\frac{1}{x}\right)^2 = y^2 \Rightarrow x^2+\frac{1}{x^2}+2 = y^2 \Rightarrow x^2+\frac{1}{x^2} = y^2-2$

$\Rightarrow$ Above equation (A) becomes

$6(y^2-2)-35y+62 = 0$

$\Rightarrow \quad 6y^2-12-35y+62 = 0$

$\Rightarrow \quad 6y^2-35y+50 = 0$

$\Rightarrow \quad 6y^2-20y-15y+50 = 0$

$\Rightarrow \quad 2y(3y-10)-5(3y-10) = 0$

$\Rightarrow \quad (3y-10)(2y-5) = 0$

$\Rightarrow$ Either $3y-10 = 0$ or $2y-5 = 0$

$\Rightarrow \quad y = \frac{10}{3}$ or $y = \frac{5}{2}$

Put $y = \frac{10}{3}$ and $y = \frac{5}{2}$ in equation (1)

$\Rightarrow \quad x+\frac{1}{x} = \frac{10}{3}$	$\Rightarrow \quad x+\frac{1}{x} = \frac{5}{2}$
$\Rightarrow \quad x^2+1 = \frac{10}{3}x$	$\Rightarrow \quad x^2+1 = \frac{5}{2}x$
$\Rightarrow \quad 3x^2+3 = 10x$	$\Rightarrow \quad 2x^2+2 = 5x$
$\Rightarrow \quad 3x^2-10x+3 = 0$	$\Rightarrow \quad 2x^2-5x+2 = 0$
$\Rightarrow \quad 3x^2-9x-x+3 = 0$	$\Rightarrow \quad 2x^2-4x-x+2 = 0$
$\Rightarrow \quad 3x(x-3)-1(x-3) = 0$	$\Rightarrow \quad 2x(x-2)-1(x-2) = 0$
$\Rightarrow \quad (x-3)(3x-1) = 0$	$\Rightarrow \quad (x-2)(2x-1) = 0$
$\Rightarrow$ Either $x-3 = 0$ or $3x-1 = 0$	$\Rightarrow$ Either $x-2 = 0$ or $2x-1 = 0$
$\Rightarrow \quad x = 3$ or $x = \frac{1}{3}$	$\Rightarrow \quad x = 2$ or $x = \frac{1}{2}$

Hence the solution set $= \left\{2, 3, \frac{1}{2}, \frac{1}{3}\right\}$

Q.24 $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Solution:

$$x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$$

$$\Rightarrow \quad x^4 + \frac{1}{x^4} - 6x^2 - \frac{6}{x^2} + 10 = 0$$

$$\Rightarrow \quad \left(x^4 + \frac{1}{x^4}\right) - 6\left(x^2 + \frac{1}{x^2}\right) + 10 = 0 \qquad \ldots\ldots\ldots (A)$$

Put $x^2 + \frac{1}{x^2} = y$ (1)

Then $\left(x^2 + \frac{1}{x^2}\right)^2 = y^2 \Rightarrow x^4 + \frac{1}{x^4} + 2 = y^2 \Rightarrow x^4 + \frac{1}{x^4} = y^2 - 2$

$\Rightarrow$ above equation (A) becomes

$$y^2 - 2 - 6y + 10 = 0$$

$$\Rightarrow \quad y^2 - 6y + 8 = 0$$

$$\Rightarrow \quad y^2 - 4y - 2y + 8 = 0$$

$$\Rightarrow \quad y(y-4) - 2(y-4) = 0$$

$$\Rightarrow \quad (y-4)(y-2) = 0$$

$\Rightarrow$ Either $y - 4 = 0$ or $y - 2 = 0$

$\Rightarrow$ $y = 4$ or $y = 2$

Put $y = 4$ and $y = 2$ in equation (1)

$\Rightarrow \quad x^2 + \frac{1}{x^2} = 4$	$\Rightarrow \quad x^2 + \frac{1}{x^2} = 2$
$\Rightarrow \quad x^4 + 1 = 4x^2$	$\Rightarrow \quad x^4 + 1 = 2x^2$
$\Rightarrow \quad x^4 - 4x^2 + 1 = 0$	$\Rightarrow \quad x^4 - 2x^2 + 1 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$	$\Rightarrow \quad (x^2 - 1)^2 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{16 - 4}}{2}$	$\Rightarrow \quad x^2 - 1 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{12}}{2}$	$\Rightarrow \quad (x+1)(x-1) = 0$

Q.24 $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Solution:

$$x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$$

$$\Rightarrow \quad x^4 + \frac{1}{x^4} - 6x^2 - \frac{6}{x^2} + 10 = 0$$

$$\Rightarrow \quad \left(x^4 + \frac{1}{x^4}\right) - 6\left(x^2 + \frac{1}{x^2}\right) + 10 = 0 \qquad \ldots\ldots\ldots (A)$$

Put $x^2 + \frac{1}{x^2} = y$ (1)

Then $\left(x^2 + \frac{1}{x^2}\right)^2 = y^2 \Rightarrow x^4 + \frac{1}{x^4} + 2 = y^2 \Rightarrow x^4 + \frac{1}{x^4} = y^2 - 2$

$\Rightarrow$ above equation (A) becomes

$$y^2 - 2 - 6y + 10 = 0$$

$$\Rightarrow \quad y^2 - 6y + 8 = 0$$

$$\Rightarrow \quad y^2 - 4y - 2y + 8 = 0$$

$$\Rightarrow \quad y(y-4) - 2(y-4) = 0$$

$$\Rightarrow \quad (y-4)(y-2) = 0$$

$\Rightarrow$ Either $y - 4 = 0$ or $y - 2 = 0$

$\Rightarrow$ $y = 4$ or $y = 2$

Put $y = 4$ and $y = 2$ in equation (1)

$\Rightarrow \quad x^2 + \frac{1}{x^2} = 4$	$\Rightarrow \quad x^2 + \frac{1}{x^2} = 2$
$\Rightarrow \quad x^4 + 1 = 4x^2$	$\Rightarrow \quad x^4 + 1 = 2x^2$
$\Rightarrow \quad x^4 - 4x^2 + 1 = 0$	$\Rightarrow \quad x^4 - 2x^2 + 1 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$	$\Rightarrow \quad (x^2 - 1)^2 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{16 - 4}}{2}$	$\Rightarrow \quad x^2 - 1 = 0$
$\Rightarrow \quad x^2 = \frac{4 \pm \sqrt{12}}{2}$	$\Rightarrow \quad (x+1)(x-1) = 0$

$\Rightarrow \quad x^2 = \dfrac{4 \pm 2\sqrt{3}}{2}$

$\Rightarrow \quad x^2 = 2 \pm \sqrt{3}$

$\Rightarrow \quad x = \pm\sqrt{2 \pm \sqrt{3}}$

$\Rightarrow$ Either $x + 1 = 0$ or $x - 1 = 0$

$\Rightarrow \quad x = -1$ or $x = 1$

Hence the solution set $= \{ -1, 1, \pm\sqrt{2 \pm \sqrt{3}} \}$

TYPE V: RADICAL EQUATIONS

Equations involving radical expressions of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical free equation but the new equation have solutions that are not solutions of the original radical equation. Such extra solutions are called **extraneous roots**.

There are four types of radical equations.

(i) The equations of the form: $l(ax^2 + bx) + m\sqrt{ax^2 + bx + c} = 0$

(ii) The equations of the form: $\sqrt{x + a} + \sqrt{x + b} = \sqrt{x + c}$

(iii) The equations of the form: $\sqrt{ax^2 + bx + c} + \sqrt{px^2 + qx + r} = \sqrt{1x^2 + mx + n}$

(iv) The equations of the form: $\sqrt{ax^2 + bx + c} + \sqrt{px^2 + qx + r} = mx + n$

EXERCISE 4.3

Solve the following equation:

Q.1 $\mathbf{3x^2 + 2x - \sqrt{3x^2 + 2x - 1} = 3}$

Solution:

$3x^2 + 2x - \sqrt{3x^2 + 2x - 1} = 3$ ..... (1)

Put $\sqrt{3x^2 + 2x - 1} = y$ ..... (2)

$\Rightarrow \quad 3x^2 + 2x - 1 = y^2$

$\Rightarrow \quad 3x^2 + 2x = y^2 + 1$

$\Rightarrow$ equation (1) becomes

$y^2 + 1 - y = 3$

$\Rightarrow \quad y^2 - y + 1 - 3 = 0$

$\Rightarrow \quad y^2 - y - 2 = 0$

$\Rightarrow \quad y^2 - 2y + y - 2 = 0$

$\Rightarrow \quad y(y - 2) + 1(y - 2) = 0$

$\Rightarrow \quad (y-2)(y+1) = 0$

$\Rightarrow \quad$ Either $y-2=0 \quad$ or $\quad y+1=0$

$\Rightarrow \quad y=2 \quad$ or $\quad y=-1$

Put $y=2$ and $y=-1$ in equation (2)

$\Rightarrow \quad \sqrt{3x^2+2x-1} = 2$

$\Rightarrow \quad 3x^2+2x-1 = 4$

$\Rightarrow \quad 3x^2+2x-1-4 = 0$

$\Rightarrow \quad 3x^2+2x-5 = 0$

$\Rightarrow \quad 3x^2+5x-3x-5 = 0$

$\Rightarrow \quad x(3x+5)-1(3x+5) = 0$

$\Rightarrow \quad (3x+5)(x-1) = 0$

$\Rightarrow \quad$ Either $3x+5=0$ or $x-1=0$

$\Rightarrow \quad x = \frac{-5}{3} \quad$ or $\quad x = 1$

$\Rightarrow \quad x = 1, \ -\frac{5}{3}, \ \frac{-1\pm\sqrt{7}}{3}$

$\Rightarrow \quad \sqrt{3x^2+2x-1} = -1$

$\Rightarrow \quad 3x^2+2x-1 = 1$

$\Rightarrow \quad 3x^2+2x-1-1 = 0$

$\Rightarrow \quad 3x^2+2x-2 = 0$

$\Rightarrow \quad x = \frac{-2\pm\sqrt{(2)^2-4(3)(-2)}}{2(3)}$

$\Rightarrow \quad x = \frac{-2\pm\sqrt{4+24}}{6}$

$\Rightarrow \quad x = \frac{-2\pm\sqrt{28}}{6}$

$\Rightarrow \quad x = \frac{-2\pm 2\sqrt{7}}{6}$

$\Rightarrow \quad x = \frac{-1\pm\sqrt{7}}{3}$

Checking:

Put $x=1$ in equation (1)

$3(1)^2+2(1)-\sqrt{3(1)^2+2(1)-1} = 3$

$3+2-\sqrt{3+2-1} = 3$

$3 = 3$

$\Rightarrow \quad$ L.H.S. = R.H.S.

Put $x = -\frac{5}{3}$ in equation (1)

$$3\left(-\frac{5}{3}\right)^2+2\left(-\frac{5}{3}\right)-\sqrt{3\left(-\frac{5}{3}\right)^2+2\left(-\frac{5}{3}\right)-1} = 3$$

$3 = 3$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = \frac{-1+\sqrt{7}}{3}$ in equation (1)

$$3\left(\frac{-1+\sqrt{7}}{3}\right)^2 + 2\left(\frac{-1+\sqrt{7}}{3}\right) - \sqrt{3\left(\frac{-1+\sqrt{7}}{3}\right)^2 + 2\left(\frac{-1+\sqrt{7}}{3}\right) - 1} = 3$$

$\Rightarrow$ $1 \neq 3$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

Put $x = \frac{-1-\sqrt{7}}{3}$ in equation (1)

$$3\left(\frac{-1-\sqrt{7}}{3}\right)^2 + 2\left(\frac{-1-\sqrt{7}}{3}\right) - \sqrt{3\left(\frac{-1-\sqrt{7}}{3}\right)^2 + 2\left(\frac{-1-\sqrt{7}}{3}\right) - 1} = 3$$

$1 \neq 3$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

On checking, we find that $\frac{-1 \pm \sqrt{7}}{3}$ are extraneous roots.

Hence the solution set $= \left\{1, \frac{-5}{3}\right\}$

Q.2 $\mathbf{x^2 - \frac{x}{2} - 7 = = x - 3\sqrt{2x^2 - 3x + 2}}$

Solution:

$$x^2 - \frac{x}{2} - 7 = = x - 3\sqrt{2x^2 - 3x + 2}$$

$\Rightarrow$ $\frac{2x^2 - x - 14}{2} = x - 3\sqrt{2x^2 - 3x + 2}$

$\Rightarrow$ $2x^2 - x - 14 = 2x - 6\sqrt{2x^2 - 3x + 2}$

$\Rightarrow$ $2x^2 - x - 14 - 2x + 6\sqrt{2x^2 - 3x + 2} = 0$

$\Rightarrow$ $2x^2 - 3x - 14 + 6\sqrt{2x^2 - 3x + 2} = 0$ ..... (1)

Put $\sqrt{2x^2 - 3x + 2} = y$ ..... (2)

Then $2x^2 - 3x + 2 = y^2$

$2x^2 - 3x = y^2 - 2$

Equation (1) becomes

$$y^2 - 2 - 14 + 6y = 0$$

$$\Rightarrow \quad y^2 + 6y - 16 = 0$$

$$\Rightarrow \quad y^2 + 8y - 2y - 16 = 0$$

$$\Rightarrow \quad y(y+8) - 2(y+8) = 0$$

$$\Rightarrow \quad (y+8)(y-2) = 0$$

$$\Rightarrow \quad \text{Either } y + 8 = 0 \quad \text{or} \quad y - 2 = 0$$

$$\Rightarrow \quad y = -8 \quad \text{or} \quad y = 2$$

Put $y = -8$ and $y = 2$ in equation (2)

$$\Rightarrow \quad \sqrt{2x^2 - 3x + 2} = -8$$

$$\Rightarrow \quad 2x^2 - 3x + 2 = 64$$

$$\Rightarrow \quad 2x^2 - 3x + 2 - 64 = 0$$

$$\Rightarrow \quad 2x^2 - 3x + -62 = 0$$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-62)}}{2(2)}$$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{9 + 496}}{4}$$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{505}}{4}$$

$$\Rightarrow \quad \sqrt{2x^2 - 3x + 2} = 2$$

$$\Rightarrow \quad 2x^2 - 3x + 2 = 0$$

$$\Rightarrow \quad 2x^2 - 3x + 2 - 4 = 0$$

$$\Rightarrow \quad 2x^2 - 3x + 2 - 4 = 0$$

$$\Rightarrow \quad 2x^2 - 4x + x - 2 = 0$$

$$\Rightarrow \quad 2x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \quad (x-2)(2x+1) = 0$$

$$\Rightarrow \quad \text{Either } x - 2 = 0 \quad \text{or} \quad 2x + 1 = 0$$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = -\frac{1}{2}$$

$$x = -\frac{1}{2}, 2, \frac{3 \pm \sqrt{505}}{4}$$

Checking:

Put $x = -\frac{1}{2}$ in L.H.S. of equation (1)

$$2\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) - 14 + 6\sqrt{2\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 2}$$

$$= 2\left(\frac{1}{4}\right) + \frac{3}{2} - 14 + 6\sqrt{2\left(\frac{1}{4}\right) + \frac{3}{2} + 2} = 0 = \text{R.H.S.}$$

Put $x = 2$ in L.H.S. of equation (1)

$$2(2)^2 - 3(2)^2 - 14 + 6\sqrt{2(2)^2 - 3(2) + 2} = 0 = \text{R.H.S.}$$

Put $x = \dfrac{3+\sqrt{505}}{4}$ in L.H.S. of equation (1)

$$2\left(\frac{3+\sqrt{505}}{4}\right)^2 - 3\left(\frac{3+\sqrt{505}}{4}\right) - 14 + 6\sqrt{2\left(\frac{3+\sqrt{505}}{4}\right)^2 - 3\left(\frac{3+\sqrt{505}}{4}\right) + 2} \neq 0$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

Put $x = \dfrac{3-\sqrt{505}}{4}$ in L.H.S. of equation (1)

$$2\left(\frac{3-\sqrt{505}}{4}\right)^2 - 3\left(\frac{3-\sqrt{505}}{4}\right) - 14 + 6\sqrt{2\left(\frac{3-\sqrt{505}}{4}\right)^2 - 3\left(\frac{3-\sqrt{505}}{4}\right) + 2} \neq 0$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

On checking, we find that $\dfrac{3 \pm \sqrt{505}}{4}$ are extraneous roots.

Hence the solution set $= \left\{-\frac{1}{2},\ 2\right\}$

Q.3 $\sqrt{2x+8} + \sqrt{x+5} = 7$

Solution:

$$\sqrt{2x+8} + \sqrt{x+5} = 7 \qquad \ldots\ldots\ldots (1)$$

Squaring on both sides

$$\left(\sqrt{2x+8} + \sqrt{x+5}\right)^2 = (7)^2$$

$\Rightarrow$ $\left(\sqrt{2x+8}\right)^2 + \left(\sqrt{x+5}\right)^2 + 2\left(\sqrt{2x+8}\right)\left(\sqrt{x+5}\right) = 49$

$\Rightarrow$ $2x + 8 + x + 5 + 2\sqrt{2x+8}\,(x+5)) = 49$

$\Rightarrow$ $3x + 13 + 2\sqrt{2x^2 + 10x + 8x + 40} = 49$

$\Rightarrow$ $2\sqrt{2x^2 + 18x + 40} = 49 - 3x - 13$

$\Rightarrow$ $2\sqrt{2x^2 + 18x + 40} = 36 - 3x$

$\Rightarrow$ $-2\sqrt{2x^2 + 18x + 40} = 3x - 36$

Squaring again

$$\left(-2\sqrt{2x^2 + 18x + 40}\right)^2 = (3x - 36)^2$$

$\Rightarrow$ $4(2x^2 + 18x + 40) = 9x^2 + 1296 - 216x$

$\Rightarrow$ $8x^2 + 72x + 160 = 9x^2 + 1296 - 216x$

$\Rightarrow \quad 9x^2 + 1296 - 216x - 8x^2 - 72x - 160 = 0$

$\Rightarrow \quad x^2 - 288x + 1136 = 0$

$\Rightarrow \quad x^2 - 284x - 4x + 1136 = 0$

$\Rightarrow \quad x(x - 284) - 4(x - 284) = 0$

$\Rightarrow \quad (x - 284)(x - 4) = 0$

$\Rightarrow \quad$ Either $x - 284 = 0$ or $x - 4 = 0$

$\Rightarrow \quad x = 284$ or $x = 4$

Checking:

Put $x = 284$ in L.H.S. of equation (1)

$\sqrt{2(284) + 8} + \sqrt{284 + 5} = 41 \neq$ R.H.S.

Put $x = 4$ in L.H.S. of equation (1)

$\sqrt{2(4) + 8} + \sqrt{4 + 5} = 7 =$ R.H.S.

$\Rightarrow \quad x = 284$ is an extraneous root.

Hence the solution set $= \{4\}$

Q.4 $\sqrt{3x + 4} = 2 + \sqrt{2x - 4}$

Solution:

$\sqrt{3x + 4} = 2 + \sqrt{2x - 4}$

$\sqrt{3x - 4} - \sqrt{2x - 4} = 2 \quad \ldots\ldots\ldots (1)$

Squaring both sides

$\left(\sqrt{3x + 4} - \sqrt{2x - 4}\right)^2 = (2)^2$

$\Rightarrow \quad \left(\sqrt{3x + 4}\right)^2 + \left(\sqrt{2x - 4}\right)^2 - 2\sqrt{3x + 4}\sqrt{2x - 4} = 4$

$\Rightarrow \quad 3x + 4 + 2x - 4 - 2\sqrt{(3x + 4)(2x - 4)} = 4$

$\Rightarrow \quad 5x - 2\sqrt{6x^2 - 12x + 8x - 16} = 4$

$\Rightarrow \quad 5x - 4 = 2\sqrt{6x^2 - 4x - 16}$

Squaring again

$\Rightarrow \quad 25x^2 + 16 - 40x = 4(6x^2 - 4x - 16)$

$\Rightarrow \quad 25x^2 + 16 - 40x = 24x^2 - 16x - 64$

$\Rightarrow \quad 25x^2 + 16 - 40x - 24x^2 + 16x + 64 = 0$

$\Rightarrow \quad x^2 - 24x + 80 = 0$

$\Rightarrow \quad x^2 - 20x - 4x + 80 = 0$

$\Rightarrow \quad x(x - 20) - 4(x - 20) = 0$

$\Rightarrow \quad$ Either $x - 20 = 0$ or $x - 4 = 0$

$\Rightarrow \quad x = 20$ or $x = 4$

Put $x = 20$ in L.H.S. of equation (1)

$\sqrt{3(20) + 4} - \sqrt{2(20) - 4} = 2 =$ R.H.S.

Put $x = 4$ in L.H.S. of equation (1)

$\sqrt{3(4) + 4} - \sqrt{2(4) - 4} = 2 =$ R.H.S.

None of them is extraneous root.

Hence the solution set $= \{4, 20\}$

Q.5 $\sqrt{x + 7} + \sqrt{x + 2} = \sqrt{6x + 13}$ **(Gujranwala Board 2006)**

Solution:

$\sqrt{x + 7} + \sqrt{x + 2} = \sqrt{6x + 13}$ ..... (1)

Squaring on both sides

$\left(\sqrt{x + 7}\right)^2 + \left(\sqrt{x + 2}\right)^2 + 2\sqrt{x + 7}\sqrt{x + 2} = \left(\sqrt{6x + 13}\right)^2$

$\Rightarrow \quad x + 7 + x + 2 + 2\sqrt{(x + 7)(x + 2)} = 6x + 13$

$\Rightarrow \quad 2x + 9 + 2\sqrt{x^2 + 2x + 7x + 14} = 6x + 13$

$\Rightarrow \quad 2\sqrt{x^2 + 9x + 14} = 6x + 13 - 2x - 9$

$\Rightarrow \quad 2\sqrt{x^2 + 9x + 14} = 4x + 4$

$\Rightarrow \quad 2\sqrt{x^2 + 9x + 14} = 2(2x + 2)$

$\Rightarrow \quad \sqrt{x^2 + 9x + 14} = 2x + 2$

Squaring again

$\Rightarrow \quad x^2 + 9x + 14 = (2x + 2)^2$

$\Rightarrow \quad x^2 + 9x + 14 = 4x^2 + 4 + 8x$

$\Rightarrow \quad 4x^2 + 4 + 8x - x^2 - 9x - 14 = 0$

$\Rightarrow \quad 3x^2 - x - 10 = 0$

$\Rightarrow \quad 3x^2 - 6x + 5x - 10 = 0$

$\Rightarrow \quad 3x(x - 2) + 5(x - 2) = 0$

$\Rightarrow \quad (x - 2) + (3x + 5) = 0$

$\Rightarrow$ Either $x - 2 = 0$ or $3x + 5 = 0$

$\Rightarrow$ $x = 2$ or $x = \frac{-5}{3}$

$\Rightarrow$ $x = 2$ and $x = \frac{-5}{3}$

Checking:

Put $x = 2$ in equation (1)

$\sqrt{(2)+7} + \sqrt{2+2} = \sqrt{6(2)+13}$

$5 = 5$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = \frac{-5}{3}$ in equation (1)

$$\sqrt{\frac{-5}{3}+7} + \sqrt{\frac{-5}{3}+2} = \sqrt{6\left(\frac{-5}{3}\right)+13}$$

$$\frac{5}{\sqrt{3}} \neq \sqrt{3}$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

$\Rightarrow$ $x = -\frac{5}{3}$ is an extraneous root.

Hence the solution set = {2}

Q.6 $\sqrt{x^2 + x + 1} - \sqrt{x^2 + x - 1} = 1$

Solution:

$$\sqrt{x^2 + x + 1} - \sqrt{x^2 + x - 1} = 1 \quad \ldots\ldots\ldots (1)$$

Let $\sqrt{x^2 + x + 1} = a$ and $\sqrt{x^2 + x - 1} = b$

Now $a^2 - b^2 = (x^2 + x + 1) - (x^2 + x - 1) = x^2 + x + 1 - x^2 - x + 1$

$a^2 - b^2 = 2 \quad \ldots\ldots\ldots (2)$

The given equation can be written as

$a - b = 1 \quad \ldots\ldots\ldots (3)$

Dividing equation (2) by equation (3)

$$\frac{a^2 - b^2}{a - b} = \frac{2}{1}$$

$$\frac{(a+b)(a-b)}{(a-b)} = 2$$

$$a + b = 2 \qquad \ldots\ldots\ldots (4)$$

adding (3) and (4)

$$\Rightarrow \quad 2a = 3 \quad \Rightarrow \quad a = \frac{3}{2}$$

$$\Rightarrow \quad \sqrt{x^2 + x + 1} = \frac{3}{2}$$

$$\Rightarrow \quad x^2 + x + 1 = \frac{9}{4}$$

$$\Rightarrow \quad 4x^2 + 4x + 4 = 9$$

$$\Rightarrow \quad 4x^2 + 4x + 4 - 9 = 0$$

$$\Rightarrow \quad 4x^2 + 4x - 5 = 0$$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(-5)}}{2(4)}$$

$$= \frac{-4 \pm \sqrt{16 + 80}}{8} = \frac{-4 \pm \sqrt{96}}{8} = \frac{-4 \pm 4\sqrt{6}}{8} = \frac{-1 \pm \sqrt{6}}{2}$$

Checking

Put $x = \frac{-1 \pm \sqrt{6}}{2}$ in L.H.S. of equation (1)

$$\Rightarrow \quad \sqrt{\left(\frac{-1+\sqrt{6}}{2}\right)^2 + \left(\frac{-1+\sqrt{6}}{2}\right) + 1} - \sqrt{\left(\frac{-1+\sqrt{6}}{2}\right)^2 + \frac{-1+\sqrt{6}}{2} - 1} = 1 = \text{R.H.S.}$$

Put $x = \frac{-1 - \sqrt{6}}{2}$ in L.H.S. of equation (1)

$$\Rightarrow \quad \sqrt{\left(\frac{-1-\sqrt{6}}{2}\right)^2 + \left(\frac{-1-\sqrt{6}}{2}\right) + 1} - \sqrt{\left(\frac{-1-\sqrt{6}}{2}\right)^2 + \frac{-1-\sqrt{6}}{2} - 1} = 1 = \text{R.H.S.}$$

Hence the solution set $= \left\{\frac{-1 \pm \sqrt{6}}{2}\right\}$

Q.7 $\sqrt{x^2 + 2x - 3} + \sqrt{x^2 + 7x - 8} = \sqrt{5(x^2 + 3x - 4)}$

Solution:

$$\sqrt{x^2 + 2x - 3} + \sqrt{x^2 + 7x - 8} = \sqrt{5(x^2 + 3x - 4)} \qquad \ldots\ldots\ldots (1)$$

$$\Rightarrow \quad \sqrt{x^2 + 3x - x - 3} + \sqrt{x^2 + 8x - x - 8} = \sqrt{5(x^2 + 4x - x - 4)}$$

$\Rightarrow \quad \sqrt{x(x+3)-1(x+3)}+\sqrt{x(x+8)-1(x+8)} = \sqrt{5(x+4)(x-1)}$

$\Rightarrow \quad \sqrt{(x+3)(x-1)}+\sqrt{(x+8)(x-1)} = \sqrt{5(x+4)(x-1)}$

$\Rightarrow \quad \sqrt{(x+3)(x-1)}+\sqrt{(x+8)(x-1)}-\sqrt{5(x+4)(x-1)} = 0$

$\Rightarrow \quad \sqrt{(x-1)}\left[\sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)}\right] = 0$

$\Rightarrow \quad$ Either $\sqrt{x-1} = 0 \quad$ or $\quad \sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)} = 0$

$\Rightarrow \quad x-1 = 0 \quad$ or $\quad \sqrt{x+3}+\sqrt{x+8} = \sqrt{5(x+4)}$

$\Rightarrow \quad \boxed{x = 1} \quad$ or $\quad$ Squaring on both sides

$\Rightarrow \quad x+3+x+8+2\sqrt{x+3}\sqrt{x+8} = 5(x+4)$

$\Rightarrow \quad 2x+11+2\sqrt{x^2+8x+3x+24} = 5x+20$

$\Rightarrow \quad 2\sqrt{x^2+11x+24} = 5x+20-2x-11$

$\Rightarrow \quad 2\sqrt{x^2+11x+24} = 3x+9$

Squaring again

$\Rightarrow \quad \left(2\sqrt{x^2+11x+24}\right)^2 = (3x+9)^2$

$\Rightarrow \quad 4(x^2+11x+24) = 9x^2+81+54x$

$\Rightarrow \quad 4x^2+44x+96 = 9x^2+81+54x$

$\Rightarrow \quad 9x^2+81+54x-4x^2-44x-96 = 0$

$\Rightarrow \quad 5x^2+10x-15 = 0$

$\Rightarrow \quad 5(x^2+2x-3) = 0$

$\Rightarrow \quad x^2+2x-3 = 0$

$\Rightarrow \quad x^2+3x-x-3 = 0$

$\Rightarrow \quad x(x+3)-1(x+3) = 0$

$\Rightarrow \quad (x+3)(x-1) = 0$

$\Rightarrow \quad$ Either $x+3 = 0 \quad$ or $\quad x-1 = 0$

$\Rightarrow \quad x = -3 \quad$ or $\quad x = 1$

Checking:

Put $x = 1$ in equation (1)

$$\sqrt{(1)^2+2(1)-3}+\sqrt{(1)^2+7(1)-8} = \sqrt{5\left((1)^2+3(1)-4\right)}$$

$$0+0 = 0$$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = -3$ in equation (1)

$$\sqrt{(-3)^2 + 2(-3) - 3} + \sqrt{(-3)^2 + 7(-3) - 8} = \sqrt{5\left((-3)^2 + 3(-3) - 4\right)}$$

$$0 + \sqrt{-20} = \sqrt{-20}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence the solution set $= \{1, -3\}$

Q.8 $\sqrt{2x^2 - 5x - 3} + 3\sqrt{2x + 1} = \sqrt{2x^2 + 25x + 12}$

Solution:

$$\sqrt{2x^2 - 5x - 3} + 3\sqrt{2x + 1} = \sqrt{2x^2 + 25x + 12} \quad \ldots\ldots\ldots (1)$$

$\Rightarrow$ $\sqrt{2x^2 - 6x + x - 3} + 3\sqrt{2x + 1} = \sqrt{2x^2 + 24x + x + 12}$

$\Rightarrow$ $\sqrt{2x(x-3) + 1(x-3)} + 3\sqrt{2x + 1} = \sqrt{2x(x+12) + 1(x+12)}$

$\Rightarrow$ $\sqrt{(2x+1)(x-3)} + 3\sqrt{2x+1} = \sqrt{(2x+1)(x+12)}$

$\Rightarrow$ $\sqrt{(2x+1)(x-3)} + 3\sqrt{2x+1} - \sqrt{(2x+1)(x+12)} = 0$

$\Rightarrow$ $\sqrt{2x+1}\left[\sqrt{x-3} + 3 - \sqrt{x+12}\right] = 0$

Either $\sqrt{2x+1} = 0$ or $\sqrt{x-3} + 3 - \sqrt{x+12} = 0$

$\Rightarrow$ $2x + 1 = 0$ $\sqrt{x-3} + 3 = \sqrt{x+12}$

$\Rightarrow$ $2x = -\frac{1}{2}$ Squaring on both sides

$x - 3 + 9 + 6\sqrt{x-3} = x + 12$

$6\sqrt{x-3} = x + 12 - x + 3 - 9$

$6\sqrt{x-3} = 6$

$\sqrt{x-3} = 1$

Squaring again

$x - 3 = 1$

$x = 4$

$\Rightarrow$ $x = -\frac{1}{2}, 4$

Checking:

Put $x = -\frac{1}{2}$ in equation (1)

$$\sqrt{2\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 3} + 3\sqrt{2\left(-\frac{1}{2}\right) + 1} = \sqrt{2\left(-\frac{1}{2}\right)^2 - 25\left(-\frac{1}{2}\right) + 12}$$

$$0 + 0 = 0$$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = 4$ in equation (1)

$\sqrt{2(4)^2 - 5(4) - 3} + 3\sqrt{2(4) + 1} = \sqrt{2(4) + 25(4) + 12}$

$3 + 9 = 12$

$\Rightarrow$ L.H.S. = R.H.S.

Hence the solution set $= \left\{-\frac{1}{2}, 4\right\}$

Q.9 $\sqrt{3x^2 - 5x + 2} + \sqrt{6x^2 - 11x + 5} = \sqrt{5x^2 - 9x + 4}$ **(Lahore Board 2003)**

Solution:

$\sqrt{3x^2 - 5x + 2} + \sqrt{6x^2 - 11x + 5} = \sqrt{5x^2 - 9x + 4}$ ..... (1)

$\Rightarrow \sqrt{3x^2 - 3x - 2x + 2} + \sqrt{6x^2 - 6x - 5x + 5} = \sqrt{5x^2 - 5x - 4x + 4}$

$\Rightarrow \sqrt{3x(x-1) - 2(x-1)} + \sqrt{6x(x-1) - 5(x-1)} = \sqrt{5x(x-1) - 4(x-1)}$

$\Rightarrow \sqrt{(x-1)(3x-2)} + \sqrt{(x-1)(6x-5)} = \sqrt{(x-1)(5x-4)}$

$\Rightarrow \sqrt{(x-1)(3x-2)} + \sqrt{(x-1)(6x-5)} - \sqrt{(x-1)(5x-4)} = 0$

$\Rightarrow \sqrt{x-1}\left[\sqrt{3x-2} + \sqrt{6x-5} - \sqrt{5x-4}\right] = 0$

$\Rightarrow$ Either $\sqrt{3x-2} + \sqrt{6x-5} - \sqrt{5x-4} = 0$ or $\sqrt{x-1} = 0$

$\Rightarrow \sqrt{3x-2} + \sqrt{6x-5} = \sqrt{5x-4}$ or $x - 1 = 0$

or $x = 1$

Squaring on both sides

$3x - 2 + 6x - 5 + 2\sqrt{3x-2}\sqrt{6x-5} = 5x - 4$

$\Rightarrow 9x - 7 + 2\sqrt{(3x-2)(6x-5)} = 5x - 4$

$\Rightarrow 2\sqrt{18x^2 - 15x - 12x + 10} = 5x - 4 - 9x + 7$

$\Rightarrow 2\sqrt{18x^2 - 27x + 10} = -4x + 3$

$\Rightarrow -2\sqrt{18x^2 - 27x + 10} = 4x - 3$

Squaring again

$\Rightarrow 4(18x^2 - 27x + 10) = 16x^2 + 9 - 24x$

$\Rightarrow 72x^2 - 108x + 40 = 16x^2 + 9 - 24x$

$\Rightarrow 72x^2 - 108x + 40 - 16x^2 - 9 + 24x = 0$

$\Rightarrow 56x^2 - 84x + 31 = 0$

$$\Rightarrow \quad x = \frac{+84 \pm \sqrt{(-84)^2 - 4(56)(31)}}{2(56)}$$

$$\Rightarrow \quad = \frac{84 \pm \sqrt{7056 - 6944}}{112} = \frac{84 \pm \sqrt{112}}{112}$$

$$\Rightarrow \quad = \frac{84 \pm 4\sqrt{7}}{112} = \frac{21 \pm \sqrt{7}}{28}$$

$$\Rightarrow \quad x = 1, \quad \frac{21 \pm \sqrt{7}}{28}$$

Checking:

Put $x = 1$ in equation (1)

$$\sqrt{3(1)^2 - 5(1) + 2} + \sqrt{6(1)^2 - 11(1) + 5} = \sqrt{5(1)^2 - 9(1) + 4}$$

$$0 + 0 = 0$$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = \frac{21 + \sqrt{7}}{28}$ in equation (1)

$$\sqrt{3\left(\frac{21+\sqrt{7}}{28}\right)^2 - 5\left(\frac{21+\sqrt{7}}{28}\right) + 2} + \sqrt{6\left(\frac{21+\sqrt{7}}{28}\right)^2 - 11\left(\frac{21+\sqrt{7}}{28}\right) + 5}$$

$$= \sqrt{5\left(\frac{21+\sqrt{7}}{28}\right)^2 - 9\left(\frac{21+\sqrt{7}}{28}\right) + 4}$$

$$\sqrt{-0.083} + \sqrt{-0.010} \neq \sqrt{-0.035}$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

Put $x = \frac{21 - \sqrt{7}}{28}$ in equation (1)

$$\sqrt{3\left(\frac{21-\sqrt{7}}{28}\right)^2 - 5\left(\frac{21-\sqrt{7}}{28}\right) + 2} + \sqrt{6\left(\frac{21-\sqrt{7}}{28}\right)^2 - 11\left(\frac{21-\sqrt{7}}{28}\right) + 5}$$

$$= \sqrt{5\left(\frac{21-\sqrt{7}}{28}\right)^2 - 9\left(\frac{21-\sqrt{7}}{28}\right) + 4}$$

L.H.S. $\neq$ R.H.S.

$\Rightarrow$ $x = \frac{21 \pm \sqrt{7}}{28}$ are extraneous roots .

Hence the solution set = {1}

Q.10 $(x + 4)(x + 1) = \sqrt{x^2 + 2x - 15} + 3x + 31$

Solution:

$$(x + 4)(x + 1) = \sqrt{x^2 + 2x - 15} + 3x + 31$$

$$\Rightarrow \quad x^2 + x + 4x + 4 = \sqrt{x^2 + 2x - 15} + 3x + 31$$

$$\Rightarrow \quad x^2 + 5x + 4 - \sqrt{x^2 + 2x - 15} - 3x - 31 = 0$$

$$\Rightarrow \quad x^2 + 2x - 27 - \sqrt{x^2 + 2x - 15} = 0 \qquad \ldots\ldots\ldots (1)$$

Put $\sqrt{x^2 + 2x - 15} = y \qquad \ldots\ldots\ldots (2)$

Then $x^2 + 2x - 15 = y^2$

$$x^2 + 2x = y^2 + 15$$

Equation (1) becomes

$$y^2 + 15 - 27 - y = 0$$

$$\Rightarrow \quad y^2 - y - 12 = 0$$

$$\Rightarrow \quad y^2 - 4y + 3y - 12 = 0$$

$$\Rightarrow \quad y(y - 4) + 3(y - 4) = 0$$

$$\Rightarrow \quad (y - 4)(y + 3) = 0$$

$$\Rightarrow \quad \text{Either } y - 4 = 0 \quad \text{or} \quad y + 3 = 0$$

$$\Rightarrow \quad y = 4 \quad \text{or} \quad y = -3$$

Put $y = 4$ and $y = -3$ in equation (2)

$$\sqrt{x^2 + 2x - 15} = 4$$

$$\Rightarrow \quad x^2 + 2x - 15 = 16$$

$$\Rightarrow \quad x^2 + 2x - 15 - 16 = 0$$

$$\Rightarrow \quad x^2 + 2x - 31 = 0$$

$$\sqrt{x^2 + 2x - 15} = -3$$

$$\Rightarrow \quad x^2 + 2x - 15 = 9$$

$$\Rightarrow \quad x^2 + 2x - 15 - 9 = 0$$

$$\Rightarrow \quad x^2 + 2x - 24 = 0$$

$$\Rightarrow \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-31)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 24}}{2}$$

$$= \frac{-2 \pm \sqrt{128}}{2}$$

$$= \frac{-2 \pm 8\sqrt{2}}{2}$$

$$= -1 \pm 4\sqrt{2}$$

$$x = -6, 4, -1 \pm 4\sqrt{2}$$

$$\Rightarrow \quad x^2 + 6x - 4x - 24 = 0$$

$$\Rightarrow \quad x(x + 6) - 4(x + 6) = 0$$

$$\Rightarrow \quad (x + 6)(x - 4) = 0$$

$$\Rightarrow \quad \text{Either } x + 6 = 0 \text{ or } x - 4 = 0$$

$$\Rightarrow \quad x = -6 \text{ or } x = 4$$

Checking:

Put $x = -6$ in L.H.S. of equation (1)

$$(-6)^2 + 2(-6) - 27 - \sqrt{(-6)^2 + 2(-6) - 15} - 6 \neq 0$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

Put $x = 4$ in L.H.S. of equation (1)

$$(4)^2 + 2(4) - 27 - \sqrt{(4)^2 + 2(4) - 15} - 6 \neq 0$$

$\Rightarrow$ L.H.S. $\neq$ R.H.S.

Put $x = -1 + 4\sqrt{2}$ in L.H.S. of equation (1)

$$(-1 + 4\sqrt{2})^2 + 2(-1 + 4\sqrt{2}) - 27 - \sqrt{(-1 + 4\sqrt{2})^2 + 2(-1 + 4\sqrt{2}) - 15} = 0 = \text{R.H.S.}$$

$\Rightarrow$ L.H.S. = R.H.S.

Put $x = -1 - 4\sqrt{2}$ in L.H.S. of equation (1)

$$(-1 - 4\sqrt{2})^2 + 2(-1 - 4\sqrt{2}) - 27 - \sqrt{(-1 - 4\sqrt{2})^2 + 2(-1 - 4\sqrt{2}) - 15} = 0 = \text{R.H.S.}$$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $x = -6$, and $x = 4$ are extraneous roots.

Hence the solution set $= \{-1 \pm 4\sqrt{2}\}$

Q.11 $\sqrt{3x^2 - 2x + 9} + \sqrt{3x^2 - 2x - 4} = 13$

Solution:

$$\sqrt{3x^2 - 2x + 9} + \sqrt{3x^2 - 2x - 4} = 13 \quad \ldots\ldots\ldots (1)$$

Let $\sqrt{3x^2 - 2x + 9} = a$ and $\sqrt{3x^2 - 2x - 4} = b$

Now $a^2 - b^2 = (3x^2 - 2x + 9) - (3x^2 - 2x - 4)$

$= 3x^2 - 2x + 9 - 3x^2 + 2x + 4$

$a^2 - b^2 = 13$ ..... (2)

Equation (1) can be written as

$a + b = 13$ ..... (3)

Dividing (2) by (3)

$$\frac{a^2 - b^2}{a + b} = \frac{13}{13}$$

$$\frac{(a + b)(a - b)}{a + b} = 1$$

$a - b = 1$ ..... (4)

Adding (3) and (4)

$2a = 14 \Rightarrow a = 7$

$\Rightarrow \sqrt{3x^2 - 2x + 9} = 7$

$\Rightarrow 3x^2 - 2x + 9 = 49$

$\Rightarrow 3x^2 - 2x + 9 - 49 = 0$

$\Rightarrow 3x^2 - 2x - 40 = 0$

$\Rightarrow 3x^2 - 12x + 10x - 40 = 0$

$\Rightarrow 3x(x - 4) + 10(x - 4) = 0$

$\Rightarrow (x - 4)(3x + 10) = 0$

$\Rightarrow$ Either $x - 4 = 0$ or $3x + 10 = 0$

$\Rightarrow x = 4$ or $x = \frac{-10}{3}$

Checking:

Put $x = 4$ in L.H.S. of equation (1)

$\sqrt{3(4)^2 - 2(4) + 9} + \sqrt{3(4)^2 - 2(4) - 4}$

$= 7 + 6 = 13 =$ R.H.S. $\Rightarrow$ L.H.S. = R.H.S.

Put $x = -\frac{10}{3}$ in L.H.S. of equation (1)

$$\sqrt{3\left(-\frac{10}{3}\right)^2 - 2\left(-\frac{10}{3}\right) + 9} + \sqrt{3\left(-\frac{10}{3}\right)^2 - 2\left(-\frac{10}{3}\right) - 4} = 7 + 6 = 13 = \text{R.H.S.}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence the solution set $= \left\{ 4 , -\frac{10}{3} \right\}$

Q.12 $\sqrt{5x^2 + 7x + 2} - \sqrt{4x^2 + 7x + 18} = x - 4$ **(Gujranwala Board 2003)**

Solution:

$$\sqrt{5x^2 + 7x + 2} - \sqrt{4x^2 + 7x + 18} = x - 4 \quad \ldots\ldots\ldots (1)$$

Let $\sqrt{5x^2 + 7x + 2} = a$ and $\sqrt{4x^2 + 7x + 18} = b$

Then $a^2 - b^2 = (5x^2 + 7x + 2) - (4x^2 + 7x + 18)$

$$= 5x^2 + 7x + 2 - 4x^2 - 7x - 18$$

$$a^2 - b^2 = x^2 - 16 \quad \ldots\ldots\ldots (2)$$

Equation (1) can be written as

$$a - b = x - 4 \quad \ldots\ldots\ldots (3)$$

Divide (2) by (3)

$$\frac{a^2 - b^2}{a - b} = \frac{x^2 - 16}{x - 4}$$

$$\frac{(a - b)(a + b)}{(a - b)} = \frac{(x + 4)(x - 4)}{(x - 4)}$$

$$a + b = x + 4 \quad \ldots\ldots\ldots (4)$$

Adding (3) and (4)

$$2a = 2x$$

$\Rightarrow$ $a = x$

$\Rightarrow$ $\sqrt{5x^2 + 7x + 2} = x$

$\Rightarrow$ $5x^2 + 7x + 2 = x^2$

$\Rightarrow$ $5x^2 - x^2 + 7x + 2 = 0$

$\Rightarrow$ $4x^2 + 7x + 2 = 0$

$\Rightarrow$ $x = \frac{-7 \pm \sqrt{(7)^2 - 4(4)(2)}}{2(4)}$

$$= \frac{-7 \pm \sqrt{49 - 32}}{8} = \frac{-7 \pm \sqrt{17}}{8}$$

$\Rightarrow$ $x = \frac{-7 + \sqrt{17}}{8}, \quad \frac{-7 - \sqrt{17}}{8}$

Checking:

Put $x = \dfrac{-7+\sqrt{17}}{8}$ in equation (1)

$$\sqrt{5\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+2}-\sqrt{4\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+18}$$

$$= \frac{-7+\sqrt{17}}{8}-4$$

$$\sqrt{5(-0.3596)^2+7(-0.3596)+2}-\sqrt{4(-0.3596)^2+7(-0.359)+18} = -0.3596-4$$

$$= -4.3596$$

$$\sqrt{-1.8706+2}-\sqrt{-1.8706+18} = -4.3596$$

$$0.3597-4.0161 = -4.3596$$

$$-3.6564 = -4.3596$$

Put, $x = \dfrac{-7-\sqrt{17}}{8}$ in equation (i)

$$\sqrt{5\left(\frac{-7-\sqrt{17}}{8}\right)^2+7\left(\frac{-7-\sqrt{17}}{8}\right)+2}-\sqrt{4\left(\frac{-7-\sqrt{17}}{8}\right)^2+7\left(\frac{-7-\sqrt{17}}{8}\right)+18}=\frac{-7-\sqrt{17}}{8}-4$$

$$\sqrt{5(-1.39)^2+7(-1.39)+2}-\sqrt{4(-1.39)^2+7(-1.39)+18} = -1.39-4$$

$$\sqrt{5(1.93)-9.73+2}-\sqrt{4(1.93)-9.73+18} = -5.39$$

$$\sqrt{1.92}-\sqrt{15.99} = -5.39$$

$$1.38-3.99 = -5.39$$

$$-2.61 = -5.39$$

$\Rightarrow$ L.H.S $\neq$ R.H.S

$\Rightarrow$ $x = \dfrac{-7\pm\sqrt{17}}{8}$ are extraneous roots

$\Rightarrow$ S.S $= \phi$

THREE CUBE ROOTS OF UNITY

As we know that square roots of one (unity) are two, 1 and – 1. Similarly cube roots of one (unity) are three and these can be calculated as:

Let 'x' be the cube root of unity, then

$1^{1/3} = x$

$\Rightarrow \quad 1 = x^3 \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad (x)^3-(1)^3 = 0$

Checking:

Put $x = \frac{-7+\sqrt{17}}{8}$ in equation (1)

$$\sqrt{5\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+2}-\sqrt{4\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+18}$$

$$= \frac{-7+\sqrt{17}}{8}-4$$

$$\sqrt{5(-0.3596)^2+7(-0.3596)+2}-\sqrt{4(-0.3596)^2+7(-0.359)+18} = -0.3596-4$$

$$= -4.3596$$

$$\sqrt{-1.8706+2}-\sqrt{-1.8706+18} = -4.3596$$

$$0.3597-4.0161 = -4.3596$$

$$-3.6564 = -4.3596$$

Put, $x = \frac{-7-\sqrt{17}}{8}$ in equation (i)

$$\sqrt{5\left(\frac{-7-\sqrt{17}}{8}\right)^2+7\left(\frac{-7-\sqrt{17}}{8}\right)+2}-\sqrt{4\left(\frac{-7-\sqrt{17}}{8}\right)^2+7\left(\frac{-7-\sqrt{17}}{8}\right)+18}=\frac{-7-\sqrt{17}}{8}-4$$

$$\sqrt{5(-1.39)^2+7(-1.39)+2}-\sqrt{4(-1.39)^2+7(-1.39)+18} = -1.39-4$$

$$\sqrt{5(1.93)-9.73+2}-\sqrt{4(1.93)-9.73+18} = -5.39$$

$$\sqrt{1.92}-\sqrt{15.99} = -5.39$$

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$\Rightarrow$ L.H.S $\neq$ R.H.S

$\Rightarrow$ $x = \frac{-7\pm\sqrt{17}}{8}$ are extraneous roots

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THREE CUBE ROOTS OF UNITY

As we know that square roots of one (unity) are two, 1 and – 1. Similarly cube roots of one (unity) are three and these can be calculated as:

Let 'x' be the cube root of unity, then

$1^{1/3} = x$

$\Rightarrow \quad 1 = x^3 \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad (x)^3-(1)^3 = 0$

$\Rightarrow \quad (x-1)(x^2+x+1) = 0$

$\Rightarrow \quad$ Either $x-1=0 \quad$ or $\quad x^2+x+1=0$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{1-4}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow \quad$ Cube roots of unity (one) are

$1, \dfrac{-1+\sqrt{-3}}{2}, \dfrac{-1-\sqrt{-3}}{2}$

or $\quad 1, \omega, \omega^2$

There are four properties of cube roots of unity.

(i) Each complex cube root of unity is square of the other.

i.e. if $\dfrac{-1+\sqrt{-3}}{2} = \omega$ then $\dfrac{-1-\sqrt{-3}}{2} = \omega^2$

and if $\dfrac{-1-\sqrt{-3}}{2} = \omega$ then $\dfrac{-1+\sqrt{-3}}{2} = \omega^2$

(ii) The sum of all the three cube roots of unity is zero.

i.e. $1 + \omega + \omega^2 = 0$

(iii) The product of all the three cube roots of unity is unity.

i.e. $1 \cdot \omega \cdot \omega^2 = 1 \quad \Rightarrow \quad \omega^3 = 1$

(iv) For any $n \in z$, ω^n is equivalent to one of the cube roots of unity.

e.g. $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$

$\omega^{-5} = \dfrac{1}{\omega^5} = \dfrac{1}{\omega^3 \cdot \omega^2} = \dfrac{1}{1 \cdot \omega^2} = \dfrac{1}{\omega^2} \cdot \dfrac{\omega}{\omega} = \dfrac{\omega}{\omega^3} = \dfrac{\omega}{1} = \omega$

FOUR FOURTH ROOTS OF UNITY

Let 'x' be the fourth root of unity then

$1^{1/4} = x \quad \Rightarrow \quad x^4 = 1$

$\Rightarrow \quad x^4 - 1 = 0$

$\Rightarrow \quad (x^2)^2 - (1)^2 = 0$

$\Rightarrow \quad (x^2 + 1)(x^2 - 1) = 0$

$\Rightarrow \quad$ Either $x^2 + 1 = 0$ or $x^2 - 1 = 0$

$\Rightarrow \quad x^2 = -1$ or $x^2 = 1$

$\Rightarrow \quad x = \pm\sqrt{-1}$ or $x = \pm\sqrt{1}$

$\Rightarrow \quad x = \pm i$ or $x = \pm 1$

$\Rightarrow \quad$ Four fourth roots of unity are $1, -1, i, -i$.

PROPERTIES OF FOUR FOURTH ROOTS OF UNITY

(i) Sum of all the four fourth roots of unity is zero.

i.e. $1 - 1 + i - i = 0$

(ii) The real fourth roots of unity are additive inverses of each other.

$+1$ and -1 are real fourth roots of unity and $+1 + (-1) = 0 = (-1) + 1$

(iii) Both the complex fourth roots of unity are conjugate of each other.

i and $-i$ are complex fourth roots of unity which are conjugate of each other.

(iv) Product of all the fourth roots of unity is -1.

i.e. $1 \times (-1) \times i \times (-i) = -1$

EXERCISE 4.4

Q.1 Find the three cube roots of: 8, – 8, 27, – 27, 64

Solution:

(i) Cube root of 8.

Let 'x' be the cube root of 8.

$\Rightarrow \quad 8^{1/3} = x$

$\Rightarrow \quad 8 = x^3 \quad \Rightarrow \quad x^3 = 8$

$\Rightarrow \quad x^3 - 8 = 0$

$\Rightarrow \quad (x)^3 - (2)^3 = 0$

$\Rightarrow \quad (x - 2)(x^2 + 2x + 4) = 0$

$\Rightarrow \quad$ Either $x - 2 = 0$ or $x^2 + 2x + 4 = 0$

$\Rightarrow \quad x = 2$ or $x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$= \dfrac{-2 \pm \sqrt{4 - 16}}{2}$

$= \dfrac{-2 \pm \sqrt{-12}}{2}$

$= \dfrac{-2 \pm 2\sqrt{-3}}{2}$

$= 2\left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)$

$= 2\omega, \ 2\omega^2$

$\Rightarrow \quad$ cube roots of 8 are $2, 2\omega, 2\omega^2$.

(ii) Cube root of -8.

Let 'x' be the cube root of -8.

$\Rightarrow \quad (-8)^{1/3} = x$

$\Rightarrow \quad -8 = x^3 \quad \Rightarrow \quad x^3 = 8 = 0$

$\Rightarrow \quad (x)^3 + (2)^3 = 0$

$\Rightarrow \quad (x + 2)(x^2 - 2x + 4) = 0$

$\Rightarrow \quad$ Either $x + 2 = 0$ or $x^2 - 2x + 4 = 0$

$\Rightarrow \quad x = -2$ or $x = \dfrac{-(-2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$$= \frac{+2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2\sqrt{-3}}{2}$$

$$= -2\left(\frac{-2 \pm \sqrt{-3}}{2}\right)$$

$$= -2\omega, \; -2\omega^2$$

$\Rightarrow \quad$ cube roots of -8 are $-2, -2\omega, -2\omega^2$.

(iii) Cube root of 27.

Let 'x' be the cube root of 27.

$\Rightarrow \quad (27)^{1/3} = x \quad \Rightarrow \quad x^3 = 27$

$\Rightarrow \quad x^3 - 27 = 0$

$\Rightarrow \quad (x)^3 - (3)^3 = 0$

$\Rightarrow \quad (x - 3)(x^2 + 3x + 9) = 0$

$\Rightarrow \quad$ Either $x - 3 = 0$ or $x^2 + 3x + 9 = 0$

$\Rightarrow \quad x = 3$ or $x = \dfrac{-3 \pm \sqrt{(3)^2 - 4(1)(9)}}{2(1)}$

$$= \frac{-3 \pm \sqrt{9 - 36}}{2}$$

$$= \frac{-3 \pm \sqrt{-27}}{2}$$

$$= \frac{-3 \pm 3\sqrt{-3}}{2}$$

$$= 3\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$

$$= 3\omega, \; 3\omega^2$$

$\Rightarrow \quad$ cube roots of 27 are $3, 3\omega, 3\omega^2$.

(iv) Cube root of -27.

Let 'x' be the cube root of -27.

$\Rightarrow \quad (-27)^{1/3} = x \quad \Rightarrow \quad -27 = x^3$

$\Rightarrow \quad x^3 + 27 = 0$

$\Rightarrow \quad (x)^3 + (3)^3 = 0$

$\Rightarrow \quad (x + 3)(x^2 - 3x + 9) = 0$

$\Rightarrow \quad$ Either $x + 3 = 0$ or $x^2 - 3x + 9 = 0$

$\Rightarrow \quad x = -3$ or $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{9 - 36}}{2}$

$= \dfrac{3 \pm \sqrt{-27}}{2}$

$= \dfrac{3 \pm 3\sqrt{-3}}{2}$

$= -3\left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)$

$= -3\omega, \; -3\omega^2$

$\Rightarrow$ cube roots of -27 are $-3, -3\omega, -3\omega^2$.

(v) Cube root of 64.

Let 'x' be the cube root of 64.

$\Rightarrow \quad (64)^{1/3} = x \quad \Rightarrow \quad 64 = x^3$

$\Rightarrow \quad x^3 - 64 = 0$

$\Rightarrow \quad (x)^3 - (4)^3 = 0$

$\Rightarrow \quad (x - 4)(x^2 + 4x + 16) = 0$

$\Rightarrow \quad$ Either $x - 4 = 0$ or $x^2 + 4x + 16 = 0$

$\Rightarrow \quad x = 4$ or $x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{16 - 64}}{2}$

$= \dfrac{-4 \pm \sqrt{-48}}{2}$

$= \dfrac{-4 \pm 4\sqrt{-3}}{2}$

$= 4\left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)$

$= 4\omega, \; 4\omega^2$

$\Rightarrow$ cube roots of 64 are $4, 4\omega, 4\omega^2$.

Q.2 Evaluate:

Solution:

(i) $(1+\omega-\omega^2)^8 = (-\omega^2-\omega^2)^8$ **(Lahore Board 2008, 2011)**

$= (-2\omega^2)^8$ ∵ $1+\omega+\omega^2 = 0$

$= (-2)^8(\omega^2)^8$

$= 256\,\omega^{16} = 256\,\omega^{15}.\omega$ ∵ $\omega^3 = 1$

$= 256\,(\omega^3)^5.\omega = 256\,(1)^5.\omega = 256\,\omega$

(ii) $\omega^{28}+\omega^{29}+1 = \omega^{27}.\omega+\omega^{27}.\omega^2+1$ **(Lahore Board 2004, 2006, 2007)**

$= (\omega^3)^9.\omega+(\omega^3)^9.\omega^2+1$

$= (1)^9.\omega+(1)^9.\omega^2+1$ ∵ $\omega^3 = 1$

$= \omega+\omega^2+1 = 0$

(iii) $(1+\omega-\omega^2)(1-\omega+\omega^2) = 1-\omega+\omega^2+\omega-\omega^2+\omega^3-\omega^2+\omega^3-\omega^4$ **(Lahore Board 2006)**

$= 1-\omega^2+\omega^3+\omega^3-\omega^4$

$= 1-\omega^2+1+1-\omega^3.\omega$ ∵ $1+\omega+\omega^2 = 0$

$= 3-\omega^2-(1).\omega$ $\omega+\omega^2 = -1$

$= 3-\omega^2-\omega$

$= 3-(\omega^2+\omega) = 3-(-1) = 3+1 = 4$

(iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^7 = \omega^9+(\omega^2)^7$

$= (\omega^3)^3+\omega^{14} = (1)^3+\omega^{12}.\omega^2$

$= 1+(\omega^3)^4.\omega^2$

$= 1+(1)^4.\omega^2$

$= 1+\omega^2$ ∵ $1+\omega+\omega^2 = 0$

$= -\omega$ $1+\omega^2 = -\omega$

(v) $\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5 = \left(2.\frac{-1+\sqrt{-3}}{2}\right)^5+\left(2.\frac{-1-\sqrt{-3}}{2}\right)^5$

$= (2\omega)^5+(2\omega^2)^5$

$= (2)^5.\omega^5+2^5.(\omega^2)^5$

$= 32.\omega^3.\omega^2+32.\omega^{10}$

$= 32.(1).\omega^2+32.\omega^9.\omega$

$= 32\,\omega^2+32.(\omega^3)^3.\omega$

$= 32\,\omega^2+32\,(1)^3.\omega$

$= 32\,\omega^2+32\,\omega$ ∵ $1+\omega+\omega^2 = 0$

$= 32\,(\omega^2+\omega)$ $\Rightarrow \omega^2+\omega = -1$

$= 32\,(-1) = -32$

Q.2 Evaluate:

Solution:

(i) $(1+\omega-\omega^2)^8 = (-\omega^2-\omega^2)^8$ **(Lahore Board 2008, 2011)**

$= (-2\omega^2)^8$ ∵ $1+\omega+\omega^2 = 0$

$= (-2)^8(\omega^2)^8$

$= 256\,\omega^{16} = 256\,\omega^{15}.\omega$ ∵ $\omega^3 = 1$

$= 256\,(\omega^3)^5.\omega = 256\,(1)^5.\omega = 256\,\omega$

(ii) $\omega^{28}+\omega^{29}+1 = \omega^{27}.\omega+\omega^{27}.\omega^2+1$ **(Lahore Board 2004, 2006, 2007)**

$= (\omega^3)^9.\omega+(\omega^3)^9.\omega^2+1$

$= (1)^9.\omega+(1)^9.\omega^2+1$ ∵ $\omega^3 = 1$

$= \omega+\omega^2+1 = 0$

(iii) $(1+\omega-\omega^2)(1-\omega+\omega^2) = 1-\omega+\omega^2+\omega-\omega^2+\omega^3-\omega^2+\omega^3-\omega^4$ **(Lahore Board 2006)**

$= 1-\omega^2+\omega^3+\omega^3-\omega^4$

$= 1-\omega^2+1+1-\omega^3.\omega$ ∵ $1+\omega+\omega^2 = 0$

$= 3-\omega^2-(1).\omega$ $\omega+\omega^2 = -1$

$= 3-\omega^2-\omega$

$= 3-(\omega^2+\omega) = 3-(-1) = 3+1 = 4$

(iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^7 = \omega^9+(\omega^2)^7$

$= (\omega^3)^3+\omega^{14} = (1)^3+\omega^{12}.\omega^2$

$= 1+(\omega^3)^4.\omega^2$

$= 1+(1)^4.\omega^2$

$= 1+\omega^2$ ∵ $1+\omega+\omega^2 = 0$

$= -\omega$ $1+\omega^2 = -\omega$

(v) $\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5 = \left(2.\frac{-1+\sqrt{-3}}{2}\right)^5+\left(2.\frac{-1-\sqrt{-3}}{2}\right)^5$

$= (2\omega)^5+(2\omega^2)^5$

$= (2)^5.\omega^5+2^5.(\omega^2)^5$

$= 32.\omega^3.\omega^2+32.\omega^{10}$

$= 32.(1).\omega^2+32.\omega^9.\omega$

$= 32\,\omega^2+32.(\omega^3)^3.\omega$

$= 32\,\omega^2+32\,(1)^3.\omega$

$= 32\,\omega^2+32\,\omega$ ∵ $1+\omega+\omega^2 = 0$

$= 32\,(\omega^2+\omega)$ $\Rightarrow \omega^2+\omega = -1$

$= 32\,(-1) = -32$

Q.3 Show that

Solution:

(i) $x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y)$ **(Gujranwala Board 2004)**

$$\text{R.H.S.} = (x - y)(x - \omega y)(x - \omega^2 y)$$

$$= (x - y)(x^2 - x y \omega^2 - x y \omega + y^2 \omega^3)$$

$$= (x - y)(x^2 - x y \omega^2 - x y \omega + y^2) \quad \because \omega^3 = 1$$

$$= (x - y)[x^2 - xy(\omega^2 + \omega) + y^2]$$

$$= (x - y)[x^2 - xy(-1) + y^2] \quad \because 1 + \omega + \omega^2 = 0$$

$$= (x - y)(x^2 + xy + y^2) \quad \omega + \omega^2 = -1$$

$$= x^3 - y^3 = \text{L.H.S.}$$

Hence proved.

(ii) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 y)(x + \omega^2 y + \omega z)$ **(Lahore Board 2004)**

$(x + y + z)(x + \omega y + \omega^2 y)(x + \omega^2 y + \omega z)$

$$= (x + y + z)\left[x^2 + xy\omega^2 + xz\omega + xy\omega + y^2\omega^3 + yz\omega^2 + xz\omega^2 + yz\omega^4 + z^2\omega^3\right]$$

$$= (x + y + z)\left[x^2 + y^2\omega^3 + z^2\omega^3 + xy\omega^2 + xy\omega + yz\omega^2 + yz\omega^4 + yz\omega^4 + xz\omega + xz\omega^2\right]$$

$$= (x + y + z)\left[x^2 + y^2 + z^2 + xy(\omega^2 + \omega) + yz(\omega^2 + \omega^4) + xz(\omega + \omega^2)\right]$$

$$= (x + y + z)\left[x^2 + y^2 + z^2 + xy(-1) + yz(\omega^2 + \omega) + xz(-1)\right] \quad \because \omega^4 = \omega^3 \cdot \omega = \omega$$

$$= (x + y + z)\left[x^2 + y^2 + z^2 - xy + yz(-1) - xz\right]$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= x^3 + y^3 + z^3 - 3xyz = \text{R.H.S.} \Rightarrow \text{Hence proved.}$$

(iii) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots\ldots$ 2n factors $= 1$ **(Lahore Board 2006)**

$$\text{L.H.S.} = (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots. \text{2n factors.}$$

$$= (1 + \omega)(1 + \omega^2)(1 + \omega^3 \cdot \omega)(1 + \omega^6 \cdot \omega^2) \ldots. \text{2n factors.}$$

$$= (-\omega^2)(-\omega)(-\omega^2)(-\omega) \ldots. \text{2n factors}$$

$$= \left[(-\omega^2)(-\omega)\right]\left[(-\omega^2)(-\omega)\right] \ldots. \text{n factors}$$

$$= \left[\omega^3\right]\left[\omega^3\right] \ldots. \text{n factors}$$

$$= 1 \cdot 1 \ldots. \text{n factors}$$

$$= (1)^n = 1 = \text{R.H.S.}$$

Hence the proof.

Q.4 If ω is a root of $x^2 + x + 1 = 0$ show that its other root is ω^2 and prove that $\omega^3 = 1$.

Solution:

If 'ω' is a root of $x^2 + x + 1 = 0$

$\Rightarrow$ It will satisfy the given equation.

$\Rightarrow$ $\omega^2 + \omega + 1 = 0$ ..... (1)

To prove ω^2 is also its root.

We will prove that

$(\omega^2)^2 + \omega^2 + 1 = 0$

i.e. $\omega^4 + \omega^2 + 1 = 0$ ..... (2)

Take L.H.S.

$\Rightarrow$ $\omega^4 + \omega^2 + 1 = (\omega^2)^2 + \omega^2 + \omega^2 - \omega^2 + 1$

$= (\omega^2)^2 + 2\omega^2 + (1)^2 - \omega^2$

$= (\omega^2 + 1)^2 - \omega^2$

$= (\omega^2 + 1 - \omega^2)(\omega^2 + 1 + \omega)$ ⌑ from equation (1)

$= (\omega^2 + 1 - \omega^2)(0)$

$= 0 =$ R.H.S.

$\Rightarrow$ ω^2 is also root of $x^2 + x + 1$.

Now to prove $\omega^3 = 1$.

Subtracting equation (1) from equation (2)

$\omega^4 + \omega^2 + 1 - \omega^2 - \omega - 1 = 0$

$\Rightarrow$ $\omega^4 - \omega = 0$

$\Rightarrow$ $\omega(\omega^3 - 1) = 0$

As $\omega \neq 0 \Rightarrow \omega^3 - 1 = 0 \Rightarrow \omega^3 = 1$

Hence proved.

Q.5 Prove that complex cube roots of -1 are $\frac{1 + \sqrt{3}\,i}{2}$ and $\frac{1 - \sqrt{3}\,i}{2}$ and hence prove that $\left(\frac{1 + \sqrt{-3}}{2}\right)^9 + \left(\frac{1 - \sqrt{-3}}{2}\right)^9 = -2$.

Solution:

Let 'x' be the cube root of -1 then $(-1)^{1/3} = x \Rightarrow x^3 = -1$.

$\Rightarrow$ $x^3 + 1 = 0$

$\Rightarrow$ $(x + 1)(x^2 - x + 1) = 0$

$\Rightarrow$ Either $x + 1 = 0$ or $x^2 - x + 1 = 0$

$\Rightarrow$ $x = -1$ $\qquad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$= \frac{1 \pm \sqrt{1 - 4}}{2}$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}$$

$\Rightarrow$ cube roots of -1 are $-1, \dfrac{1 \pm \sqrt{3}\, i}{2}$

$\Rightarrow$ complex cube roots of -1 are $\dfrac{1 \pm \sqrt{3}\, i}{2}$ and $\dfrac{1 - \sqrt{3}\, i}{2}$.

Now to prove

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9 = -2$$

Take L.H.S.

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9 = \left[-\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^9 + \left[-\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^9$$

$$= (-\omega^2)^9 + (-\omega)^9$$

$$= -\omega^{18} - \omega^9 = -(\omega^3)^6 - (\omega^3)^3$$

$$= -(1)^6 - (1)^3$$

$$= -1 - 1 = -2 = \text{R.H.S.}$$

$\Rightarrow$ Hence proved.

Q.6 It 'ω' is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$. (Lahore Board 2003)

Solution:

Let the required equation is

$$x^2 - Sx + P = 0 \qquad \ldots\ldots\ldots (1)$$

where S = Sum of roots

$$= 2\omega + 2\omega^2 = 2(\omega + \omega^2) = 2(-1) = -2 \qquad \because\ 1 + \omega + \omega^2 = 0$$

$$\omega + \omega^2 = -1$$

P = Product of roots

$$= 2\omega \cdot 2\omega^2 = 4\omega^3 = 4(1) = 4 \qquad \because\ \omega^3 = 1$$

$\Rightarrow$ equation (1) becomes

$$x^2 - (-2)x + 4 = 0$$

$\Rightarrow$ $x^2 + 2x + 4 = 0$

Q.7 Find the four fourth roots of 16, 81; 625.

Solution:

(i) Fourth roots of 16

Let 'x' be the fourth root of 16

$\Rightarrow \quad (16)^{1/4} = x \quad \Rightarrow \quad x^4 = 16$

$\Rightarrow \quad x^4 - 16 = 0$

$\Rightarrow \quad (x^2)^2 - (4)^2 = 0$

$\Rightarrow \quad (x^2 + 4)(x^2 - 4) = 0$

$\Rightarrow \quad$ Either $x^2 + 4 = 0$ or $x^2 - 4 = 0$

$\Rightarrow \quad x^2 = -4 \qquad x^2 = 4$

$\Rightarrow \quad x = \pm\sqrt{-4} \qquad x = \pm\sqrt{4}$

$\Rightarrow \quad x = \pm 2i \qquad x = \pm 2$

$\Rightarrow \quad$ four fourth roots of 16 are $-2, 2, -2i, 2i$.

(ii) Fourth roots of 81

Let 'x' be the fourth root of 81

$\Rightarrow \quad (81)^{1/4} = x \quad \Rightarrow \quad x^4 = 81$

$\Rightarrow \quad x^4 - 81 = 0$

$\Rightarrow \quad (x^2)^2 - (9)^2 = 0$

$\Rightarrow \quad (x^2 + 9)(x^2 - 9) = 0$

$\Rightarrow \quad$ Either $x^2 + 9 = 0$ or $x^2 - 9 = 0$

$\Rightarrow \quad x^2 = -9 \qquad x^2 = 9$

$\Rightarrow \quad x = \pm\sqrt{-9} \qquad x = \pm\sqrt{9}$

$\Rightarrow \quad x = \pm 3i \qquad x = \pm 3$

$\Rightarrow \quad$ four fourth roots of 81 are $3, -3, 3i, -3i$.

(iii) Fourth roots of 625

Let 'x' be the fourth root of 625

$\Rightarrow \quad (625)^{1/4} = x \quad \Rightarrow \quad x^4 = 625$

$\Rightarrow \quad x^4 - 625 = 0$

$\Rightarrow \quad (x^2)^2 - (25)^2 = 0$

$\Rightarrow \quad (x^2 + 25)(x^2 - 25) = 0$

$\Rightarrow \quad$ Either $x^2 + 25 = 0$ or $x^2 - 25 = 0$

$\Rightarrow$ $x^2 = -25$ $x^2 = 25$

$\Rightarrow$ $x = \pm\sqrt{-25}$ $x = \pm\sqrt{25}$

$\Rightarrow$ $x = \pm 5\,i$ $x = \pm 5$

$\Rightarrow$ four fourth roots of 625 are $5, -5, 5i, -5i$.

Q.8 Solve the following equation:

Solution:

(i) $\mathbf{2x^4 - 32 = 0}$

$2(x^4 - 16) = 0$

$\Rightarrow$ $x^4 - 16 = 0$

$\Rightarrow$ $(x^2)^2 - (4)^2 = 0$

$\Rightarrow$ $(x^2 + 4)(x^2 - 4) = 0$

$\Rightarrow$ Either $x^2 + 4 = 0$ or $x^2 - 4 = 0$

$\Rightarrow$ $x^2 = -4$ $x^2 = 4$

$\Rightarrow$ $x = \pm\sqrt{-4}$ $x = \pm\sqrt{4}$

$\Rightarrow$ $x = \pm 2\,i$ $x = \pm 2$

$\Rightarrow$ $x = \pm 2, \pm 2i$

Hence the solution set $= \{\pm 2, \pm 2i\}$

(ii) $\mathbf{3y^5 - 243y = 0}$

$3y^5 - 243y = 0$

$3y(y^4 - 81) = 0$

$\Rightarrow$ Either $y = 0$ or $y^4 - 81 = 0$

$\Rightarrow$ $y = 0$ $\Rightarrow$ $(y^2)^2 - (9)^2 = 4$

$\Rightarrow$ $(y^2 + 9)(y^2 - 9) = 0$

$\Rightarrow$ Either $y^2 + 9 = 0$ or $y^2 - 9 = 0$

$\Rightarrow$ $y^2 = -9$ $y^2 = +9$

$\Rightarrow$ $y = \pm\sqrt{-9}$ $y = \pm\sqrt{9}$

$\Rightarrow$ $y = \pm 3i$ $y = \pm 3$

$\Rightarrow$ $y = 0, \pm 3, \pm 3i$

Hence the solution set $= \{0, \pm 3, \pm 3i\}$

(iii) $\mathbf{x^3 + x^2 + x + 1 = 0}$

$x^3 + x^2 + x + 1 = 0$

$\Rightarrow \quad x^2 (x + 1) + 1 (x + 1) = 0$

$\Rightarrow \quad (x + 1) (x^2 + 1) = 0$

$\Rightarrow$ Either $x + 1 = 0$ or $x^2 + 1 = 0$

$x = -1$ $\quad x^2 = -1$

$x = \pm\sqrt{-1}$

$x = \pm i$

Hence the solution set $= \{ -1, \pm i \}$

(iv) $\mathbf{5x^5 - 5x = 0}$

$5x^5 - 5x = 0$

$\Rightarrow \quad 5x (x^4 - 1) = 0$

$\Rightarrow$ Either $x = 0$ or $x^4 - 1 = 0$

$\Rightarrow \quad x = 0$ or $(x^2)^2 - (1)^2 = 0$

$(x^2 + 1) (x^2 - 1) = 0$

Either $x^2 + 1 = 0$ or $x^2 - 1 = 0$

$\Rightarrow \quad x^2 = -1$ or $x^2 = 1$

$\Rightarrow \quad x = \pm\sqrt{-1}$ or $x = \pm\sqrt{1}$

$\Rightarrow \quad x = \pm i$ or $x = \pm 1$

Hence the solution set $= \{ 0, \pm 1, \pm i \}$

POLYNOMIAL FUNCTION

A polynomial in 'x' is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$ where 'n' is a non–negative integer and the coefficients $a_n, a_{n-1}, \ldots a_1$ and a_0 are real numbers. The highest power of 'x' is called the degree of the polynomial. So the above expression is a polynomial of degree n.

REMAINDER THEOREM

If a polynomial $f(x)$ of degree $n \geq 1$, 'n' is non–negative integer is divided by $x - a$ till no x–term exists in the remainder, then $f(a)$ is the remainder i.e. $f(a) = R$.

FACTOR THEOREM

The polynomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

SYNTHETIC DIVISION

There is a shortcut method for long division of a polynomial $f(x)$ by a polynomial of the form $x - a$. This process of division is called synthetic division.

EXERCISE 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

Q.1 $x^2 + 3x + 7,\quad x + 1$ **(Gujranwala Board 2005)**

Solution:

$x^2 + 3x + 7,\quad x + 1$

Let $f(x) = x^2 + 3x + 7$ ..... (1) $\qquad x + 1 = 0$

$\Rightarrow \quad f(-1) = (-1)^2 + 3(-1) + 7 \qquad x = -1$

$= 1 - 3 + 7 \qquad$ Put in equation (1)

$= -2 + 7 = 5 = R$

$\Rightarrow$ Remainder $= R = 5$

Q.2 $x^3 - x^2 + 5x + 4,\quad x - 2$

Solution:

Let $f(x) = x^3 - x^2 + 5x + 4$ ..... (1) $\qquad x - 2 = 0$

$f(2) = (2)^3 - (2)^2 + 5(2) + 4 \qquad x = 2$

$= 8 - 4 + 10 + 4 \qquad$ Put in (1)

$= 18 = R$

$\Rightarrow$ Remainder $= R = 18$

Q.3 $3x^4 + 4x^3 + x - 5,\ x + 1$

Solution:

Let $f(x) = 3x^4 + 4x^3 + x - 5$ ..... (1) $\qquad x + 1 = 0$

$f(-1) = 3(-1)^4 + 4(-1)^3 + (-1) - 5 \qquad x = -1$

$= 3 - 4 - 1 - 5 \qquad$ Put in (1)

$= -7 = R$

$\Rightarrow$ Remainder $= R = -7$

Q.4 $x^3 - 2x^2 + 3x + 3,\ x - 3$ **(Gujranwala Board 2007)**

Solution:

Let $f(x) = x^3 - 2x^2 + 3x + 3$ (1) $\qquad x - 3 = 0$

$f(3) = (3)^3 - 2(3)^2 + 3(3) + 3$ $\quad x = 3$

$= 27 - 2(9) + 9 + 3$ $\quad$ Put in (1)

$= 27 - 28 + 9 + 3 = 21 = R$

$\Rightarrow$ Remainder $= R = 21$

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

Q.5 $\mathbf{x - 1, \quad x^2 + 4x - 5}$

Solution:

Let $f(x) = x^2 + 4x - 5$ ..... (1) $\quad x - 1 = 0$

$f(1) = (1)^2 + 4(1) - 5$ $\quad x = 1$

$= 1 + 4 - 5$ $\quad$ Put in (1)

$= 0 =$ Remainder

$\Rightarrow$ $x - 1$ is a factor of $x^2 + 4x - 5$.

Q.6 $\mathbf{(x - 2), \ (x^3 + x^2 - 7x - 1)}$ **(Lahore Board 2006)**

Solution:

Let $f(x) = x^3 + x^2 - 7x + 1$ ..... (1) $\quad x - 2 = 0$

$f(2) = (2)^3 + (2)^2 - 7(2) + 1$ $\quad x = 2$

$= 8 + 4 - 14 + 1$ $\quad$ Put in (1)

$= -1 = 0$

$\Rightarrow$ $(x - 2)$ is not a factor of $x^3 + x^2 - 7x - 1$.

Q.7 $\mathbf{\omega + 2, \ 2\omega^3 + \omega^2 - 4\omega + 7}$

Solution:

Let $f(x) = 2\omega^3 + \omega^2 - 4\omega + 7$ ..... (1) $\quad \omega + 2 = 0$

$f(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7$ $\quad \omega = -2$

$= 2(-8) + 4 + 8 + 7$ $\quad$ Put in (1)

$= 3 \neq 0$

$\Rightarrow$ $(\omega + 2)$ is not a factor of $2\omega^3 + \omega^2 - 4\omega + 7$.

Q.8 $\mathbf{x - a, \ x^n - a^n}$**, where 'n' is a positive integer.** **(Lahore Board 2007)**

Solution:

Let $f(x) = x^n - a^n$ ..... (1) $\quad x - a = 0$

$f(a) = (a)^n - (a)^n$ $\quad x = a$

$= 0 = R$ $\quad$ Put in (1)

$\Rightarrow$ $x - a$ is a factor of $x^n - a^n$.

Q.9 $x + a$, $x^n + a^n$, where 'n' is an odd integer. (Lahore Board 2009)

Solution:

Let $f(x) = x^n + a^n$ ..... (1) $x + a = 0$

$f(-a) = (-a)^n + a^n$ $x = -a$

$= -a^n + a^n$ ∵ 'n' is odd Put in (1)

$= 0 = R$

$\Rightarrow$ $x + a$ is a factor of $x^n + a^n$.

Q.10 When $x^4 + 2x^3 + kx^2 + 3$ is divided by $x - 2$, the remainder is 1. Find the value of K.

Solution:

Let $f(x) = x^4 + 2x^3 + kx^2 + 3$ ..... (1) $x - 2 = 0$

$f(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3$ $x = 2$

$= 16 + 16 + 4k + 3$ Put in (1)

$= 35 + 4k =$ Remainder

$\Rightarrow$ Remainder $= 35 + 4k$

But given that remainder $= 1$.

$\Rightarrow$ $35 + 4k = 1$

$\Rightarrow$ $4k = 1 - 35$

$\Rightarrow$ $4k = -34$

$\Rightarrow$ $k = \frac{-34}{4}$

$\Rightarrow$ $k = \frac{-17}{2}$

Q.11 When the polynomial $x^3 + 2x^2 + kx + 4$ is divided by $x - 2$ the remainder is 14? Find the value of k. (Lahore Board 2006)

Solution:

Let $f(x) = x^3 + 2x^2 + kx + 4$ ..... (1) $x - 2 = 0$

$f(2) = (2)^3 + 2(2)^2 + k(2) + 4$ $x = 2$

$= 8 + 8 + 2k + 4$ Put in (1)

$= 20 + 2k =$ Remainder

$\Rightarrow$ Remainder $= 20 + 2k$

but given that remainder $= 14$

$\Rightarrow$ $14 = 20 + 2k$

$\Rightarrow$ $14 - 20 = 2k$

$\Rightarrow$ $-6 = 2k$

$\Rightarrow$ $k = -3$

Use synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

Q.12 $x^3 - 7x + 6 = 0 \quad x = 2$ **(Lahore Board 2005)**

Solution:

Let $f(x) = x^3 - 7x + 6, \quad x = 2$

By synthetic division

	1	0	-7	6
2	0	2	4	-6
	1	2	-3	0

$\rightarrow$ Remainder

As remainder $= 0$

$\Rightarrow$ $x = 2$ is the solution of the polynomial $x^3 - 7x + 6 = 0$

Now

$$\text{Quotient} = x^2 + 2x - 3$$
$$= x^2 + 3x - x - 3$$
$$= x(x + 3) - 1(x + 3)$$
$$= (x + 3)(x - 1)$$

$\Rightarrow$ other factors are $x + 3$ and $x - 1$.

Q.13 $x^3 - 28x - 48 = 0, \; x = -4$

Solution:

Let $f(x) = x^3 - 28x - 48, \quad x = -4$

By synthetic division

	1	0	-28	-48
-4	0	-4	16	48
	1	-4	-12	0

$\rightarrow$ Remainder

As remainder $= 0$

$\Rightarrow$ $x = -4$ is the solution of the polynomial $x^3 - 28x - 48$.

Now

$$\text{Quotient} = x^2 - 4x - 12$$
$$= x^2 - 6x + 2x - 12$$
$$= x(x - 6) + 2(x - 6)$$
$$= (x - 6)(x + 2)$$

$\Rightarrow$ other factors are $x - 6$ and $x + 2$.

Q.14 $2x^4 + 7x^3 - 4x^2 - 27x - 18$, $x = 2$, $x = -3$.

Solution:

Let $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, $x = 2$, $x = -3$

By synthetic division

	2	7	– 4	– 27	– 18	
2	0	4	22	36	18	
	2	11	18	9	0	→ Remainder
– 3	0	– 6	– 15	– 9		
	2	5	3	0	→	Remainder

As remainder = 0

$\Rightarrow$ $x = 2$, $x = -3$ are solutions of the polynomial $2x^4 + 7x^3 - 4x^2 - 27x - 18$.

Now

$$\text{Quotient} = 2x^2 + 5x + 3$$
$$= 2x^2 + 3x + 2x + 3$$
$$= x(2x + 3) + 1(2x + 3)$$
$$= (2x + 3)(x + 1)$$

$\Rightarrow$ other factors are $2x + 3$ and $x + 1$.

Q.15 Use synthetic division to find the values of p and q if $x + 1$ and $x - 2$ are the factors of the polynomial $x^3 + px^2 + qx + 6$. (Gujranwala Board 2006)

Solution:

Let $f(x) = x^3 + px^2 + qx + 6$, $x + 1 = 0 \Rightarrow x = -1$

$x - 2 = 0 \Rightarrow x = 2$

By synthetic division

	1	p	q	6	
– 1	0	– 1	– p + 1	– q + p – 1	
	1	p – 1	q – p + 1	p – q + 5	→ Remainder
2	0	2	2p + 2		
	1	p + 1	p + q + 3	→	Remainder

As $x = -1$ and $x = 2$ are factors of the given polynomial.

$\Rightarrow$ $p - q + 5 = 0$ ..... (1)

and $p + q + 3 = 0$ ..... (2)

Adding (1) and (2)

$2p + 8 = 0$

$2p = -8 \Rightarrow p = -4$

Put $p = -4$ in equation (1)

$-4 - q + 5 = 0$

$\Rightarrow -q + 1 = 0$

$\Rightarrow q = 1$

Q.16 Find the values of 'a' and 'b' if -2 and 2 are the roots of the polynomial $x^3 - 4x^2 + ax + b$. (Gujranwala Board 2007)

Solution:

Let $f(x) = x^3 - 4x^2 + ax + b$

As -2 and 2 are the roots of the given polynomial.

$\Rightarrow f(-2) = 0$ and $f(2) = 0$

$\Rightarrow (-2)^3 - 4(-2)^2 + a(-2) + b = 0$ $\Rightarrow (2)^3 - 4(2)^2 + a(2) + b = 0$

$\Rightarrow -8 - 4(4) - 2a + b = 0$ $\Rightarrow 8 - 4(4) + 2a + b = 0$

$\Rightarrow -24 - 2a + b = 0$ ..... (1) $\Rightarrow -8 + 2a + b = 0$ ..... (2)

Adding (1) and (2)

$-24 - 2a + b = 0$

$-8 - 2a + b = 0$

$-32 + 2b = 0$

$\Rightarrow 2b = 32$

$\Rightarrow b = 16$

Put $b = 16$ in equation (1)

$-24 - 2a + 16 = 0$

$\Rightarrow -8 - 2a = 0$

$\Rightarrow +2a = -8$

$\Rightarrow a = -4$

$\Rightarrow a = -4$ and $b = 16$

RELATION BETWEEN THE ROOTS AND THE COEFFICIENTS OF A QUADRATIC EQUATION

Let α, β be the roots of $ax^2 + bx + c = 0,\ a \neq 0$ such that

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then $$\alpha + \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a}$$

and $$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 + (\sqrt{b^2 - 4ac})}{(2a)^2} = \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\Rightarrow$ $$\text{Sum of roots} = S = \frac{-b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

FORMATION OF AN EQUATION WHOSE ROOTS ARE GIVEN

As $(x - \alpha)(x - \beta) = 0$ has roots 'α' and 'β'.

$\Rightarrow$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has the roots α and β

$\Rightarrow$ $x^2 - Sx + p = 0$ has the roots α and β

where

$$S = \text{Sum of the roots} = -\frac{b}{a}$$

$$P = \text{Product of the roots} = \frac{c}{a}$$

EXERCISE 4.6

Q.1 If α, β are the roots of $3x^2 - 2x + 4 = 0$. Find the values of

(i) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution:

As α, β are roots of $3x^2 - 2x + 4 = 0$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} = \frac{-(-2)}{3} = \frac{2}{3} \qquad \ldots\ldots\ldots (1)$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{4}{3} \qquad \ldots\ldots\ldots (2)$$

$$\text{Now} \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2 \beta^2}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)}{\left(\frac{4}{3}\right)^2}$$

$$= \frac{\frac{4}{9} - \frac{8}{3}}{\frac{16}{9}} = \frac{\frac{4 - 24}{9}}{\frac{16}{9}} = \frac{-20}{16} = \frac{-5}{4}$$

(ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\,\alpha\beta}{\alpha\,\beta} = \frac{(\alpha+\beta)^2 - 2\,\alpha\,\beta}{\alpha\,\beta}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)}{\frac{4}{3}}$$

$$= \frac{\frac{4}{9} - \frac{8}{3}}{\frac{4}{3}} = \frac{\frac{4-24}{9}}{\frac{4}{3}}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{-20}{9} \times \frac{3}{4} = \frac{-5}{3}$$

(iii) $\alpha^4 + \beta^4$

$$\alpha^4 + \beta^4 = (\alpha^2)^2 + (\beta^2)^2 + 2\alpha^2\beta^2 - 2\alpha^2\beta^2$$

$$= (\alpha^2 + \beta^2)^2 - 2\,(\alpha\beta)^2$$

$$= (\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta)^2 - 2\,(\alpha\beta)^2$$

$$= [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\,(\alpha\beta)^2$$

Putting values from (1) and (2)

$$= \left[\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)\right]^2 - 2\left(\frac{4}{3}\right)^2$$

$$= \left[\frac{4}{9} - \frac{8}{3}\right]^2 - 2\left(\frac{16}{9}\right)$$

$$= \left[\frac{4-24}{9}\right]^2 - \left(\frac{32}{9}\right)$$

$$= \left[\frac{-20}{9}\right]^2 - \left(\frac{32}{9}\right)$$

$$= \frac{400}{81} - \frac{32}{9} = \frac{400-288}{81} = \frac{112}{81}$$

(iv) $\alpha^3 + \beta^3$

$$\alpha^3 + \beta^3 = (\alpha+\beta)\,(\alpha^2 - \alpha\beta + \beta^2)$$

$$= (\alpha+\beta)\,(\alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta - \alpha\beta)$$

$$= (\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]$$

Putting values from (1) and (2)

$$= \left(\frac{2}{3}\right)\left[\left(\frac{2}{3}\right)^2 - 3\left(\frac{4}{3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{4}{9} - 4\right] = \frac{2}{3}\left[\frac{4 - 36}{9}\right]$$

$$= \frac{2}{3}\left[\frac{-32}{9}\right] = \frac{-64}{27}$$

(v) $\dfrac{\mathbf{1}}{\boldsymbol{\alpha^3}} + \dfrac{\mathbf{1}}{\boldsymbol{\beta^3}}$

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\,\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\,\beta)^3}$$

$$= \frac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha + \beta)(\alpha^2 + 2\alpha\beta + \beta^2 - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)\left[\left(\frac{2}{3}\right)^2 - 3\left(\frac{4}{3}\right)\right]}{\left(\frac{4}{3}\right)^3} = \frac{\frac{2}{3}\left[\frac{4}{9} - 4\right]}{\frac{64}{27}}$$

$$= \frac{\left(\frac{2}{3}\right)\left(\frac{-32}{9}\right)}{\frac{64}{27}} = -\frac{2}{3}\left(\frac{32}{9}\ \frac{27}{64}\right) = -1$$

(vi) $\boldsymbol{\alpha^2 - \beta^2}$

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$$

$$= (\alpha + \beta)\sqrt{(\alpha - \beta)^2}$$

$$= (\alpha + \beta)\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$$

$$= (\alpha + \beta)\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - 2\alpha\beta}$$

$= (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

Putting values from (1) and (2)

$$= \frac{2}{3}\sqrt{\left(\frac{2}{3}\right)^2 - 4\frac{4}{3}} = \frac{2}{3}\sqrt{\frac{4}{9} - \frac{16}{3}}$$

$$= \frac{2}{3}\sqrt{\frac{-44}{9}} = \frac{2}{3}\frac{2\sqrt{-11}}{3} = \frac{4\sqrt{11}}{9}\,i$$

Q.2 If 'α' β are the roots of $x^2 - px - p - c = 0$.
Prove that $(1+\alpha)(1+\beta) = 1 - c$ (Lahore Board 2008, 2010, 2011)

Solution:

As α, β are the roots of $x^2 - px - p - c = 0$

$\therefore$ $\alpha + \beta = \frac{-b}{a} = \frac{-(-p)}{1} = p$ ..... (1)

$\alpha\beta = \frac{c}{a} = \frac{-p-c}{1} = -p-c$ ..... (2)

To prove $(1+\alpha)(1+\beta) = 1 - c$

Take L.H.S. $= (1+\alpha)(1+\beta) = 1 + \beta + \alpha + \alpha\beta$

$= 1 + (\alpha + \beta) + \alpha\beta$

Put values from (1) and (2)

$= 1 + p - p - c$

$= 1 - c$ = R.H.S.

$\Rightarrow$ Hence proved.

Q.3 Find the condition that one root of $x^2 + px + q = 0$ is

Solution:

(i) Double the other

Let $\alpha, 2\alpha$ be roots of $x^2 + px + q = 0$ then

Sum of the roots $= \alpha + 2\alpha = \frac{-b}{a} = \frac{-p}{1} = -p$

Product of roots $= \alpha \cdot 2\alpha = \frac{c}{a} = \frac{q}{1} = q$

$\Rightarrow$ $\alpha + 2\alpha = -p \Rightarrow 3\alpha = -p$ ..... (1)

$\alpha \cdot 2\alpha = q \Rightarrow 2\alpha^2 = q$ ..... (2)

Putting $\alpha = \frac{-p}{3}$ from (1) and (2)

$\Rightarrow \quad 2\left(\frac{-p}{3}\right)^2 = q \quad \Rightarrow \quad 2 \cdot \frac{p^2}{9} = q$

$\Rightarrow \quad 2p^2 = 9q$ is the required condition.

(ii) Square of the other

Let α, α^2 be roots of $x^2 + px + q = 0$ then

$$\alpha + \alpha^2 = \frac{-b}{a} = \frac{-p}{1} = -p$$

$\Rightarrow \quad \alpha + \alpha^2 = -p \qquad \ldots\ldots\ldots (1)$

and $\quad \alpha \cdot \alpha^2 = \frac{c}{a} = \frac{q}{1}$

$$\alpha^3 = q \qquad \ldots\ldots\ldots (2)$$

Cubing equation (1)

$$(\alpha + \alpha^2)^3 = (-p)^3$$

$\Rightarrow \quad \alpha^3 + (\alpha^2)^3 + 3(\alpha)(\alpha^2)(\alpha + \alpha^2) = -p^3$

$\Rightarrow \quad \alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -p^3$

Put $\alpha^3 = q$ and $\alpha + \alpha^2 = -p$ from equation (1) and (2)

$\Rightarrow \quad q + q^2 + 3q(-p) = -p^3$

$\Rightarrow \quad q^2 + q - 3pq + p^3 = 0$

$\Rightarrow \quad p^3 + q^2 + q = 3pq$

is required condition.

(iii) Additive inverse of the other: **(Gujranwala Board 2005)**

Let $\alpha, -\alpha$ be roots of $x^2 + px + q = 0$ then

$$\alpha + (-\alpha) = \frac{-b}{a} = \frac{-p}{1}$$

$\Rightarrow \quad \alpha - \alpha = -p$

$\Rightarrow \quad -p = 0 \quad \Rightarrow \quad p = 0$

and $\quad \alpha \cdot (-\alpha) = \frac{c}{a} = \frac{q}{1}$

$\Rightarrow \quad -\alpha^2 = q \quad \Rightarrow \quad \alpha^2 = -q$

$\Rightarrow \quad p = 0$ is the required condition.

Q.5 Find the condition that $\frac{a}{x-a}+\frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.

Solution:

$$\frac{a}{x-a}+\frac{b}{x-b} = 5$$

$$\frac{a(x-b)+b(x-a)}{(x-a)(x-b)} = 5$$

$$\Rightarrow \quad \frac{ax-ab+bx-ab}{x^2-bx-ax+ab} = 5$$

$$\Rightarrow \quad ax+bx-2ab = 5(x^2-bx-ax+ab)$$

$$\Rightarrow \quad ax+bx-2ab = 5x^2-5bx-5ax+5ab$$

$$\Rightarrow \quad 5x^2-5bx-5ax+5ab-ax-bx+2ab = 0$$

$$\Rightarrow \quad 5x^2-6bx-6ax+7ab = 0$$

$$\Rightarrow \quad 5x^2-6x(a+b)+7ab = 0$$

Let $\alpha, -\alpha$ be the roots of this equation.

$$\Rightarrow \quad \alpha+(-\alpha) = -\frac{-6(a+b)}{5}$$

$$\Rightarrow \quad \alpha-\alpha = \frac{6(a+b)}{5}$$

$$\Rightarrow \quad 0 = \frac{6(a+b)}{5} \quad \Rightarrow \quad a+b = 0$$

is the required condition.

Q.6 If the roots of $px^2+qx+q = 0$ are α and β then prove that

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}} = 0$$

Solution:

Since α, β be the roots of $px^2+qx+q = 0$, then

$$\alpha+\beta = \frac{-q}{p} \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\beta = \frac{q}{p} \qquad \ldots\ldots\ldots (2)$$

To prove

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}} = 0$$

Take

$$\text{L.H.S.} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\sqrt{\alpha}}{\sqrt{\beta}} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\alpha + \beta}{\sqrt{\beta}\sqrt{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{q}{p}}$$

Put values from (1) and (2)

$$= \frac{-\frac{q}{p}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}} \qquad \because \frac{q}{p} = \sqrt{\frac{q}{p}} \cdot \sqrt{\frac{q}{p}}$$

$$= -\frac{\sqrt{\frac{q}{p}} \cdot \sqrt{\frac{q}{p}}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}$$

$$= -\sqrt{\frac{q}{p}} + \sqrt{\frac{q}{p}} = 0 = \text{R.H.S.}$$

Hence proved.

Q.7 If α, β are the roots the equation $ax^2 + bx + c = 0$, form the equations whose roots are.

Solution:

(i) α^2, β^2 **(Lahore Board 2006)**

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} \qquad \ldots\ldots\ldots (1)$

and $\quad \alpha\beta = \frac{c}{a} \qquad \ldots\ldots\ldots (2)$

Let the required equation is

$$y^2 - sy + p = 0 \qquad \ldots\ldots\ldots (3)$$

whose roots are α^2, β^2

$\Rightarrow \quad s = \alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$ from equations (1) and (2)

$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

and $p = \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}$ from equation (2)

Put values of s and p in equation (3)

$\Rightarrow \quad y^2 - \left(\frac{b^2 - 2ac}{a^2}\right) y + \frac{c^2}{a^2} = 0$

$\Rightarrow \quad a^2y^2 - (b^2 - 2ac)\, y + c^2 = 0$

is required equation.

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}$ ..... (1)

and $\alpha\beta = \frac{c}{a}$ ..... (2)

Let the required equation is

$y^2 - sy + p = 0$ ..... (3)

whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$\Rightarrow \quad s = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$= \frac{-\frac{b}{a}}{\frac{c}{a}}$ from equations (1) and (2)

$= -\frac{b}{a}$

and $\quad p = \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$ $\quad$ from equation (2)

Put these values in equation (3)

$$y^2 - \left(-\frac{b}{c}\right) y + \frac{a}{c} = 0$$

$\Rightarrow \quad y^2 + \frac{b}{c} y + \frac{a}{c} = 0$

$\Rightarrow \quad cy^2 + by + a = 0$

is required condition.

(iii) $\quad \mathbf{\frac{1}{\alpha^2}, \frac{1}{\beta^2}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}$ $\quad$ ..... (1)

and $\quad \alpha\beta = \frac{c}{a}$ $\quad$ ..... (2)

Let the required equation is

$$y^2 - sy + p = 0 \quad \text{..... (3)}$$

whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

$\Rightarrow \quad s = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

Put values from equation (1) and (2)

$$= \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{\frac{b^2 - 2ac}{a^2}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}$$

and $p = \frac{1}{\alpha^2} \cdot \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{\left(\frac{c}{a}\right)^2} = \frac{a^2}{c^2}$ from equation (2)

Put these values of s and p in equation (3)

$$y^2 - \left(\frac{b^2 - 2ac}{c^2}\right) y + \frac{a^2}{c^2} = 0$$

$\Rightarrow$ $c^2 y^2 - (b^2 - 2ac)\, y + a^2 = 0$ is required equation.

(iv) α^3, β^3 **(Lahore Board 2005)**

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow$ $\alpha + \beta = -\frac{b}{a}$ ..... (1)

and $\alpha\beta = \frac{c}{a}$ ..... (2)

Let the required equation is

$$y^2 - sy + p = 0 \quad \text{..... (3)}$$

whose roots are α^3, β^3

$\Rightarrow$ $s = \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (\alpha + \beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta)$

$= (\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]$

Put values from equations (1) and (2)

$$= -\frac{b}{a}\left[\left(-\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)\right]$$

$$= -\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = -\frac{b}{a}\left[\frac{b^2 - 3ac}{a^2}\right] = \frac{-b(b^2 - 3ac)}{a^3}$$

and $p = \alpha^3 \cdot \beta^3 = (\alpha\beta)^3 = \left(\frac{c}{a}\right)^3$ from equation (2)

$$= \frac{c^3}{a^3}$$

$\Rightarrow$ equation (3) becomes

$$y^2 - \frac{-b(b^2 - 3ac)}{a^3} y + \frac{c^3}{a^3} = 0$$

$$\Rightarrow \quad y^2 + \frac{b\,(b^2 - 3ac)}{a^3}\,y + \frac{c^3}{a^3} = 0$$

$$\Rightarrow \quad a^3 y^2 + b\,(b^2 - 3ac)\,y + c^3 = 0$$

is required equation.

(v) $\dfrac{1}{\alpha^3}, \dfrac{1}{\beta^3}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\,\beta = \frac{c}{a} \qquad \ldots\ldots\ldots (2)$$

Let the required equation is

$$y^2 - sy + p = 0 \qquad \ldots\ldots\ldots (3)$$

whose roots are $\dfrac{1}{\alpha^3}, \dfrac{1}{\beta^3}$

$$\Rightarrow \quad s = \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$$

$$= \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} = \frac{(\alpha + \beta)\,(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha + \beta)\,(\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha + \beta)\,[(\alpha + \beta)^2 - 3\alpha\beta]}{(\alpha\beta)^3}$$

Put values from equations (1) and (2)

$$= \frac{\dfrac{-b}{a}\left[\left(\dfrac{-b}{a}\right)^2 - 3\left(\dfrac{c}{a}\right)\right]}{\left(\dfrac{c}{a}\right)^3}$$

$$= \frac{\dfrac{-b}{a}\left[\dfrac{b^2}{a^2} - \dfrac{3c}{a}\right]}{\dfrac{c^3}{a^3}}$$

$$= \frac{a\left[\quad\quad a\quad\quad\right]}{\frac{c^3}{a^3}}$$

$$s = \frac{-b\,(b^2 - 3ac)}{a^3}\,\frac{a^3}{c^3} = \frac{-b\,(b^2 - 3ac)}{c^3}$$

and $\quad p = \frac{1}{\alpha^3} \cdot \frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{c}{a}\right)^3} = \frac{a^3}{c^3}$ from equation (2)

Put these values in equation (3)

$$y^2 - \left[\frac{-b\,(b^2 - 3ac)}{c^3}\right] y + \frac{a^3}{c^3} = 0$$

$\Rightarrow \quad y^2 + \frac{b\,(b^2 - 3ac)}{c^3}\, y + \frac{a^3}{c^3} = 0$

$$c^3 y^2 + b\,(b^2 - 3ac)\, y + a^3 = 0$$

(vi) $\quad \boldsymbol{\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = \frac{-b}{a}$ ..... (1)

and $\quad \alpha\beta = \frac{c}{a}$ ..... (2)

Let the required equation is

$$y^2 - sy + p = 0 \quad \text{..... (3)}$$

whose roots are $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$, then

$\Rightarrow \quad s = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = (\alpha + \beta) + \frac{\beta + \alpha}{\alpha\beta}$

Put values from equations (1) and (2)

$$s = \frac{-b}{a} + \frac{\frac{-b}{a}}{\frac{c}{a}} = -\frac{b}{a} - \frac{b}{c} = \frac{-bc - ab}{ac} = \frac{-b\,(a + c)}{ac}$$

and $p = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

$$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

Put values from equations (1) and (2)

$$p = \frac{c}{a} + \frac{1}{\frac{c}{a}} + \frac{\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\frac{c}{a}}$$

$$= \frac{c}{a} + \frac{a}{c} + \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} = \frac{c}{a} + \frac{a}{c} + \frac{b^2 - 2ac}{a^2}\ \frac{a}{c}$$

$$= \frac{c}{a} + \frac{a}{c} + \frac{b^2 - 2ac}{ac} = \frac{c^2 + a^2 + b^2 - 2ac}{ac} = \frac{(a - c)^2 + b^2}{ac}$$

Put these values in equation (3)

$$y^2 - \left[\frac{-b\,(a + c)}{ac}\right] y + \frac{(a - c)^2 + b^2}{ac} = 0$$

$$y^2 + \frac{b\,(a + c)}{ac}\, y + \frac{(a - c)^2 + b^2}{ac} = 0$$

is the required equation.

(vii) $\mathbf{(\alpha - \beta)^2,\ (\alpha + \beta)^2}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow$ $\alpha + \beta = \frac{-b}{a}$ ..... (1)

and $\alpha\beta = \frac{c}{a}$ ..... (2)

Let the required equation is

$$y^2 - sy + p = 0 \quad \text{..... (3)}$$

whose roots are $(\alpha - \beta)^2$, $(\alpha + \beta)^2$, then

$$\Rightarrow \quad s = (\alpha - \beta)^2 + (\alpha + \beta)^2$$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\alpha + \beta)^2$$

$$= (\alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta) + (\alpha + \beta)^2$$

$$= \left[(\alpha + \beta)^2 - 4\alpha\beta\right] + (\alpha + \beta)^2$$

Put values from equations (1) and (2)

$$s = \left[\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}\right] + \left(\frac{-b}{a}\right)^2$$

$$= \left[\frac{b^2}{a^2} - \frac{4c}{a}\right] + \frac{b^2}{a^2} = \frac{b^2 - 4ac}{a^2} + \frac{b^2}{a^2} = \frac{b^2 - 4ac + b^2}{a^2} = \frac{2b^2 - 4ac}{a^2}$$

and $p = (\alpha - \beta)^2 (\alpha + \beta)^2$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta)(\alpha + \beta)^2$$

$$= (\alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta)(\alpha + \beta)^2$$

$$= \left[(\alpha + \beta)^2 - 4\alpha\beta\right](\alpha + \beta)^2$$

Put values from equations (1) and (2)

$$= \left[\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}\right]\left(\frac{-b}{a}\right)^2 = \left[\frac{b^2}{a^2} - \frac{4c}{a}\right]\frac{b^2}{a^2}$$

$$p = \frac{b^2 - 4ac}{a^2}\,\frac{b^2}{a^2} = \frac{b^2(b^2 - 4ac)}{a^4}$$

Put these values in equation (3)

$$y^2 - \frac{2b^2 - 4ac}{a^2}y + \frac{b^2(b^2 - 4ac)}{a^4} = 0$$

$$a^4 y^2 - 2a^2(b^2 - 2ac)y + b^2(b^2 - 4ac) = 0$$

is required equation.

(viii) $-\frac{1}{\alpha^3} - \frac{1}{\beta^3}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}$ ..... (1)

and $\quad \alpha\beta = \frac{c}{a}$ ..... (2)

Let the required equation is

$y^2 - sy + p = 0$ ..... (3)

whose roots are $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

$$\Rightarrow \quad s = -\frac{1}{\alpha^3} - \frac{1}{\beta^3} = \frac{-\beta^3 - \alpha^3}{\alpha^3\beta^3}$$

$$= -\frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} = -\frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= -\frac{(\alpha+\beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{-(\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}$$

Put values from equations (1) and (2)

$$= \frac{-\left(-\frac{b}{a}\right)\left[\left(-\frac{b}{a}\right)^2 - 3\frac{c}{a}\right]}{\left(\frac{c}{a}\right)^3}$$

$$= \frac{\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right]}{\frac{c^3}{a^3}} = \frac{b}{a}\left[\frac{b^2 - 3ac}{a^2}\right]\frac{a^3}{c^3}$$

$$s = \frac{b(b^2 - 3ac)}{c^3}$$

and $\quad p = \left(-\frac{1}{\alpha^3}\right)\left(-\frac{1}{\beta^3}\right) = \frac{1}{\alpha^3\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{c}{a}\right)^3} = \frac{a^3}{c^3}$ from equation (2)

Put these values in equation (3)

$$y^2 - \frac{b(b^2 - 3ac)}{c^3})\, y + \frac{a^3}{c^3} = 0$$

$$\Rightarrow \quad c^3 y^2 - b(b^2 - 3ac)\, y + a^3 = 0$$

is the required equation.

Q.8 If α, β are the roots of $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$. (Lahore Board 2007)

Solution:

Since α, β are the roots of the equation $5x^2 - x - 2 = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-(-1)}{5} = \frac{1}{5} \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\beta = \frac{-2}{5} \qquad \ldots\ldots\ldots (2)$$

Let the required equation is

$$y^2 - sy + p = 0 \qquad \ldots\ldots\ldots (3)$$

whose roots are $\frac{3}{\alpha}, \frac{3}{\beta}$

$$\Rightarrow \quad s = \frac{3}{\alpha} + \frac{3}{\beta} = \frac{3\beta + 3\alpha}{\alpha\beta} = \frac{3(\alpha + \beta)}{\alpha\beta}$$

Put values from equations (1) and (2)

$$s = \frac{3\left(\frac{1}{5}\right)}{\frac{-2}{5}} = -3 \cdot \frac{1}{5} \cdot \frac{5}{2} = \frac{-3}{2}$$

$$p = \frac{3}{\alpha}\,\frac{3}{\beta} = \frac{9}{\alpha\beta} = \frac{9}{\frac{-2}{5}} = -9 \cdot \frac{5}{2} = -\frac{45}{2} \quad \text{from equation (2)}$$

Put these values in equation (3)

$$\Rightarrow \quad y^2 - \left(\frac{-3}{2}\right) y - \frac{45}{2} = 0$$

$$\Rightarrow \quad y^2 + \frac{3}{2} y - \frac{45}{2} = 0$$

$$\Rightarrow \quad 2y^2 + 3y - 45 = 0 \quad \text{is required equation.}$$

Q.9 **If α, β are the roots of $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$.** **(Gujranwala Board 2003)**

Solution:

Since α, β are the roots of the equation $x^2 - 3x + 5 = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-(-3)}{1} = 3 \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\beta = \frac{5}{1} = 5 \qquad \ldots\ldots\ldots (2)$$

Let the required equation be

$$y^2 - sy + p = 0 \qquad \ldots\ldots\ldots (3)$$

whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

$$\Rightarrow \quad s = \frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}$$

$$= \frac{(1-\alpha)(1+\beta) + (1+\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$$

$$= \frac{1+\beta-\alpha-\alpha\beta+1-\beta+\alpha-\alpha\beta}{1+\beta+\alpha+\alpha\beta} = \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

Put values from equations (1) and (2)

$$s = \frac{2-2(5)}{1+3+5} = \frac{2-10}{9} = \frac{-8}{9}$$

and $$p = \frac{1-\alpha}{1+\alpha}\,\frac{1-\beta}{1+\beta}$$

$$= \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)} = \frac{1-\beta-\alpha+\alpha\beta}{1+\alpha+\beta+\alpha\beta} = -\frac{1-(\alpha+\beta)+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

Put these values in equations (1) and (2)

$$= \frac{1-3+5}{1+3+5} = \frac{3}{9}$$

$\Rightarrow$ equation (3) becomes

$$y^2 - \left(-\frac{8}{9}\right)y + \frac{3}{9} = 0 \quad \Rightarrow \quad 9y^2 + 8y + 3 = 0$$

NATURE OF THE ROOTS OF A QUADRATIC EQUATION

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$, which is called its Discriminate.

Case 1: If $b^2 - 4ac = 0$ then the roots will be $-\frac{b}{2a}$ and $\frac{-b}{2a}$. So the roots are real and repeated equal.

Case 2: If $b^2 - 4ac < 0$ then $\sqrt{b^2 - 4ac}$ will be imaginary so, the roots are complex and distinct/unequal.

Case 3: If $b^2 - 4ac > 0$ then $\sqrt{b^2 - 4ac}$ will be real. So, the roots are real and distinct/unequal.

NOTE: If $b^2 - 4ac$ is a perfect square then $\sqrt{b^2 - 4ac}$ will be rational, and the roots are rational otherwise irrational.

EXERCISE 4.7

Q.1 Discuss the nature of the roots of the following equations:

Solution:

(i) $\mathbf{4x^2 + 6x + 1 = 0}$

Comparing with $ax^2 + bx + c = 0$

We have $a = 4,\ b = 6,\ c = 1$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac = (6)^2 - 4(4)(1) = 36 - 16 = 20 > 0$

$\Rightarrow$ $D > 0$

and 'D' is not a perfect square.

$\Rightarrow$ roots are real unequal and irrational.

(ii) $\mathbf{x^2 - 5x + 6 = 0}$ **(Lahore Board 2005, Gujranwala Board 2007)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 \quad b = -5 \quad c = 6$

$\Rightarrow$ Discriminant $= b^2 - 4ac = (-5)^2 - 4(1)(6) = 25 - 24 = 1 > 0$

$\Rightarrow$ $D > 0$

and D is a perfect square also

$\Rightarrow$ roots are real unequal and rational.

(iii) $\mathbf{2x^2 - 5x + 1 = 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 2 \quad b = -5 \quad c = 1$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac = (-5)^2 - 4(2)(1) = 25 - 8 = 17 > 0$

$\Rightarrow$ $D > 0$ and 'D' is not a perfect square.

$\Rightarrow$ roots are real, unequal and irrational.

(iv) $\mathbf{25x^2 - 30x + 9 = 0}$ **(Lahore Board 2007)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = 25 \quad b = -30 \quad c = 9$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac = (-30)^2 - 4(25)(9) = 900 - 900 = 0$

$\Rightarrow$ $D > 0$ and D is a perfect square also

$\Rightarrow$ roots are real, equal and rational.

Q.2 Show that the roots of the following equations will be real.

Solution:

(i) $\mathbf{x^2 - 2\left(m + \frac{1}{m}\right) + 3 = 0, \quad m \neq 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 \quad b = -2\left(m + \frac{1}{m}\right), \quad c = 3$

Now Discriminant $= D = b^2 - 4ac$

$$= \left[-2\left(m + \frac{1}{m}\right)\right]^2 - 4(1)(3)$$

$$= \left[4\left(m + \frac{1}{m}\right)^2\right] - 12$$

$$= \left[4\left(m^2 + \frac{1}{m^2} + 2\right)\right] - 12$$

$$= 4\left[m^2 + \frac{1}{m^2} + 2 - 3\right]$$

$$= 4\left[m^2 + \frac{1}{m^2} - 1\right]$$

$$= 4\left[m^2 + \frac{1}{m^2} - 2 + 1\right]$$

$$= 4\left[\left(m - \frac{1}{m}\right)^2 + 1\right] > 0$$

As 'D' is always positive because $\left(m - \frac{1}{m}\right)^2 > 0 \quad \forall\ m \in R$ and $m \neq 0$.

$\Rightarrow$ roots are real.

(ii) $\mathbf{(b-c)\,x^2 + (c-a)\,x + (a-b) = 0; \quad a, b, c \in Q}$

$(b-c)\,x^2 + (c-a)\,x + (a-b) = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = b-c, \quad b = c-a, \quad c = a-b$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac$

$= (c-a)^2 - 4\,(b-c)\,(a-b)$

$= c^2 + a^2 - 2ac - 4\,(ab - b^2 - ac + bc)$

$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$

$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac$

$= (a - 2b + c)^2 > 0$

As square is always positive.

$\Rightarrow$ 'D' $> 0 \Rightarrow$ roots are real.

Q.3 Show that the roots of the following equations will be rational.

Solution:

(i) $\mathbf{(p+q)\,x^2 - px - q = 0}$ **(Lahore Board 2008)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = p+q, \quad b = -p, \quad c = -q$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac$

$= (-p)^2 - 4\,(p+q)\,(-q)$

$= p^2 - 4q\,(p+q)$

$= p^2 + 4pq + 4q^2$

$= (p + 2q)^2$

As 'D' is a perfect square.

$\Rightarrow$ roots are rational.

(ii) $\mathbf{px^2 - (p-q)\,x - q = 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = p, \quad b = -(p-q), \quad c = -q$

Now Discriminant $= D = b^2 - 4ac$

$= [(-p-q)]^2 - 4\,(p)\,(-q)$

$= (p-q)^2 + 4pq$

$= p^2 + q^2 - 2pq + 4pq$

$= p^2 + q^2 + 2pq$

$= (p+q)^2$

As D is a perfect square

$\Rightarrow$ roots are rational.

Q.4 For what values of m will the roots of the following equations be equal.

Solution:

(i) $(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$ **(Lahore Board 2011)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = m + 1 \quad b = 2(m + 3) \quad c = m + 8$

Discriminant $= D = b^2 - 4ac$

$= [2(m + 3)]^2 - 4(m + 1)(m + 8)$

$= 4(m + 3)^2 - 4(m^2 + 8m + m + 8)$

$= 4(m^2 + 9 + 6m)^2 - 4(m^2 + 9m + 8)$

$= 4m^2 + 36 + 24m - 4m^2 - 36m - 32$

$= -12m + 4$

But it is given that the roots of the equation are equal.

$\Rightarrow \quad D = 0$

$\Rightarrow \quad -12m + 4 = 0 \quad \Rightarrow \quad 12m = 4 \quad \Rightarrow \quad m = \frac{1}{3}$

(ii) $x^2 - 2(1 + 3m)x + 7(3 + 2m) = 0$ **(Lahore Board 2009)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1, \quad b = -2(1 + 3m), \quad c = 7(3 + 2m)$

Discriminant $= D = b^2 - 4ac$

$= [-2(1 + 3m)]^2 - 4(1)7(3 + 2m)$

$= 4(1 + 3m)^2 - 28(3 + 2m)$

$= 4(1 + 9m^2 + 6m) - 84 - 56m$

$= 4 + 36m^2 + 24m - 84 - 56m$

$= 36m^2 - 32m - 80$

But it is given that the roots of the equation are equal

$\Rightarrow \quad D = 0$

$\Rightarrow \quad 36m^2 - 32m - 80 = 0$

$\Rightarrow \quad 4(9m^2 - 8m - 20) = 0$

$\Rightarrow \quad 9m^2 - 8m - 20 = 0$

$\Rightarrow \quad 9m^2 - 18m + 10m - 20 = 0$

$\Rightarrow$ $9m(m-2) + 10(m-2) = 0$

$\Rightarrow$ $(m-2)(9m+10) = 0$

$\Rightarrow$ Either $m - 2 = 0$ or $9m + 10 = 0$

$\Rightarrow$ $m = 2$ or $m = \frac{-10}{9}$

(iii) $\mathbf{(1+m)x^2 - 2(1+3m)x + (1+8m) = 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 + m, \quad b = -2(1+3m), \quad c = 1 + 8m$

Discriminant $= D = b^2 - 4ac$

$= [-2(1+3m)]^2 - 4(1+m)(1+8m)$

$= 4(1+3m)^2 - 4(1+8m+m+8m^2)$

$= 4(1+9m^2+6m) - 4(1+9m+8m^2)$

$= 4 + 36m^2 + 24m - 4 - 36m - 32m^2$

$= 4m^2 - 12m$

But it is given that the roots of the equation are equal.

$\Rightarrow$ $D = 0$

$\Rightarrow$ $4m^2 - 12m = 0$

$\Rightarrow$ $4m(m-3) = 0$

$\Rightarrow$ $m(m-3) = 0$

$\Rightarrow$ Either $m = 0$ or $m - 3 = 0$

or $m = 3$

Q.5 Show that the roots of $x^2 + (mx + c)^2 = a^2$ will be equal, if $c^2 = a^2(1 + m^2)$.

(G.B 2004, 2006, L.B. 2007)

Solution:

Given equation

$x^2 + (mx + c)^2 = a^2$

$\Rightarrow$ $x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$

$\Rightarrow$ $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 + m^2, \quad b = 2mc, \quad c = c^2 - a^2$

Discriminant $= D = b^2 - 4ac$

$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$

$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$

$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2$

$= 4a^2 + 4m^2a^2 - 4c^2$

Now roots of the given equation will be equal if $D = 0$, i.e.

$4a^2 + 4m^2a^2 - 4c^2 = 0$

or $4(a^2 + m^2a^2 - c^2) = 0$

or $a^2 + m^2a^2 - c^2 = 0$

or $c^2 = a^2 + m^2a^2$

or $c^2 = a^2(1 + m^2)$

Hence proved.

Q.6 Show that the roots of the $(mx + c)^2 = 4ax$ will be equal if $C = \frac{a}{m}$; $m \neq 0$.

(Lahore Board 2010)

Solution:

Given equation

$(mx + c)^2 = 4ax$

$\Rightarrow$ $m^2x^2 + c^2 + 2mcx - 4ax = 0$

$\Rightarrow$ $m^2x^2 + 2mcx - 4ax + c^2 = 0$

$\Rightarrow$ $m^2x^2 + (2mc - 4a)x + c^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = m^2, \quad b = 2mc - 4a, \quad c = c^2$

Discriminant $= D = b^2 - 4ac$

$= (2mc - 4a)^2 - 4(m^2)(c^2)$

$= 4m^2c^2 + 16a^2 - 16amc - 4m^2c^2$

$= 16a^2 - 16amc$

roots of the given equation will be equal if $D = 0$ i.e.

$16a^2 - 16amc = 0$

or $16a(a - mc) = 0$

or $a - mc = 0$

or $a = mc$

or $c = \frac{a}{m}$

Hence proved.

Q.7 Prove that $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$ will have equal roots, if $c^2 = a^2 m^2 + b^2$; $a \neq b \neq 0$. **(Lahore Board 2008)**

Solution:

Given that

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{b^2 x^2 + a^2 (mx + c)^2}{a^2 b^2} = 1$$

$$\Rightarrow \quad b^2x^2 + a^2 (m^2x^2 + c^2 + 2mcx) = a^2b^2$$

$$\Rightarrow \quad b^2x^2 + a^2m^2x^2 + a^2c^2 + 2mca^2x - a^2b^2 = 0$$

$$\Rightarrow \quad b^2x^2 + a^2m^2x^2 + a^2c^2 + 2mca^2x - a^2b^2 = 0$$

$$\Rightarrow \quad (b^2 + a^2m^2) x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$$

Comparing with $ax^2 + bx + c = 0$, we have

$$a = b^2 + a^2m^2, \quad b = 2mca^2, \quad c = a^2c^2 - a^2b^2$$

$$\text{Discriminant} = D = b^2 - 4ac$$

$$= (2mca^2)^2 - 4 (b^2 + a^2m^2) (a^2c^2 - a^2b^2)$$

$$= 4m^2c^2a^4 - 4 (a^2b^2c^2 - a^2b^4 + a^4m^2c^2 - a^4b^2m^2)$$

$$= 4m^2c^2a^4 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2$$

$$= 4 (a^2b^4 - 4a^4b^2m^2 - 4a^2b^2m^2)$$

As the roots of the given equation will be equal if $D = 0$

or $4a^2b^4 + 4a^4b^2m^2 - 4a^2b^2c^2 = 0$

or $4a^2b^2 (b^2 + a^2m^2 - c^2) = 0$

or $b^2 + a^2m^2 - c^2 = 0$

or $c^2 = c^2m^2 + b^2$

Hence proved.

Q.8 Show that the roots of the equation

$(a^2 - bc)\,x^2 + 2\,(b^2 - ca)\,x + c^2 - ab = 0$ will be equal, if either

$a^3 + b^3 + c^3 = 3abc$ or $b = 0$.

Solution:

Given equation

$$(a^2 - bc)\,x^2 + 2\,(b^2 - ca)\,x + c^2 - ab = 0$$

Comparing with $ax^2 + bx + c = 0$, we have

$$a = a^2 - bc, \quad b = 2\,(b^2 - ac), \quad c = c^2 - ab$$

$$\text{Discriminant} = D = b^2 - 4ac$$

$$= \left[2\,(b^2 - ac)\right]^2 - 4\,(a^2 - bc)\,(c^2 - ab)$$

$$= 4\,(b^2 - ac)^2 - 4\,(a^2c^2 - a^3b - bc^3 + ab^2c)$$

$$= 4\,(b^4 + a^2c^2 - 2ab^2c) - 4a^2c^2 + 4a^3b + 4bc^3 - 4ab^2c$$

$$= 4b^4 + 4a^2c^2 - 8ab^2c - 4a^2c^2 + 4a^3b + 4bc^3 - 4ab^2c$$

$$= 4b^4 - 12ab^2c + 4a^3b + 4bc^3$$

As, the roots of the given equation will be equal if $D = 0$. i.e.

$$4b^4 - 12ab^2c + 4a^3b + 4bc^3 = 0$$

or $\quad 4b\,(b^3 - 3abc + a^3 + c^3) = 0$

or $\quad 4b\,(b^3 - 3abc + a^3 + c^3) = 0$

or $\quad b\,(b^3 - 3abc + a^3 + c^3) = 0$

$\Rightarrow$ Either $b = 0$ or $b^3 - 3abc + a^3 + c^3 = 0$

or $a^3 + b^3 + c^3 = 3abc$

Hence proved.

SYSTEM OF TWO EQUATIONS INVOLVING TWO VARIABLES

Here we shall solve the equations in two variables when at least one of them is quadratic.

To determine the value of two variables, we need a pair of equations. Such a pair of equations is called a system of Simultaneous Equations.

To solve One Linear Equation and One Quadratic Equation

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it.

NOTE: Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtraction.

EXERCISE 4.8

Solve the following system of equations:

Q.1 $\mathbf{2x - y = 4;\ \ 2x^2 - 4xy - y^2 = 6}$

Solution:

$2x^2 - 4xy - y^2 = 6$ ..... (1)

$2x - y = 4$

$\Rightarrow \quad y = 2x - 4$ ..... (2)

Put equation (2) in equation (1)

$\Rightarrow \quad 2x^2 - 4x\,(2x - 4) - (2x - 4)^2 = 6$

$\Rightarrow \quad 2x^2 - 8x^2 + 16x - (4x^2 + 16 - 16x) = 6$

$\Rightarrow \quad -6x^2 + 16x - 4x^2 - 16 + 16x - 6 = 0$

$\Rightarrow \quad -10x^2 + 32x - 22 = 0$

$\Rightarrow \quad -2\,(5x^2 - 16x + 11) = 0$

$\Rightarrow \quad 5x^2 - 16x + 11 = 0$

$\Rightarrow \quad 5x^2 - 11x - 5x + 11 = 0$

$\Rightarrow \quad x\,(5x - 11) - 1\,(5x - 11) = 0$

$\Rightarrow \quad (5x - 11)\,(x - 1) = 0$

$\Rightarrow \quad$ Either $5x - 11 = 0$ or $x - 1 = 0$

$\Rightarrow \quad x = \frac{11}{5}$ or $x = 1$

Put $x = \frac{11}{5}$ and $x = 1$ in equation (2)

when $x = \frac{11}{5}$ then

$y = 2\left(\frac{11}{5}\right) - 4$

$= \frac{22}{5} - 4$

$= \frac{22 - 20}{5}$

$y = \frac{2}{5}$

when $x = 1$ then

$y = 2\,(1) - 4$

$y = 2 - 4$

$y = -2$

$\Rightarrow \quad (1, -2)$

$\Rightarrow \quad \left(\frac{11}{5}, \frac{2}{5}\right)$

Hence the solution set $= \left\{(1, -2), \left(\frac{11}{5}, \frac{2}{5}\right)\right\}$

Q.2 $x + y = 5,\ x^2 + 2y^2 = 17$

Solution:

$x^2 + 2y^2 = 17$ ..... (1)

and $x + y = 5$

$\Rightarrow \quad x = 5 - y$ ..... (2)

Put $x = 5 - y$ in equation (1)

$\Rightarrow \quad (5 - y)^2 + 2y^2 = 17$

$\Rightarrow \quad 25 + y^2 - 10y + 2y^2 = 17$

$\Rightarrow \quad 3y^2 - 10y + 25 - 17 = 0$

$\Rightarrow \quad 3y^2 - 10y + 8 = 0$

$\Rightarrow \quad 3y^2 - 6y - 4y + 8 = 0$

$\Rightarrow \quad 3y(y - 2) - 4(y - 2) = 0$

$\Rightarrow \quad (y - 2)(3y - 4) = 0$

$\Rightarrow \quad$ Either $y - 2 = 0$ or $3y - 4 = 0$

$\Rightarrow \quad y = 2$ or $y = \frac{4}{3}$

Put $y = 2$ and $y = \frac{4}{3}$ in equation (2)

when $y = 2$ then

$\Rightarrow \quad x = 5 - 2$

$\Rightarrow \quad x = 3$

$\Rightarrow \quad (3, 2)$

when $y = \frac{4}{3}$ then

$\Rightarrow \quad x = 5 - \frac{4}{3}$

$\Rightarrow \quad x = \frac{15 - 4}{3} = \frac{11}{3}$

$\Rightarrow \quad \left(\frac{11}{3}, \frac{4}{3}\right)$

Hence the solution set $= \left\{(3, 2), \left(\frac{11}{3}, \frac{4}{3}\right)\right\}$

Q.3 $3x + 2y = 7;\ 3x^2 - 2y^2 = 25.$

Solution:

$3x^2 = 25 + 2y^2$ ..... (1)

and $3x + 2y = 7$

$3x = 7 - 2y$

$\Rightarrow \quad x = \frac{7 - 2y}{3}$ ..... (2)

Put value of x from equation (2) in equation (1)

$$\Rightarrow \quad 3\left(\frac{7-2y}{3}\right)^2 = 25 + 2y^2$$

$$\Rightarrow \quad \frac{3(7-2y)^2}{9} = 25 + 2y^2$$

$$\Rightarrow \quad \frac{49 + 4y^2 - 28y}{3} = 25 + 2y^2$$

$$\Rightarrow \quad 49 + 4y^2 - 28y = 3(25 + 2y^2)$$

$$\Rightarrow \quad 49 + 4y^2 - 28y = 75 + 6y^2$$

$$\Rightarrow \quad 6y^2 + 75 - 49 - 4y^2 + 28y = 0$$

$$\Rightarrow \quad 2y^2 + 28y + 26 = 0$$

$$\Rightarrow \quad 2(y^2 + 14y + 13) = 0$$

$$\Rightarrow \quad 2(y^2 + 14y + 13) = 0$$

$$\Rightarrow \quad y^2 + 14y + 13 = 0$$

$$\Rightarrow \quad y^2 + 13y + y + 13 = 0$$

$$\Rightarrow \quad y(y + 13) + 1(y + 13) = 0$$

$$\Rightarrow \quad (y + 13) + (y + 1) = 0$$

$$\Rightarrow \quad \text{Either} \quad y + 13 = 0 \quad \text{or} \quad y + 1 = 0$$

$$\Rightarrow \quad y = -13 \quad \text{or} \quad y = -1$$

Put $y = -13$ and $y = -1$ in equation (2)

when $y = -13$

then $x = \frac{7 - 2(-13)}{3}$

$= \frac{7 + 26}{3} = \frac{33}{3} = 11$

$\Rightarrow \quad (11, -13)$

when $y = -1$

then $x = \frac{7 - 2(-1)}{3}$

$= \frac{7 + 2}{3} = \frac{9}{3} = 3$

$\Rightarrow \quad (3, -1)$

Hence the solution set $= \{(3, -1)\ (11, -13)\}$

Q.4 $\mathbf{x + y = 5,\ \frac{2}{x} + \frac{3}{y} = 2,\ x \neq 0,\ y \neq 0.}$

Solution:

$$\frac{2}{x} + \frac{3}{y} = 2 \qquad \ldots\ldots\ldots (1)$$

and $x + y = 5$

$\Rightarrow \quad y = 5 - x \qquad \ldots\ldots\ldots (2)$

Put $y = 5 - x$ from equation (2) in equation (1)

$$\frac{2}{x} + \frac{3}{5-x} = 2$$

$$\Rightarrow \quad \frac{2(5-x) + 3x}{x(5-x)} = 2$$

$\Rightarrow \quad 2(5-x) + 3x = 2x(5-x)$

$\Rightarrow \quad 10 - 2x + 3x = 10x - 2x^2$

$\Rightarrow \quad 10 - 2x + 3x - 10x + 2x^2 = 0$

$\Rightarrow \quad 2x^2 - 9x + 10 = 0$

$\Rightarrow \quad 2x^2 - 5x - 4x + 10 = 0$

$\Rightarrow \quad x(2x-5) - 2(2x-5) = 0$

$\Rightarrow \quad (2x-5)(x-2) = 0$

$\Rightarrow \quad$ Either $2x - 5 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = \frac{5}{2}$ or $x = 2$

Put $x = \frac{5}{2}$ and $x = 2$ in equation (2)

when $x = \frac{5}{2}$		when $x = 2$
$\Rightarrow \quad y = 5 - \frac{5}{2}$	$\Rightarrow$	$y = 5 - 2$
$\Rightarrow \quad y = \frac{10-5}{2}$	$\Rightarrow$	$y = 3$
$\Rightarrow \quad y = \frac{5}{2}$	$\Rightarrow$	$(2, 3)$
$\Rightarrow \quad \left(\frac{5}{2}, \frac{5}{2}\right)$		

Hence the solution set $= \left\{ (2, 3), \left(\frac{5}{2}, \frac{5}{2}\right) \right\}$

Q.5 $x + y = a + b, \quad \frac{a}{x} + \frac{b}{y} = 2$ **(Lahore Board 2003, 2005)**

Solution:

$$\frac{a}{x} + \frac{b}{y} = 2 \qquad \ldots\ldots\ldots (1)$$

and $x + y = a + b$

$$y = a + b - y \qquad \ldots\ldots\ldots (2)$$

Put $y = a + b - x$ from equation (2) in equation (1)

$$\frac{a}{x} + \frac{b}{a + b - x} = 2$$

$$\frac{a(a + b - x) + bx}{x(a + b - x)} = 2$$

$$a(a + b - x) + bx = 2x(a + b - x)$$

$$a^2 + ab - ax + bx = 2ax + 2bx - 2x^2$$

$$2x^2 - 2ax - 2bx + a^2 + ab - ax + bx = 0$$

$$2x^2 - 3ax - bx + a^2 + ab = 0$$

$$2x^2 - (3a + b)x + a^2 + ab = 0$$

Comparing with $ax^2 + bx + c = 0$, we have

$$a = 2, \quad b = -(3a + b), \quad c = a^2 + ab$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-[-(3a + b)] \pm \sqrt{[-(3a + b)]^2 - 4(2)(a^2 + ab)}}{2(2)}$$

$$= \frac{(3a + b) \pm \sqrt{(3a + b)^2 - 8(a^2 + ab)}}{4}$$

$$= \frac{(3a + b) \pm \sqrt{(9a^2 + b^2 + 6ab) - 8a^2 - 8ab}}{4}$$

$$= \frac{(3a + b) \pm \sqrt{9a^2 + b^2 + 6ab - 8a^2 - 8ab}}{4}$$

$$= \frac{(3a + b) \pm \sqrt{a^2 + b^2 - 2ab}}{4}$$

$$x = \frac{(3a+b) \pm \sqrt{(a-b)^2}}{4} = \frac{(3a+b) \pm (a-b)}{4}$$

$\Rightarrow$ $x = \frac{(3a+b)+(a-b)}{4}$ or $x = \frac{(3a+b)-(a-b)}{4}$

$\Rightarrow$ $x = \frac{3a+b+a-b}{4}$ or $x = \frac{3a+b-a+b}{4}$

$\Rightarrow$ $x = \frac{4a}{4}$ or $x = \frac{2a+2b}{4}$

$\Rightarrow$ $x = a$ or $x = \frac{2(a+b)}{4}$

or $x = \frac{a+b}{2}$

Now put these values in equation (2)

when $x = a$ equation (2) $\Rightarrow$

$\Rightarrow$ $y = a + b - a$

$\Rightarrow$ $y = b$

$\Rightarrow$ (a, b)

when $x = \frac{a+b}{2}$ equation (2) $\Rightarrow$

$\Rightarrow$ $y = a + b - \frac{a+b}{2}$

$= \frac{2a+2b-a-b}{2}$

$= \frac{a+b}{2}$

$\Rightarrow$ $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$

Hence the solution set $= \left\{(a, b), \left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right\}$

Q.6 $\mathbf{3x + 4y = 25,\ \frac{3}{x} + \frac{4}{y} = 2}$

Solution:

$\frac{3}{x} + \frac{4}{y} = 2$ ..... (1)

and $3x + 4y = 25$

$\Rightarrow$ $4y = 25 - 3x$

$y = \frac{25-3x}{4}$ ..... (2)

Put value of 'y' from equation (2) in equation (1)

$$\Rightarrow \quad \frac{3}{x} + \frac{4}{\frac{25 - 3x}{4}} = 2$$

$$\Rightarrow \quad \frac{3}{x} + \frac{16}{25 - 3x} = 2$$

$$\Rightarrow \quad \frac{3\,(25 - 3x) + 16x}{x\,(25 - 3x)} = 2$$

$$\Rightarrow \quad 3\,(25 - 3x) + 16x = 2x\,(25 - 3x)$$

$$\Rightarrow \quad 75 - 9x + 16x = 50x - 6x^2$$

$$\Rightarrow \quad 6x^2 - 50x + 75 - 9x + 16x = 0$$

$$\Rightarrow \quad 6x^2 - 43x + 75 = 0$$

$$\Rightarrow \quad x = \frac{-(-43) \pm \sqrt{(-43)^2 - 4\,(6)\,(75)}}{2\,(6)}$$

$$= \frac{43 \pm \sqrt{1849 - 1800}}{12} = \frac{43 \pm \sqrt{49}}{12} = \frac{43 \pm 7}{12}$$

$$\Rightarrow \quad x = \frac{43 + 7}{12} \quad \text{or} \quad x = \frac{43 - 7}{12}$$

$$\Rightarrow \quad x = \frac{50}{12} \quad \text{or} \quad x = \frac{36}{12}$$

$$\Rightarrow \quad x = \frac{25}{6} \quad \text{or} \quad x = 3$$

Put these values in equation (2)

when $x = \frac{25}{6}$ equation (2) $\Rightarrow$

$$\Rightarrow \quad y = \frac{25 - 3\left(\frac{25}{6}\right)}{4}$$

$$\Rightarrow \quad y = \frac{25 - \frac{25}{2}}{4}$$

$$\Rightarrow \quad \frac{\frac{50 - 25}{2}}{4}$$

$$\Rightarrow \quad y = \frac{25}{2}\,\frac{1}{4} = \frac{25}{8}$$

$$\Rightarrow \quad \left(\frac{25}{6}, \frac{25}{8}\right)$$

when $x = 3$ equation (2) $\Rightarrow$

$$\Rightarrow \quad y = \frac{25 - 3\,(3)}{4}$$

$$\Rightarrow \quad y = \frac{25 - 9}{4}$$

$$\Rightarrow \quad = \frac{16}{4} = 4$$

$$\Rightarrow \quad (3, 4)$$

Hence the solution set $= \left\{ (3\,,\,4)\,,\, \left(\frac{25}{6}\,,\,\frac{25}{8}\right) \right\}$

Q.7 $(x-3)^2 + y^2 = 5, \quad 2x = y + 6$

Solution:

$(x-3)^2 + y^2 = 5 \quad \ldots\ldots\ldots (1)$

and $2x = y + 6$

$\Rightarrow \quad y = 2x - 6 \quad \ldots\ldots\ldots (2)$

Put value of 'y' from equation (2) in equation (1)

$\Rightarrow \quad (x-3)^2 + (2x-6)^2 = 5$

$\Rightarrow \quad x^2 + 9 - 6x + 4x^2 + 36 - 24x - 5 = 0$

$\Rightarrow \quad 5x^2 - 30x + 40 = 0$

$\Rightarrow \quad 5(x^2 - 6x + 8) = 0$

$\Rightarrow \quad x^2 - 6x + 8 = 0$

$\Rightarrow \quad x^2 - 4x - 2x + 8 = 0$

$\Rightarrow \quad x(x-4) - 2(x-4) = 0$

$\Rightarrow \quad (x-4)(x-2) = 0$

$\Rightarrow \quad$ Either $x - 4 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = 4$ or $x = 2$

Put these values in equation (2)

when $x = 4$ equation (2) $\Rightarrow$

$y = 2(4) - 6 = 8 - 6 = 2$

$\Rightarrow \quad (4, 2)$

when $x = 2$ equation (2) $\Rightarrow$

$y = 2(2) - 6 = 4 - 6 = -2$

$\Rightarrow \quad (2, -2)$

Hence the solution set $= \{(4, 2), (2, -2)\}$

Q.8 $(x+3)^2 + (y-1)^2 = 5, \quad x^2 + y^2 + 2x = 9$

Solution:

Let $x^2 + y^2 + 2x = 9 \quad \ldots\ldots\ldots (1)$

and $(x+3)^2 + (y-1)^2 = 5$

$\Rightarrow \quad x^2 + 9 + 6x + y^2 + 1 - 2y = 5$

$\Rightarrow \quad x^2 + y^2 + 6x - 2y + 10 = 5$

$\Rightarrow \quad x^2 + y^2 + 6x - 2y = -5 \quad \ldots\ldots\ldots (2)$

Subtracting equation (1) from equation (2)

$x^2 + y^2 + 6x - 2y = -5$

$x^2 + y^2 + 2x = 9$

$- \quad - \quad - \quad -$

$4x - 2y = -14$

$2(2x - y) = -14$

$2x - y = -7$

$\Rightarrow \quad y = 2x + 7 \quad \ldots\ldots\ldots (3)$

Put value of y from equation (3) in equation (1)

$x^2 + (2x + 7)^2 + 2x = 9$

$\Rightarrow \quad x^2 + 4x^2 + 49 + 28x + 2x - 9 = 0$

$\Rightarrow \quad 5x^2 + 30x + 40 = 0$

$\Rightarrow \quad 5(x^2 + 6x + 8) = 0$

$\Rightarrow \quad x^2 + 6x + 8 = 0$

$\Rightarrow \quad x^2 + 4x + 2x + 8 = 0$

$\Rightarrow \quad x(x + 4) + 2(x + 4) = 0$

$\Rightarrow \quad (x + 4)(x + 2) = 0$

$\Rightarrow \quad$ Either $x + 4 = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = -4$ or $x = -2$

Put these values in equation (3)

when $x = -4$ equation (3) $\Rightarrow$

$y = 2(-4) + 7$

$y = -8 + 7$

$y = -1$

$\Rightarrow \quad (-4, -1)$

when $x = -2$ equation (3) $\Rightarrow$

$y = 2(-2) + 7$

$y = -7 + 7$

$y = 3$

$\Rightarrow \quad (-2, 3)$

Hence the solution set $= \{(-4, -1), (-2, 3)\}$

Q.9 $\mathbf{x^2 + (y + 1)^2 = 18, \quad (x + 2)^2 + y^2 = 21}$

Solution:

Given equations

$x^2 + (y + 1)^2 = 18$ and $(x + 2)^2 + y^2 = 21$

$x^2 + y^2 + 1 + 2y = 18$ $\quad x^2 + 4 + 4x + y^2 - 21 = 0$

$x^2 + y^2 + 2y - 17 = 0 \quad \ldots\ldots\ldots (1)$ $\quad x^2 + y^2 + 4x - 17 = 0 \quad \ldots\ldots\ldots (2)$

Subtracting equation (1) from equation (2)

$$x^2 + y^2 + 4x - 17 = 0$$
$$x^2 + y^2 - 17 + 2y = 0$$
$$- \quad - \quad + \quad -$$

$$4x - 2y = 0$$
$$2(2x - y) = 0$$
$$2x - y = 0$$

$\Rightarrow \quad y = 2x \quad \ldots\ldots\ldots (3)$

Put value of y in equation (3) in equation (1)

$\Rightarrow \quad x^2 + (2x)^2 + 2(2x) - 17 = 0$

$$x^2 + 4x^2 + 4x - 17 = 0$$
$$5x^2 + 4x - 17 = 0$$
$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(5)(-17)}}{2(5)} = \frac{-4 \pm \sqrt{16 + 340}}{10}$$
$$= \frac{-4 \pm \sqrt{356}}{10} = \frac{-4 \pm 2\sqrt{89}}{10} = \frac{2(-2 \pm \sqrt{89})}{10} = \frac{-2 \pm \sqrt{89}}{5}$$

$\Rightarrow \quad x = \frac{-2 + \sqrt{89}}{5}$ or $x = \frac{-2 - \sqrt{89}}{5}$

Put these values in equation (3)

when $x = \frac{-2 + \sqrt{89}}{5}$ equation (3) $\Rightarrow$

$$y = 2\left(\frac{-2 + \sqrt{89}}{5}\right)$$
$$y = \frac{-4 + 2\sqrt{89}}{5}$$

$\Rightarrow \quad \left(\frac{-2 + \sqrt{89}}{5}, \frac{-4 + 2\sqrt{89}}{5}\right)$

when $x = \frac{-2 - \sqrt{89}}{5}$ equation (3) $\Rightarrow$

$$y = 2\left(\frac{-2 - \sqrt{89}}{5}\right)$$
$$y = \frac{-4 - 2\sqrt{89}}{5}$$

$\Rightarrow \quad \left(\frac{-2 - \sqrt{89}}{5}, \frac{-4 - 2\sqrt{89}}{5}\right)$

Hence the solution set $= \left\{\left(\frac{-2 + \sqrt{89}}{5}, \frac{-4 + 2\sqrt{89}}{5}\right), \left(\frac{-2 - \sqrt{89}}{5}, \frac{-4 - 2\sqrt{89}}{5}\right)\right\}$

Q.10 $x^2 + y^2 + 6x = 1,\ x^2 + y^2 + 2(x + y) = 3$

Solution:

Given equations

$x^2 + y^2 + 6x = 1$ and $x^2 + y^2 + 2(x + y) = 3$

$x^2 + y^2 + 6x - 1 = 0$ ..... (1) $x^2 + y^2 + 2x + 2y - 3 = 0$ ..... (2)

Subtracting equation (1) from equation (2)

$$\begin{array}{rrrr} x^2 + y^2 + 2x & + 2y & - 3 & = 0 \\ x^2 + y^2 + 6x & & - 1 & = 0 \\ - & - & - & + \end{array}$$

$$-4x + 2y - 2 = 0$$

$2(-2x + y - 1) = 0$

$-2x + y - 1 = 0$

$y = 2x + 1$ ..... (3)

Put $y = 2x + 1$ in equation (1)

$\Rightarrow \quad x^2 + (2x + 1)^2 + 6x - 1 = 0$

$\Rightarrow \quad x^2 + 4x^2 + 1 + 4x + 6x - 1 = 0$

$\Rightarrow \quad 5x^2 + 10x = 0$

$\Rightarrow \quad 5x(x + 2) = 0 \quad \Rightarrow \quad x(x + 2) = 0$

$\Rightarrow$ Either $x = 0$ or $x + 2 = 0 \quad \Rightarrow \quad x = -2$

Put $x = 0$ and $x = -2$ in equation (3)

when $x = 0$ equation (3) $\Rightarrow$

$y = 1$

$\Rightarrow \quad (0, 1)$

when $x = -2$ equation (3) $\Rightarrow$

$y = 2(-2) + 1$

$y = -4 + 1 = -3$

$\Rightarrow \quad (-2, -3)$

Hence the solution set $= \{(0, 1), (-2, -3)\}$

BOTH THE EQUATIONS ARE QUADRATIC IN TWO VARIABLES

The equations in this case are classified as:

(i) Both the equations contain only x^2 and y^2 terms

To solve these equations we eliminate one of x^2 or y^2, which yields two linear equations.

Example: $2x^2 + y^2 = 13,\quad x^2 + y^2 = 9$

Subtracting: $x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$\Rightarrow$ two linear equations $x = 2$ and $x = -2$.

(ii) One of the equations is homogeneous in x and y

If every term in an equation is of the same degree then it is called homogeneous equation.

For example $x^2 - 3xy + 2y^2 = 0$ is homogeneous in x and y.

We shall factorize its L.H.S. and get two linear equations as

$x^2 - 3xy + 2y^2 = 0$ gives

$(x - y)(x - 2y) = 0$

$\Rightarrow$ $x - y = 0$ and $x - 2y = 0$ are the two linear equations.

(iii) Both equations are non–homogeneous

To solve these equations we eliminate the constants and then get a homogeneous equation.

For example: $y^2 - 2xy = 7$ ..... (1) $2x^2 - xy = -3$ ..... (2)

Multiply (1) by 3 and (2) by 7 and adding to eliminate constants, gives

$\Rightarrow$ $14x^2 - 13xy + 3y^2 = 0$ $\Rightarrow$ $(2x - y)(7x - 3y) = 0$ linear factors.

EXERCISE 4.9

Q.1 $2x^2 = 6 + 3y^2$; $3x^2 - 5y^2 = 7$.

Solution:

Given equations

$2x^2 = 6 + 3y^2$ ..... (1)

$3x^2 - 5y^2 = 7$ ..... (2)

$\Rightarrow$ $2x^2 - 3y^2 = 6$

Multiplying equation (1) by (3) and equation (2) by (2) and subtracting

$6x^2 - 9y^2 = 18$

$6x^2 - 10y^2 = 14$

$- \quad + \quad -$

$y^2 = -4$

$y^2 = 4$

$\Rightarrow$ $y = \pm 2$

Now put $y = \pm 2$ in equation (1)

when $y = 2$

equation (1) $\Rightarrow$

$2x^2 - 3(2)^2 = 6$

$\Rightarrow \quad 2x^2 - 12 = 6$

$\Rightarrow \quad 2x^2 = 18$

$\Rightarrow \quad x^2 = 9$

$\Rightarrow \quad x = \pm 3$

$\Rightarrow \quad (\pm 3, 2)$

or $\quad (3, 2), (-3, 2)$

when $y = -2$

equation (1) $\Rightarrow$

$2x^2 - 3(-2)^2 = 6$

$\Rightarrow \quad 2x^2 - 12 = 6$

$\Rightarrow \quad 2x^2 = 18$

$\Rightarrow \quad x^2 = 9$

$\Rightarrow \quad x = \pm 3$

$\Rightarrow \quad (\pm 3, -2)$

or $\quad (3, -2), (-3, -2)$

Hence solution set $= \{3, 2), (-3, 2), (3, -2), (-3, -2)\}$

Q.2 $8x^2 = y^2$; $x^2 + 2y^2 = 19$ **(Gujranwala Board 2007)**

Solution:

Given equations

$8x^2 = y^2$ ..... (1)

and $8x^2 - y^2 = 0$

$x^2 + 2y^2 = 19$ ..... (2)

multiplying equation (1) by 2 and adding in equation (2)

$16x^2 - 2y^2 = 0$

$x^2 + 2y^2 = 19$

$17x^2 = 19$

$x^2 = \frac{19}{17}$

$x = \pm\sqrt{\frac{19}{17}}$

Put these values in equation (1)

when $x = \sqrt{\frac{19}{17}}$

equation (1) $\Rightarrow$

$8 \cdot \frac{19}{17} - y^2 = 0$

when $x = -\sqrt{\frac{19}{17}}$

equation (1) $\Rightarrow$

$8 \cdot \frac{19}{17} - y^2 = 0$

$\Rightarrow \quad y^2 = 8 \cdot \frac{19}{17}$ $\qquad\qquad$ $\Rightarrow \quad y^2 = 8 \cdot \frac{19}{17}$

$\Rightarrow \quad y = \pm\sqrt{\frac{8 \times 19}{19}}$ $\qquad\qquad$ $\Rightarrow \quad y = \pm\sqrt{\frac{8 \times 19}{19}}$

$y = \pm 2\sqrt{\frac{38}{17}}$ $\qquad\qquad$ $\Rightarrow \quad y = \pm 2\sqrt{\frac{38}{17}}$

$\Rightarrow \quad \left(\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right)$ $\qquad\qquad$ $\Rightarrow \quad \left(-\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right)$

Hence solution set $= \left\{\left(\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right), \left(-\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right)\right\}$

Q.3 $2x^2 - 8 = 5y^2$, $x^2 - 13 = -2$ y2.

Solution:

Given equations

$2x^2 - 8 = 5y^2$

$2x^2 - 5y^2 = 8$ ..... (1)

and $x^2 - 13 = -2y^2$

$x^2 + 2y^2 = 13$ ..... (2)

multiplying equation (2) by 2 and subtracting from equation (1)

$2x^2 - 5y^2 = 8$

$2x^2 + 4y^2 = 26$

$- \quad - \quad -$

$-9y^2 = -18$

$y^2 = 2$ $\qquad\qquad$ $\Rightarrow \quad y = \pm\sqrt{2}$

Put these values in equation (1)

when $y = \sqrt{2}$ $\qquad\qquad$ $y = -\sqrt{2}$

equation (1) $\Rightarrow$ $\qquad\qquad$ equation (1) $\Rightarrow$

$2x^2 - 5\left(\sqrt{2}\right)^2 = 8$ $\qquad\qquad$ $2x^2 - 5\left(-\sqrt{2}\right)^2 = 8$

$\Rightarrow \quad 2x^2 - 5(2) = 8$ $\qquad\qquad$ $\Rightarrow \quad 2x^2 - 5(2) = 8$

$\Rightarrow \quad 2x^2 = 18$ $\qquad\qquad$ $\Rightarrow \quad 2x^2 - 10 = 8$

$\Rightarrow \quad x^2 = 9$ $\qquad\qquad$ $\Rightarrow \quad x^2 = 9$

$\Rightarrow \quad x = \pm 3$ $\qquad\qquad$ $\Rightarrow \quad x = \pm 3$

$\left(\pm 3, \sqrt{2}\right)$ $\qquad\qquad$ $\Rightarrow \quad \left(\pm 3, -\sqrt{2}\right)$

Hence solution set $= \left\{\left(\pm 3, \sqrt{2}\right), \left(\pm 3, -\sqrt{2}\right)\right\}$

Q.4 $x^2 - 5xy + 6y^2 = 0;\ x^2 + y^2 = 45$

Solution:

Given equations

$x^2 - 5xy + 6y^2 = 0$, $x^2 + y^2 = 45$

$x^2 - 3xy - 2xy + 6y^2 = 0$

$x(x - 3y) - 2y(x - 3y) = 0$

$(x - 3y)(x - 2y) = 0$

$\Rightarrow$ Either $x - 3y = 0$ or $x - 2y = 0$

(i) First we solve

$x - 3y = 0$ ..... (1)

and $x^2 + y^2 = 45$ ..... (2)

from equation (1)

$x = 3y$ ..... (3)

Put this value in equation (2)

$\Rightarrow (3y)^2 + y^2 = 45$

$\Rightarrow 9y^2 + y^2 = 45$

$\Rightarrow 10y^2 = 45 \Rightarrow y^2 = \frac{45}{10} = \frac{9}{2}$

$\Rightarrow y^2 = \frac{9}{2} \Rightarrow y = \frac{\pm 3}{\sqrt{2}}$

Put these values in equation (3)

when $y = \frac{3}{\sqrt{2}}$ when $y = -\frac{3}{\sqrt{2}}$

equation (3) $\Rightarrow$ equation (3) $\Rightarrow$

$x = 3 \cdot \frac{3}{\sqrt{2}} = \frac{9}{\sqrt{2}}$ $x = -3 \cdot \frac{3}{\sqrt{2}}$

$x = -\frac{9}{\sqrt{2}}$

$\Rightarrow \left(\frac{9}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ $\Rightarrow \left(\frac{-3}{\sqrt{2}}, \frac{-9}{\sqrt{2}}\right)$

(ii) Now we solve

$x - 2y = 0$ ..... (4)

$x^2 + y^2 = 45$ ..... (5)

Put this value in equation (5)

$\Rightarrow (2y)^2 + y^2 = 45$

$\Rightarrow 4y^2 + y^2 = 45 \Rightarrow 5y^2 = 45$

$y^2 = \frac{45}{5}$ $y^2 = 9 \Rightarrow y = \pm 3$

Put these values in equation (5)

When $y = 3$ | when $y = -3$

equation (5) $\Rightarrow$ | equation (5) $\Rightarrow$

$x = 2(3)$ | $x = 2(-3)$

$x = 6$ | $x = -6$

$\Rightarrow$ $(6, 3)$ | $\Rightarrow$ $(-6, -3)$

Hence solution set $\left\{ \left(\frac{9}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right) \left(\frac{-3}{\sqrt{2}}, \frac{-9}{\sqrt{2}} \right), (6, 3)(-3, -6) \right\}$

Q.5 $\mathbf{12x^2 - 25xy + 12y^2 = 0;\ \ 4x^2 + 7y^2 = 148.}$

Solution:

Given equations

$12x^2 - 25xy + 12y^2 = 0$ and $4x^2 + 7y^2 = 148$

$12x^2 - 16xy - 9xy + 12y^2 = 0$

$4x(3x - 4y) - 3y(3x - 4y) = 0$

$(3x - 4y)(4x - 3y) = 0$

Either $3x - y = 0$ or $4x - 3y = 0$

First we solve.

$3x - 4y = 0$ ..... (1)

and $4x^2 + 7y^2 = 148$ ..... (2)

from equation (1)

$3x = 4y$

$\Rightarrow$ $x = \frac{4}{3}y$ ..... (3)

Put this value in equation (2)

$\Rightarrow$ $4\left(\frac{4}{3}y^2\right) + 7y^2 = 148$

$\Rightarrow$ $4\left(\frac{16}{9}y^2\right) + 7y^2 = 148$

$\Rightarrow$ $\frac{64}{9}y^2 + 7y^2 = 148$

$\Rightarrow$ $\frac{64y^2 + 63y^2}{9} = 148$

$\Rightarrow$ $127y^2 = 148 \times 9$

$\Rightarrow$ $127y^2 = 1332 \Rightarrow y^2 = \frac{1332}{127} \Rightarrow y = \pm\sqrt{\frac{1332}{127}}$

$\Rightarrow y = \pm 6\sqrt{\frac{37}{127}}$

Put these values in equation (3)

when $y = 6\sqrt{\frac{37}{127}}$

equation (3) $\Rightarrow$

$$x = \frac{4}{3}\left(6\sqrt{\frac{37}{127}}\right)$$

$$x = 8\sqrt{\frac{37}{127}}$$

$$\Rightarrow \quad \left(8\sqrt{\frac{37}{127}}, 6\sqrt{\frac{37}{127}}\right)$$

when $y = -6\sqrt{\frac{37}{127}}$

equation (3) $\Rightarrow$

$$\frac{4}{3}\left(-6\sqrt{\frac{37}{127}}\right)$$

$$x = -8\sqrt{\frac{37}{127}}$$

$$\left(-8\sqrt{\frac{37}{127}}, -6\sqrt{\frac{37}{127}}\right)$$

(ii) Now we solve

$$4x - 3y = 0 \qquad (4)$$

and $4x^2 + 7y^2 = 148 \qquad (5)$

from equation (4)

$$4x = 3y$$

$$x = \frac{3y}{4} \qquad (6)$$

Put this value in equation (5)

$$\Rightarrow \quad 4\left(\frac{3y}{4}\right)^2 + 7y^2 = 148$$

$$\Rightarrow \quad 4\left(\frac{9y}{16}\right)^2 + 7y^2 = 148$$

$$\Rightarrow \quad \frac{9y^2}{4} + 7y^2 = 148$$

$$\Rightarrow \quad \frac{9y^2 + 28y^2}{4} = 148$$

$$\Rightarrow \quad 37y^2 = 148 \times 4 \quad \Rightarrow \quad 37y^2 = 592$$

$$\Rightarrow \quad y^2 = \frac{592}{37} = 16 \quad \Rightarrow \quad y = \pm 4$$

put these values in equation (6)

when $y = 4$

equation (6) $\Rightarrow$

$$x = \frac{3(4)}{4}$$

$$x = 3$$

$$\Rightarrow \quad (3, 4)$$

when $y = -4$

equation (6) $\Rightarrow$

$$x = \frac{3(-4)}{4}$$

$$x = -3$$

$$\Rightarrow \quad (-3, -4)$$

Hence solution set $= \left\{(3, 4)\ (-3, -4)\left(8\sqrt{\frac{37}{3}}, 6\sqrt{\frac{37}{127}}\right), \left(-8\sqrt{\frac{37}{3}}, -6\sqrt{\frac{37}{127}}\right)\right\}$

Q.6 $12x^2 - 11xy + 2y^2 = 0,\ \ 2x^2 + 7xy = 60.$

Solution:

Given equations

$12x^2 - 11xy + 2y^2 = 0$ and $2x^2 + 7xy = 60$

$\Rightarrow \quad 12x^2 - 8xy - 3xy + 2y^2 = 0$

$\Rightarrow \quad 4x(3x - 2y) - y(3x - 2y) = 0$

$\Rightarrow \quad (3x - 2y)(4x - y) = 0$

Either $3x - 2y = 0$ or $4x - y = 0$

(i) First we solve

$3x - 2y = 0 \quad \ldots\ldots\ldots (1)$

and $2x^2 + 7xy = 60 \quad \ldots\ldots\ldots (2)$

from equation (1)

$3x = 2y$

$\Rightarrow \quad x = \frac{2y}{3} \quad \ldots\ldots\ldots (3)$

Put this value in equation (2)

$\Rightarrow \quad 2\left(\frac{2y}{3}\right)^2 + 7\left(\frac{2y}{3}\right)y = 60$

$\Rightarrow \quad 2\left(\frac{4y^2}{9}\right) + \frac{14}{3}y^2 = 60$

$\Rightarrow \quad \frac{8y^2}{9} + \frac{14}{3}y^2 = 60$

$\Rightarrow \quad \frac{8y^2 + 42y^2}{9} = 60$

$\Rightarrow \quad 50y^2 = 9 \times 60$

$\Rightarrow \quad 50y^2 = 540$

$\Rightarrow \quad y^2 = \frac{540}{50}$

$\Rightarrow \quad y^2 = \frac{54}{5} \Rightarrow y = \pm\sqrt{\frac{54}{5}}$

$\Rightarrow \quad y = \pm 3\sqrt{\frac{6}{5}}$

put these values in equation (3)

when $y = 3\sqrt{\frac{6}{5}}$ when $y = -3\sqrt{\frac{6}{5}}$

equation (3) $\Rightarrow$ equation (3) $\Rightarrow$

$x = \frac{2}{3} 3\sqrt{\frac{6}{5}}$ $x = \frac{2}{3}\left(-3\sqrt{\frac{6}{5}}\right)$

$= 2\sqrt{\frac{6}{5}}$ $= -2\sqrt{\frac{6}{5}}$

$\Rightarrow \left(2\sqrt{\frac{5}{6}}, 3\sqrt{\frac{6}{5}}\right)$ $\left(-2\sqrt{\frac{6}{5}}, -3\sqrt{\frac{6}{5}}\right)$

(ii) Now we solve

$4x - y = 0$ ..... (4)

and $2x^2 + 7xy = 60$ ..... (5)

from equation (4)

$y = 4x$ ..... (6)

put this value in equation (5)

$\Rightarrow 2x^2 + 7x(4x) = 60 \Rightarrow 2x^2 + 28x^2 = 60$

$30x^2 = 60 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

put these values in equation (6)

when $x = \sqrt{2}$ when $x = -\sqrt{2}$

equation (6) $\Rightarrow$ equation (6) $\Rightarrow$

$y = 4\sqrt{2}$ $y = -4\sqrt{2}$

$\Rightarrow (\sqrt{2}, 4\sqrt{2})$ $\Rightarrow (-\sqrt{2}, -4\sqrt{2})$

Hence solutions set is

$$\left\{(\sqrt{2}, 4\sqrt{2}), (-\sqrt{2}, -4\sqrt{2}), \left(2\sqrt{\frac{6}{5}}, 3\sqrt{\frac{6}{5}}\right), \left(-2\sqrt{\frac{6}{5}}, -3\sqrt{\frac{6}{5}}\right)\right\}$$

Q.7 $x^2 - y^2 = 16$, $xy = 15$.

Solution:

Given equations

$x^2 - y^2 = 16$ ..... (1)

$xy = 15$ ..... (2)

from equation (2)

$y = \frac{15}{x}$ ..... (3)

Put this value in equation (1)

$$x^2 - \left(\frac{15}{x}\right)^2 = 16$$

$$\Rightarrow \quad x^2 - \frac{225}{x^2} = 16$$

$$\Rightarrow \quad \frac{x^4 - 225}{x^2} = 16$$

$$\Rightarrow \quad x^4 - 225 = 16x^2$$

$$\Rightarrow \quad x^4 - 16x^2 - 225 = 0$$

By quadratic formula

$$x^2 = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(-225)}}{2(1)}$$

$$= \frac{16 \pm \sqrt{256 + 900}}{2} = \frac{16 \pm \sqrt{1156}}{2} = \frac{16 \pm 34}{2}$$

$$\Rightarrow \quad x^2 = \frac{16 + 34}{2} \qquad \text{or} \qquad x^2 = \frac{16 - 34}{2}$$

$$\Rightarrow \quad x^2 = \frac{50}{2} = 25 \qquad x^2 = \frac{-18}{2} = -9$$

$$\Rightarrow \quad x = \pm 5 \qquad x \;\; x = \pm 3i$$

put these values in equation (3)

when $x = 5 \Rightarrow y = \frac{15}{5} = 3 \Rightarrow (5, 3)$

when $x = -5 \Rightarrow y = \frac{15}{-5} = -3 \Rightarrow (-5, -3)$

when $x = 3i \Rightarrow y = \frac{15}{3i} = \frac{5}{i} \cdot \frac{i}{i} = -5i \Rightarrow (3i, -5i)$

when $x = -3i \Rightarrow y = \frac{15}{-3i} = -\frac{5}{i} \cdot \frac{i}{i} = 5i \Rightarrow (-3i, 5i)$

Hence solution set = $\{(5, -3), (-5, 3), (3i, -5i), (-3i, 5i)\}$

Q.8 $x^2 + xy = 9, \quad x^2 - y^2 = 2.$

Solution:

Given equations

$$x^2 + xy = 9 \qquad \ldots\ldots\ldots (1)$$

$$x^2 - y^2 = 2 \qquad \ldots\ldots\ldots (2)$$

Multiplying equ. (1) by 2 and equ. (2) by 9 and subtracting.

$$2x^2 + 2xy = 18$$
$$9x^2 - 9y^2 = 18$$
$$- \quad + \quad -$$
$$-7x^2 + 2xy + 9y^2 = 0$$

$\Rightarrow \quad 7x^2 - 2xy - 9y^2 = 0$

$\Rightarrow \quad 7x^2 - 9xy + 7xy - 9y^2 = 0$

$\Rightarrow \quad x(7x - 9y) + y(7x - 9y) = 0$

$\Rightarrow \quad (7x - 9y)(x + y) = 0$

Either $7x - 9y = 0$ or $x + y = 0$

(i) First we solve

$7x - 9y = 0 \quad \ldots\ldots\ldots (3)$

and $x^2 - y^2 = 2 \quad \ldots\ldots\ldots (4)$

from equation (3)

$$7x = 9y \Rightarrow x = \frac{9}{7}y \quad \ldots\ldots\ldots (5)$$

Put this value in equation (4)

$$\left(\frac{9}{7}y\right)^2 - y^2 = 2$$

$$\Rightarrow \quad \frac{81}{49}y^2 - y^2 = 2$$

$$\Rightarrow \quad \frac{81y^2 - 49y^2}{49} = 2$$

$$\Rightarrow \quad 32y^2 = 98 \Rightarrow y^2 = \frac{98}{32} = \frac{49}{16}$$

$$\Rightarrow \quad y = \pm\frac{7}{4}$$

put these values in equation (5)

when $y = \frac{7}{4}$

equation (5) $\Rightarrow$

$$x = \frac{9}{7} \cdot \frac{7}{4}$$

$$x = \frac{9}{4}$$

$$\Rightarrow \quad \left(\frac{9}{4}, \frac{7}{4}\right)$$

when $y = -\frac{7}{4}$

equation (5) $\Rightarrow$

$$x = \frac{9}{7}\left(-\frac{7}{4}\right)$$

$$x = -\frac{9}{4}$$

$$\Rightarrow \quad \left(-\frac{9}{4}, -\frac{7}{4}\right)$$

(ii) Now we solve

$x + y = 0$ ..... (6)

and $x^2 - y^2 = 2$ ..... (7)

from equation (6)

$x = -y$ ..... (8)

Put in equation (7)

$(-y)^2 - y^2 = 2$

$y^2 - y^2 = 2$

$0 = 2$ No solution.

Hence solution set $= \left\{\left(\frac{9}{4}, \frac{7}{4}\right), \left(-\frac{9}{4}, -\frac{7}{4}\right)\right\}$

Q.9 $\mathbf{y^2 - 7 = 2xy \quad ; \quad 2x^2 + 3 = xy.}$

Solution:

Given equations

$y^2 - 7 = 2xy$ ..... (1)

$2x^2 + 3y = xy$ ..... (2)

Multiplying equation (1) by 3 and equation (2) by 7 and adding

$3y^2 - 21 = 6xy$

$14x^2 + 21 = 7xy$

$14x^2 + 3y^2 = 0$

$\Rightarrow \quad 14x^2 - 13xy + 3y^2 = 0$

$\Rightarrow \quad 14x^2 - 7xy - 6xy + 3y^2 = 0$

$\Rightarrow \quad 7x(2x - y) - 3y(2x - y) = 0$

$\Rightarrow \quad (2x - y)(7x - 3y) = 0$

Either $2x - y = 0$ or $7x - 3y = 0$

(i) First we solve

$2x - y = 0$ ..... (3)

and $2x^2 + 3 = xy$ ..... (4)

from equation (3)

$y = 2x$ ..... (5)

Put this value in equation (4)

$2x^2 + 3 = x(2x)$

$2x^2 + 3 = 2x^2$

$2x^2 - 2x^2 + 3 = 0$

$3 = 0$ No Solution

(ii) Now we solve

$7x - 3y = 0$ ..... (6)

and $2x^2 + 3 = xy$ ..... (7)

from equation (6)

$7x = 3y$

$x = \frac{3y}{7}$ ..... (8)

Put this value in equation (7)

$\Rightarrow \quad 2\left(\frac{3y}{7}\right)^2 + 3 = \frac{3y}{7} \cdot y$

$\Rightarrow \quad 2 \cdot \frac{9y^2}{49} + 3 = \frac{3y^2}{7}$

$\Rightarrow \quad \frac{18y^2}{49} + 3 = \frac{3y^2}{7}$

$\Rightarrow \quad \frac{18y^2 + 147}{49} = \frac{3y^2}{7}$

$\Rightarrow \quad 7(18y^2 + 147) = 147y^2$

$\Rightarrow \quad 126y^2 + 1029 = 147y^2$

$\Rightarrow \quad 147y^2 - 126y^2 = 1029$

$\Rightarrow \quad 21y^2 = 1029 \Rightarrow y^2 = \frac{1029}{21}$

$\Rightarrow \quad y^2 = 49 \Rightarrow y = \pm 7$

put these values in equation (8)

when $y = 7$

equation (8) $\Rightarrow$

$x = \frac{3}{7}(7)$

$x = -3$

$\Rightarrow \quad (3, 7)$

when $y = -7$

equation (8) $\Rightarrow$

$x = \frac{3}{7}(-7)$

$x = -3$

$\Rightarrow \quad (-3, -7)$

Hence solution set $= \{(3, 7), (-3, -7)\}$

Q.10 $x^2 + y^2 = 5, \quad xy = 2$ **(Lahore Board 2003, Gujranwala Board 2005)**

Solution:

Given equations

$$x^2 + y^2 = 5 \qquad \ldots\ldots\ldots (1)$$

and $$xy = 2 \qquad \ldots\ldots\ldots (2)$$

from equation (2)

$$y = \frac{2}{x} \qquad \ldots\ldots\ldots (3)$$

Put this value in equation (1)

$$\Rightarrow \quad x^2 + \left(\frac{2}{x}\right)^2 = 5$$

$$\Rightarrow \quad x^2 + \frac{4}{x^2} = 5$$

$$\Rightarrow \quad \frac{x^4 + 4}{x^2} = 5$$

$$\Rightarrow \quad x^4 + 4 = 5x^2$$

$$\Rightarrow \quad x^4 - 5x^2 + 4 = 0$$

using quadratic formula

$$x^2 = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\,(1)\,(4)}}{2\,(1)} = \frac{5 \pm \sqrt{25 - 16}}{2}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2}$$

$$\Rightarrow \quad x^2 = \frac{5 + 3}{2} \quad \text{or} \quad x^2 = \frac{5 - 3}{2}$$

$$\Rightarrow \quad x^2 = 4 \quad \text{or} \quad x^2 = 1$$

$$\Rightarrow \quad x = \pm 2 \quad \text{or} \quad x = \pm 1$$

Put these values in equation (3)

when $x = 2 \Rightarrow y = \frac{2}{2} = 1 \Rightarrow (2, 1)$

when $x = -2 \Rightarrow y = \frac{2}{-2} = -1 \Rightarrow (-2, -1)$

when $x = 1 \Rightarrow y = \frac{2}{1} = 2 \Rightarrow (1, 2)$

when $x = -1 \Rightarrow y = \frac{2}{-1} = -2 \Rightarrow (-1, -2)$

Hence solution set = $\{(2, 1), (-2, -1), (1, 2), (-1, -2)\}$

WORD PROBLEMS ON QUADRATIC EQUATIONS

Now we solve the problems which, when expressed symbolically, lead to quadratic equations in one or two variables.

In order to solve such problems, we must;

(1) Suppose the unknown quantities to be x or y etc.

(2) Translate the problem into symbols and form the equations satisfying the given conditions.

EXERCISE 4.10

Q.1 The product of one less than a certain positive number and two less than three times the number is 14. Find the number.

Solution:

Let the required positive number is = x

then according to the given condition

$(x - 1)(3x - 2) = 14$

$\Rightarrow \quad 3x^2 - 2x - 3x + 2 - 14 = 0$

$\Rightarrow \quad 3x^2 - 5x - 12 = 0$

$\Rightarrow \quad 3x^2 - 9x + 4x - 12 = 0$

$\Rightarrow \quad 3x(x - 3) + 4(x - 3) = 0$

$\Rightarrow \quad (x - 3)(3x + 4) = 0$

$\Rightarrow$ Either $x - 3 = 0$ or $3x + 4 = 0$

$\Rightarrow \quad x = 3$ or $x = \frac{-4}{3}$

$\Rightarrow$ required positive number = 3

Q.2 The sum of a positive number and its square is 380. Find the number.

(Gujranwala Board 2007)

Solution:

Let the required positive number = x

Then according to the given condition

$x + x^2 = 380$

$\Rightarrow \quad x^2 + x - 380 = 0$

$\Rightarrow \quad x^2 + 20x - 19x - 380 = 0$

$\Rightarrow \quad x(x + 20) - 19(x + 20) = 0$

$\Rightarrow \quad (x + 20)(x - 19) = 0$

$\Rightarrow$ Either $x + 20 = 0$ or $x - 19 = 0$

$x = -20$ or $x = 19$

Hence, the required positive number = `9

Q.3 Divide 40 into two parts such that the sum of their square is greater than 2 times their product by 100.

Solution:

Let one part is $= x$

then other part $= 40 - x$

$\Rightarrow \quad x^2 + (40 - x)^2 = 2x(40 - x) + 100$

$\Rightarrow \quad x^2 + 1600 + x^2 - 80x = 80x - 2x^2 + 100$

$\Rightarrow \quad 2x^2 + 1600 - 80x - 80x + 2x^2 - 100 = 0$

$\Rightarrow \quad 4x^2 - 160x + 1500 = 0$

$\Rightarrow \quad 4(4x^2 - 40x + 375) = 0$

$\Rightarrow \quad x^2 - 40x + 375 = 0$

$\Rightarrow \quad x^2 - 25x - 15x + 375 = 0$

$\Rightarrow \quad x(x - 25) - 15(x - 25) = 0$

$\Rightarrow \quad (x - 25)(x - 15) = 0$

Either $x - 25 = 0$ or $x - 15 = 0$

$x = 15$ or $x = 15$

$\Rightarrow$ If one part is 25 then other is $40 - 25 = 15$ and if one part is 15 then the other is $40 - 15 = 25$

$\Rightarrow$ Required parts are 15, 25.

Q.4 The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

(Gujranwala Board 2007)

Solution:

Let the required positive number $= x$

According to the given condition

$$x + \frac{1}{x} = \frac{26}{5}$$

$$\Rightarrow \quad \frac{x^2 + 1}{x} = \frac{26}{5}$$

$\Rightarrow \quad 5(x^2 + 1) = 26x$

$\Rightarrow \quad 5x^2 + 5 - 26x = 0$

$\Rightarrow \quad 5x^2 - 26x + 5 = 0$

$\Rightarrow \quad 5x^2 - 25x - x + 5 = 0$

$\Rightarrow \quad 5x(x-5) - 1(x-5) = 0$

$\Rightarrow \quad (x-5)(5x-1) = 0$

$\Rightarrow \quad$ Either $x - 5 = 0$ or $5x - 1 = 0$

$\Rightarrow \quad x = 5$ or $x = \frac{1}{5}$

Hence the required numbers $= 5, \frac{1}{5}$

Q.5 A number exceeds its square root by 56. Find the number.

Solution:

Let the required number $= x$

According to the given condition

$x = \sqrt{x} + 56$

$\Rightarrow \quad \sqrt{x} = x - 56$

$\Rightarrow \quad (\sqrt{x})^2 = (x-56)^2$

$\Rightarrow \quad x = x^2 + 3136 - 112x$

$\Rightarrow \quad x^2 + 3136 - 112x - x = 0$

$\Rightarrow \quad x^2 - 113x + 3136 = 0$

$\Rightarrow \quad x^2 - 64x - 49x + 3136 = 0$

$\Rightarrow \quad x(x-64) - 49(x-64) = 0$

$\Rightarrow \quad (x-64)(x-49) = 0$

$\Rightarrow \quad$ Either $x - 64 = 0$ or $x - 49 = 0$

$\Rightarrow \quad x = 64$ or $x = 49$

As $x = 64$ obeys the given condition.

$\Rightarrow \quad$ Required number is 64.

Q.6 Find two consecutive numbers, whose product is 132. (Lahore Board 2010)

Solution:

Let two consecutive numbers are $x, x + 1$

According to the given condition,

$x(x+1) = 132$

$\Rightarrow \quad x^2 + x - 132 = 0$

$\Rightarrow \quad x^2 + 12x - 11x - 132 = 0$

$\Rightarrow \quad x(x+12) - 11(x+12) = 0$

$\Rightarrow$ $(x + 12)(x - 11) = 0$

$\Rightarrow$ Either $x + 12 = 0$ or $x - 11 = 0$

$x = -12$ or $x = 11$

If $x = -12 \Rightarrow x + 1 = -12 + 1 = -11$

If $x = 11 \Rightarrow x + 1 = 11 + 1 = 12$

$\Rightarrow$ Required numbers are $-12, -11$ or $11, 12$.

Q.7 The difference between the cubes of two consecutive even numbers is 296. Find them.

Solution:

Let the required numbers are $x, x + 2$

According to the given condition,

$(x + 2)^3 - (x)^3 = 296$

$\Rightarrow$ $x^3 + 6x^2 + 12x + 8 - x^3 = 296$

$\Rightarrow$ $6x^2 + 12x + 8 - 296 = 0$ ¤ $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow$ $6x^2 + 12x - 288 = 0$

$\Rightarrow$ $6(x^2 + 2x - 48) = 0$

$\Rightarrow$ $x^2 + 2x - 48 = 0$

$\Rightarrow$ $x^2 + 8x - 6x - 48 = 0$

$\Rightarrow$ $x(x + 8) - 6(x + 8) = 0$

$\Rightarrow$ $(x + 8)(x - 6) = 0$

$\Rightarrow$ Either $x + 8 = 0$ or $x - 6 = 0$

$\Rightarrow$ $x = -8$ or $x = 6$

If $x = -8$ then $x + 2 = -8 + 2 = -6$

If $x = 6$ then $x + 2 = 6 + 2 = 8$

Hence required numbers are $-8, -6$ or $6, 8$.

Q.8 A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

Solution:

Suppose no. of sheep = x

Price of x sheep = Rs. 9000

Price of 1 sheep = Rs. $\frac{9000}{x}$ ..... (1)

If no. of sheep = $x + 3$

then the price of 1 sheep = Rs. $\frac{9000}{x+3}$ ..... (2)

then according to the given condition.

$\Rightarrow \quad \frac{9000}{x} - 100 = \frac{9000}{x+3}$

$\Rightarrow \quad \frac{9000 - 100x}{x} = \frac{9000}{x+3}$

$\Rightarrow \quad (x + 3)(9000 - 100x) = 9000x$

$\Rightarrow \quad 9000x - 100x^2 + 27000 - 300x - 9000x = 0$

$\Rightarrow \quad -100x^2 - 300x + 27000 = 0$

$\Rightarrow \quad -100(x^2 + 3x - 270) = 0$

$\Rightarrow \quad x^2 + 3x - 270 = 0$

$\Rightarrow \quad x^2 + 18x - 15x - 270 = 0$

$\Rightarrow \quad x(x + 18) - 15(x + 18) = 0$

$\Rightarrow \quad (x + 18)(x - 15) = 0$

$\Rightarrow$ Either $x + 18 = 0$ or $x - 15 = 0$

$x = -18$ or $x = 15$

$x = -18$ neglecting as it is negative

$\Rightarrow$ Required number of sheep = 15

Q.9 A man sold his stock of eggs for Rs. 240. He has 2 dozen, more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

Solution:

Suppose the number of eggs = x dozen

Price of x dozen eggs = Rs. 240

Price of 1 dozen eggs = Rs. $\frac{240}{x}$ ..... (1)

And if, Price of $x + 2$ dozen eggs = Rs. 240

Price of 1 dozen eggs = Rs. $\frac{240}{x+2}$ ..... (2)

According to the given condition

$$\Rightarrow \quad \frac{240}{x} - 0.50 = \frac{240}{x+2}$$

$$\Rightarrow \quad \frac{240}{x} - \frac{50}{100} = \frac{240}{x+2}$$

$$\Rightarrow \quad \frac{240}{x} - \frac{1}{2} = \frac{240}{x+2}$$

$$\Rightarrow \quad \frac{480-x}{2x} = \frac{240}{x+2}$$

$$240\,(2x) = (480-x)\,(x+2)$$

$$\Rightarrow \quad 480x = 480x + 960 - x^2 - 2x$$

$$\Rightarrow \quad x^2 + 2x - 480x - 960 + 480x = 0$$

$$\Rightarrow \quad x^2 + 2x - 960 = 0$$

$$\Rightarrow \quad x^2 + 32x - 30x - 960 = 0$$

$$\Rightarrow \quad x\,(x+32) - 30\,(x+32) = 0$$

$$\Rightarrow \quad (x-32)\,(x+30) = 0$$

$\Rightarrow$ Either $x + 32 = 0$ or $x - 32 = 0$

$x = -32$ (neglect) or $x = 30$

$\Rightarrow$ Required dozen of eggs = 30

Q.10 A cyclist travelled 48km at a uniform speed. Had he travelled 2km/h slower, he would have taken two hours more to perform journey, how long he did take 48 km?

Solution:

Let the Speed of the cyclist = v km/h

and time taken for 48 km = t hours

by first condition

$vt = 48$ ..... (1)

If the speed of cyclist is $(v - 2)$ km/hour, then he will take $(t + 2)$ hours to cover 48 km.

Now by 2nd condition

$(v - 2)(t + 2) = 48$ ..... (2)

from equation (1)

$$v = \frac{48}{t}$$

Put this value in equation (2)

$\Rightarrow \quad \left(\frac{48}{t} - 2\right)(t + 2) = 48$

$\Rightarrow \quad \left(\frac{48 - 2t}{t}\right)(t + 2) = 48$

$\Rightarrow \quad (48 - 2t)(t + 2) = 48t$

$\Rightarrow \quad 48t + 96 - 2t^2 - 4t - 48t = 0$

$\Rightarrow \quad -2t^2 - 4t + 96 = 0$

$\Rightarrow \quad -2(t^2 + 2t - 48) = 0$

$\Rightarrow \quad t^2 + 2t - 48 = 0$

$\Rightarrow \quad t^2 + 8t - 6t - 48 = 0$

$\Rightarrow \quad t(t + 8) - 6(t + 8) = 0$

$\Rightarrow \quad (t + 8)(t - 6) = 0$

$\Rightarrow$ Either $t + 8 = 0$ or $t - 6 = 0$

$t = -8$ or $t = 6$

$t = -8$ (neglecting)

$\Rightarrow$ Required time = 6 hours.

Q.11 The area of rectangular region 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3square meter more. Find its length and breath.

Solution:

Let length of rectangular region = x

and breath of rectangular region = y

By 1st condition

$xy = 297 \quad \ldots\ldots\ldots (1)$

By 2nd condition

$(x + 3)(y - 1) = 297 + 3$

$\Rightarrow \quad xy - x + 3y - 3 = 300$

$xy - x + 3y = 303 \quad \ldots\ldots\ldots (2)$

Now from equation (1)

$x = \frac{297}{y} \quad \ldots\ldots\ldots (3)$

Put this value in equation (2)

$\Rightarrow \quad \frac{297}{y} \cdot y - 3y = 303$

$\Rightarrow \quad 297 - \frac{297}{y} + 3y = 303$

$\Rightarrow \quad 297 - 303 = \frac{297}{y} - 3y$

$\Rightarrow \quad -6 = \frac{297 - 3y^2}{y}$

$\Rightarrow \quad -6y = -3y^2 + 297$

$\Rightarrow \quad 3y^2 - 6y - 297 = 0$

$\Rightarrow \quad 3(y^2 - 2y - 99) = 0$

$\Rightarrow \quad y^2 - 2y - 99 = 0$

$\Rightarrow \quad y^2 - 11y + 9y - 99 = 0$

$\Rightarrow \quad y(y - 11) + 9(y - 11) = 0$

$\Rightarrow \quad (y - 11)(y + 9) = 0$

$\Rightarrow$ Either $y - 11 = 0$ or $y + 9 = 0$

$\Rightarrow \quad y = 11$ or $y = -9$ (neglecting)

Put $y = 11$ in equation (3)

$x = \frac{297}{11} = 27$

$\Rightarrow$ length of rectangle = 27 m

breadth of rectangle = 11m

Q.12 The length of a rectangular piece of paper exceeds its breath by 5cm. If a strip 0.5cm wide be cut all around the piece of paper, the area of the remaining part would be 500 square cms. Find its original dimensions.

Solution:

Let length of rectangular paper = x

then Breath of rectangular paper = x – 5

If a strip of 0.5 cm wide be cut all around the piece of paper. Then

Length of paper $= x - 0.5 - 0.5 = x - 1$

and breadth of paper $= x - 5 - 0.5 - 0.5 = x - 6$

According to the given condition

$\Rightarrow (x - 1)(x - 6) = 500$

$\Rightarrow x^2 - 6x - x + 6 = 500$

$\Rightarrow x^2 - 7x + 6 - 500 = 0$

$\Rightarrow x^2 - 7x - 494 = 0$

$\Rightarrow x^2 - 26x + 19x - 494 = 0$

$\Rightarrow x(x - 26) + 19(x - 26) = 0$

$\Rightarrow (x - 26)(x + 19) = 0$

$\Rightarrow$ Either $x - 26 = 0$ or $x + 19 = 0$

$\Rightarrow$ $x = 26$ or $x = -19$ (neglecting)

$\Rightarrow$ length of rectangular paper $= 26$

breadth of rectangular paper $= x - 5 = 26 - 5 = 21$

Q.13 A number consists of two digits whose product is 18. If digits are interchanged. New number becomes 27 less than original number. Find the number.

Solution:

Suppose tens digit $= x$

and unit digit $= y$

$\therefore$ The number $= 10x + y$

By interchanging the digits, the new number $= 10y + x$

Product of the digits $= xy$

By the condition of question

$xy = 18$ ..... (1)

and $10x + y = 10y + x + 27$

or $10x + y - 10y - x - 27 = 0$

or $9x - 9y - 27 = 0$

or $9(x - y - 3) = 0$

$x - y - 3 = 0$

$y = x - 3$ ..... (2)

Put this value in equation (1)

$\Rightarrow \quad x(x-3) = 18$

$\Rightarrow \quad x^2 - 3x - 18 = 0$

$\Rightarrow \quad x^2 - 6x + 3x - 18 = 0$

$\Rightarrow \quad x(x-6) + 3(x-6) = 0$

$\Rightarrow \quad (x-6)(x+3) = 0$

$\Rightarrow \quad$ Either $x - 6 = 0$ or $x + 3 = 0$

$\Rightarrow \quad x = 6$ or $x = -3$ (neglecting)

Put $x = 6$ in equation (2)

$y = x - 3 = 6 - 3 = 3$

$\Rightarrow \quad$ tens digit $= x = 6$

$\Rightarrow \quad$ unit digit $= y = 3$

$\Rightarrow \quad$ required number $= 63$

Q.14 A number consists of two digits whose product is 14. If digits are interchanged the resulting number will exceed the original number by 45. Find the number.

Solution:

Suppose tens digit $= x$

and unit digit $= y$

$\therefore \quad$ the number $= 10x + y$

By interchanging the digits, the new number $= 10y + x$

Product of the digits $= xy$

By the condition of question

$xy = 14 \quad \ldots\ldots\ldots (1)$

and $10y + x - 45 = 10x + y$

$\Rightarrow \quad 10x + y - 10y - x - 45 = 0$

$\Rightarrow \quad 9x - 9y + 45 = 0$

$\Rightarrow \quad 9(x - y + 5) = 0$

$\Rightarrow \quad x - y + 5 = 0$

$\Rightarrow \quad y = x + 5 \quad \ldots\ldots\ldots (2)$

Put this value in equation (1)

$\Rightarrow \quad x(x+5) = 14$

$\Rightarrow \quad x^2 + 5x - 14 = 0$

$\Rightarrow \quad x^2 + 7x - 2x - 14 = 0$

$\Rightarrow \quad x(x+7) - 2(x+7) = 0$

$\Rightarrow \quad (x+7)(x-2) = 0$

$\Rightarrow \quad$ Either $x + 7 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = -7$ (neglect) or $x = 2$

Put $x = 2$ in equation (2)

$y = 2 + 5 = 7$

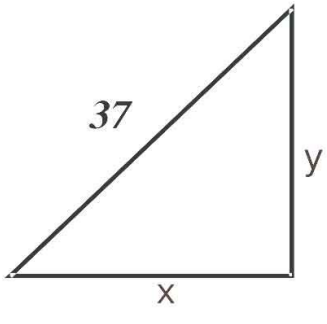
$\Rightarrow \quad$ tens digit $= x = 2$

$\Rightarrow \quad$ unit digit $= y = 7$

$\Rightarrow \quad$ required number $= 27$

Q.15 The area of right triangle is 210 square meters. If its hypotenuse is 37 meters long. Find length of the base and altitude.

Solution:



Suppose length of base of right triangle $= x$

and its altitude $= y$

As area of right triangle $= \frac{1}{2} \times$ base $\times$ altitude

$\Rightarrow \quad A = \frac{1}{2} \cdot x \cdot y$

According to the given condition

$A = 210$

$\Rightarrow \quad \frac{1}{2} xy = 210 \quad \Rightarrow \quad xy = 420 \qquad \ldots\ldots\ldots (1)$

Also given that hypotenuse $= 37$

$\Rightarrow \quad \sqrt{x^2 + y^2} = 37$

$\Rightarrow \quad x^2 + y^2 = 1369 \qquad \ldots\ldots\ldots (2)$

From equation (1)

$x = \frac{420}{y} \qquad \ldots\ldots\ldots (3)$

Put this value in equation (2)

$$\Rightarrow \quad \left(\frac{420}{y}\right)^2 + y^2 = 1369$$

$$\Rightarrow \quad \frac{176400}{y^2} + y^2 = 1369$$

$$\Rightarrow \quad \frac{176400 + y^4}{y^2} = 1369$$

$$\Rightarrow \quad y^4 + 176400 = 1369y^2$$

$$\Rightarrow \quad y^4 - 1369y2 + 176400 = 0$$

$$\Rightarrow \quad y^2 = \frac{1369 \pm \sqrt{(-1369)^2 - 4\,(1)\,(176400)}}{2\,(1)}$$

$$\Rightarrow \quad = \frac{1369 \pm \sqrt{1874161 - 705600}}{2} = \frac{1369 \pm \sqrt{1168561}}{2}$$

$$\Rightarrow \quad y^2 = \frac{1369 \pm 1081}{2}$$

$$\Rightarrow \quad y^2 = \frac{1369 + 1081}{2} \qquad \text{or} \qquad y^2 = \frac{1369 - 1081}{2}$$

$$\Rightarrow \quad y^2 = \frac{2450}{2} \qquad\qquad y^2 = \frac{288}{2}$$

$$\Rightarrow \quad y^2 = 1225 \qquad\qquad y^2 = 144$$

$$\Rightarrow \quad y = \pm 35 \qquad\qquad y = \pm 12$$

$$\Rightarrow \quad y = 35 \qquad\qquad y = 12$$

neglecting –ve values

Put these values in equation (3)

when $y = 35 \quad \Rightarrow \quad x = \frac{420}{y} = \frac{420}{35} = 12$

when $y = 12 \quad \Rightarrow \quad x = \frac{420}{12} = 35$

$\Rightarrow$ If length of base = 35 meters

then length of altitude = 12 meters

and if length of base = 12 meters

then length of altitude = 35 meters

Q.16 The area of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and the breadth of the rectangle.

Solution:

Suppose length of rectangle = x

and breadth of rectangle = y

Area of rectangle = xy = 1680 ..... (1)

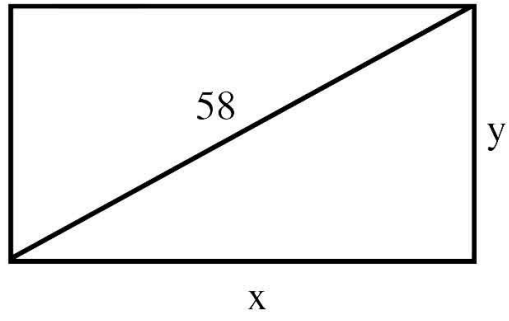
and $x^2 + y^2 = (58)^2$

$\Rightarrow$ $x^2 + y^2 = 3364$ ..... (2)

From equation (1)

$$x = \frac{1680}{y} \quad \text{..... (3)}$$

Put this value in equation (2)



$\Rightarrow$ $\left(\frac{1680}{y}\right)^2 + y^2 = 3364$

$\Rightarrow$ $\frac{2822400}{y^2} + y^2 = 3364$

$\Rightarrow$ $\frac{2822400 + y^4}{y^2} = 3364$

$\Rightarrow$ $y^4 + 2822400 = 3364y^2$

$\Rightarrow$ $y^4 - 3364y^2 + 2822400 = 0$

$\Rightarrow$ $y^2 = \frac{-(-3364) \pm \sqrt{(-3364)^2 - 4\,(1)\,(2822400)}}{2\,(1)}$

$\Rightarrow$ $= \frac{3364 \pm \sqrt{11316496 - 11289600}}{2}$

$\Rightarrow$ $= \frac{3364 \pm \sqrt{26896}}{2} = \frac{3364 \pm 64}{2}$

$\Rightarrow$ $y^2 = \frac{3364 + 164}{2}$ or $y^2 = \frac{3364 - 164}{2}$

$\Rightarrow$ $y^2 = 1764$ $y^2 = 1600$

$\Rightarrow$ $y = 42$ $y = 40$

$\Rightarrow$ Put these values in equation (3)

$x = \frac{1680}{42} = 40$ or $x = \frac{1680}{40} = 42$

$\Rightarrow$ If breadth of rectangle $= y = 40$ meters

$\Rightarrow$ Then length of rectangle $= x = 42$ meters and if

breadth of rectangle $= y = 42$ meters

then length of rectangle $= x = 40$ meters

Q.17 To do a piece of work, a takes 10 days more than B. Together they finish work in 12 days. How long would B take to finish it alone?

Solution:

Suppose B takes 'x' days and A takes $x + 10$ days to do the work.

Then one day's work of B $= \frac{1}{x}$

one day's work of A $= \frac{1}{x+10}$

One day's work of both $= \frac{1}{x} + \frac{1}{x+10}$ ..... (1)

If they do work in 12 days together then

one day's work of both $= \frac{1}{12}$ ..... (2)

Equating (1) and (2)

$\Rightarrow \frac{1}{x} + \frac{1}{x+10} = \frac{1}{12}$

$\Rightarrow \frac{x+10+x}{x(x+10)} = \frac{1}{12}$

$\Rightarrow \frac{2x+10}{x^2+10x} = \frac{1}{12}$

$\Rightarrow x^2 + 10x = 12(2x + 10)$

$\Rightarrow x^2 + 10x = 24x + 120$

$\Rightarrow x^2 + 10x - 24x - 120 = 0$

$\Rightarrow x^2 - 14x - 120 = 0$

$\Rightarrow x^2 - 20x + 6x - 120 = 0$

$\Rightarrow x(x - 20) + 6(x - 20) = 0$

$\Rightarrow (x - 20)(x + 6) = 0$

Either $x - 20 = 0$ or $x + 6 = 0$

$\Rightarrow x = 20$ or $x = -6$ (neglecting)

$\Rightarrow$ 'B' will take 20 days to do the work.

Q.18 To complete a job, A and B take 4 days working together. A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

Solution:

As A and B together finish a job in 4 days.

$\Rightarrow$ one day's work together $= \frac{1}{4}$ ..... (1)

Suppose B takes 'x' days then A will take 2x days.

Now one day's work of A $= \frac{1}{2x}$

and one day's work of B $= \frac{1}{x}$

one day's work of both $= \frac{1}{x} + \frac{1}{2x}$ ..... (2)

From equations (1) and (2)

$\Rightarrow \quad \frac{1}{x} + \frac{1}{2x} = \frac{1}{4}$

$\Rightarrow \quad \frac{2+1}{2x} = \frac{1}{4}$

$\Rightarrow \quad \frac{3}{2x} = \frac{1}{4}$

$\Rightarrow \quad 2x = 12 \quad \Rightarrow \quad 2x - 12 = 0$

$\Rightarrow \quad 2(x-6) = 0 \quad \Rightarrow \quad x - 6 = 0 \quad \Rightarrow \quad x = 6$

$\Rightarrow$ B will take 6 days to do the work and A will take 12 days to do the work.

Q.19 An open box is to be made from a square piece of tin by cutting a piece 2 dm square from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be 128 c.dm, find the length of the side of the piece.

Solution:

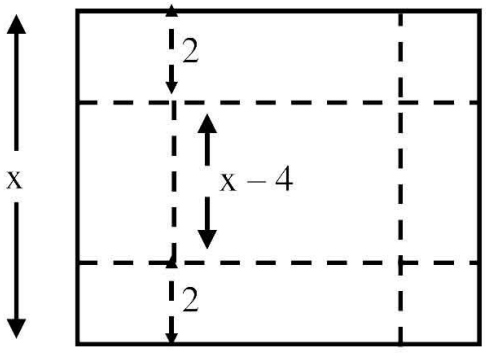
Let 'x' be the length of the square piece.

Then length of box $= x - 4$

Breadth of box $= x - 4$

Height of box $= 2$

Volume of box $= 2(x-4)(x-4)$



But given that volume = 128

$\Rightarrow$ $2(x-4)(x-4) = 128$

$\Rightarrow$ $(x-4)^2 = \frac{128}{2}$

$\Rightarrow$ $(x-4)^2 = 64$

$\Rightarrow$ $x - 4 = 8$

$\Rightarrow$ $x = 8 + 4 = 12$

$\Rightarrow$ required length of side = 12dm

Q.20 A man invest Rs. 100,000 in two companies. His total profit is Rs. 3080. If he receives Rs. 1980 from one company at the rate 1% more from the other. Find the amount of each investment.

Solution:

Suppose the amount invested in 1st company = x

then the amount invested in 2nd company = 100,000 – x

Let the profit from 1st company = y% of x = $\frac{yx}{100}$

but it is given that profit from 1st company = 1980

$\Rightarrow$ $\frac{y\,x}{100} = 1980$

$\Rightarrow$ $xy = 198000$ ..... (1)

As he gets profit 1% more from the other company

$\Rightarrow$ Profit from 2nd company = (y + 1)% of (100,000 – x) = $\frac{y+1}{100}(100{,}000 - x)$

but it is given that profit from 2nd company = 3080

$\Rightarrow$ $\frac{y+1}{100}(100{,}000 - x) = 3080$

$\Rightarrow$ $(y+1)(100{,}000 - x) = 308000$

$\Rightarrow$ $100{,}000y - xy + 100000 - x - 308000 = 0$

Put $xy = 198000$ from equation (1) in above equation

$\Rightarrow$ $100000y - 198000 + 100000 - x - 308000 = 0$

$\Rightarrow$ $100000y - x - 406000 = 0$

$\Rightarrow$ $x = 100000y - 406000$ ..... (2)

Put this value in equation (1)

$\Rightarrow \quad (100000y - 406000)\, y = 198000$

$\Rightarrow \quad 100000y^2 - 406000y - 198000 = 0$

$\Rightarrow \quad 2000\,(50y^2 - 203y - 99) = 0$

$\Rightarrow \quad 50y^2 - 203y - 99 = 0$

$\Rightarrow \quad y = \dfrac{-(-203) \pm \sqrt{(-203)^2 - 4\,(50)\,(-99)}}{2\,(50)}$

$\Rightarrow \quad y = \dfrac{203 \pm \sqrt{41209 + 19800}}{100} = 203 \pm \sqrt{61009} = \dfrac{203 \pm 247}{100}$

$\Rightarrow \quad y = \dfrac{203 + 247}{100}$ or $y = \dfrac{203 - 247}{100}$

$\Rightarrow \quad y = \dfrac{450}{100} = 4.5$ $\qquad y = -0.44$ (neglecting)

Put $y = 4.5$ in equation (2)

$x = 100000\,(4.5) - 406000 = 44000$

$\Rightarrow$ Amount invested in 1st company = Rs. 44000

and Amount invested in 2nd company = 100000 – 44000 = Rs. 56000

CHAPTER 4

QUADRATIC EQUATIONS

MULTIPLE CHOICE QUESTIONS

(1) If $x^3 + 3x^2 - 6x + 2$ is divided by $x + 2$ then remainder is:

(a) -18 (b) 9

(c) -9 (d) 18

[Lahore Board 2005]

(2) Product of roots of equation $x^2 - kx + k^2 = 0$ is equals:

(a) k (b) k^2

(c) $-k$ (d) $-k^2$

[Lahore Board 2005]

(3) Two square roots of unity are:

(a) $1, -1$ (b) $1, \omega$

(c) $1, -\omega$ (d) $-i, i^2$

[Lahore Board 2005]

(4) Roots of a quadratic equation are imaginary if:

(a) $b^2 - 4ac < 0$ (b) $b^2 - 4ac > 0$

(c) $b^2 - 4ac = 0$ (d) $b^2 - 4ac = 1$

[Lahore Board 2005]

(5) A quadratic equation has degree:

(a) 1 (b) 2

(c) 3 (d) 4

[Gujranwala Board 2005]

(6) If S and P are the sum and product of roots of the quadratic equation, then quadratic equation is:

(a) $x^2 + Sx + P = 0$ (b) $x^2 - Sx + P = 0$

(c) $x^2 - Sx - P = 0$ (d) $x^2 + Sx - P = 0$

[Gujranwala Board 2005]

(7) The discriminant of quadratic equation $ax^2 + bx + c = 0$ is:

(a) $4ac - b^2$ (b) $b^2 + 4ac$

(c) $a^2 - 4ac$ (d) $b^2 - 4ac$

[Gujranwala Board 2006]

(8) The product of the roots of equation $5x^2 - x + 2 = 0$ is:

(a) $\frac{5}{2}$ (b) $-\frac{5}{2}$

(c) $\frac{2}{5}$ (d) 2

[Gujranwala Board 2006]

(9) $3^x + 4 = 0$ is:

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) None of these

[Gujranwala Board 2006]

(10) If $f(x) = x^3 + 4x^2 - 2x + 5$ is divided by $x - 1$ then remainder is:

(a) 8 (b) 10

(c) -8 (d) None of these

[Lahore Board 2006-07]

(11) The roots of the equation $ax^2 + bx + c = 0$ are rational if:

(a) $b^2 - 4ac < 0$ (b) $b^2 - 4ac > 0$

(c) $b^2 - 4ac = 0$ (d) $b^2 - 4ac > 0$ and a perfect square

[Lahore Board 2006]

(12) $\omega^{-5} = ?$

(a) ω (b) ω^2

(c) ω^3 (d) 0

[Lahore Board 2006]

(13) If a polynomial $f(x)$ of degree $n \geq 1$ is divided by $(x - a)$, then remainder R is:

(a) $f(-a)$ (b) $f(a)$

(c) $af(a)$ (d) $f(0)$

[Gujranwala Board 2007]

(14) One root of the equation $x^2 - 3x + a = 0$ is 2, then 'a' is:

(a) -2 (b) 2

(c) 3 (d) -3

[Gujranwala Board 2007]

(15) The solution of the equation $ax^2 + bx + c = 0$ is:

(a) $x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$ (b) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$ (d) $x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$

[Gujranwala Board 2007]

(16) The roots of the equation $25x^2 - 30x + 9 = 0$ are:

(a) Imaginary (b) Rational and unequal

(c) Rational and equal (d) Irrational and equal

[Gujranwala Board 2007]

(17) For what value of k will equation $x^2 - kx + 4 = 0$ have sum of roots equal to product of roots:

(a) 3 (b) -2

(c) -4 (d) 4

[Lahore Board 2007]

(18) For equal roots of $ax^2 + bx + c = 0$, $(b^2 - 4ac)$ will be:

(a) Negative (b) 0

(c) 1 (d) 2

[Lahore Board 2007]

(19) Sum of four fourth roots of unity is:

(a) 1 (b) -1

(c) 0 (d) i

[Lahore Board 2014, Lahore Board 2007, Lahore Board 2015]

(20) Equations having same solutions are called:

(a) Exponential equations (b) Radical equations

(c) Simultaneous equations (d) Reciprocal equations

[Lahore Board 2007]

(21) If $4^x = \frac{1}{2}$ then x equals:

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) 2 (d) −2

[Lahore Board 2008]

(22) If polynomial $x^2 + 3x + 7$ is divided by $x + 1$, then remainder is:

(a) −5 (b) 11

(c) 5 (d) −11

[Lahore Board 2008-10]

(23) If $4^x = 2$ then x equals:

(a) $-\frac{1}{2}$ (b) 2

(c) −2 (d) $\frac{1}{2}$

[Lahore Board 2008-09]

(24) If α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha\beta$ is equal to:

(a) $-\frac{a}{b}$ (b) $\frac{c}{a}$

(c) $\frac{b}{a}$ (d) $-\frac{b}{a}$

[Lahore Board 2008]

(25) The quadratic equation with roots $2 - \sqrt{3}$ and $2 + \sqrt{3}$ is:

(a) $x^2 + 4x + 1 = 0$ (b) $x^2 - 4x + 1 = 0$

(c) $x^2 - 6x + 6 = 0$ (d) $x^2 - 6x - 6 = 0$

[Gujranwala Board 2008]

(26) The sum and product of roots of equation $4x^2 + 7x - 3 = 0$ are:

(a) $\frac{3}{4}, -\frac{7}{4}$ (b) $-\frac{7}{4}, -\frac{3}{4}$

(c) $-\frac{7}{4}, \frac{3}{4}$ (d) $-\frac{3}{4}, -\frac{4}{3}$

[Gujranwala Board 2008]

(27) Roots of $x^2 - x - 2 = 0$ are:

(a) 2, −1 (b) −2, 1

(c) −2, −1 (d) 2, 1

[Gujranwala Board 2008]

(28) Product of all three cube roots of unity is:

(a) 1 (b) 0

(c) -1 (d) ω

[Lahore Board 2014, Gujranwala Board 2008]

(29) The product of all four fourth roots of unity equals:

(a) 1 (b) 0

(c) -1 (d) 2

[Lahore Board 2009]

(30) If α, β are the roots of $3x^2 - 2x + 4 = 0$, then the equation whose roots are $2\alpha, 2\beta$ is:

(a) $2x^2 + 6x + 8 = 0$ (b) $4x^2 - 2x + 3 = 0$

(c) $3x^2 - 4x + 16 = 0$ (d) $3x^2 + 16x - 4 = 0$

[Lahore Board 2009]

(31) The roots of the equation $x^2 + px + q = 0$, are additive inverse of one another, then:

(a) $p = 1$ (b) $q = 1$

(c) $q = 0$ (d) $p = 0$

[Lahore Board 2010]

(32) If 'ω' is the imaginary cube root of unity then $\omega^2 =$

(a) 1 (b) -1

(c) ω^{-1} (d) ω^{-2}

[Lahore Board 2010]

(33) $1 + \omega + \omega^2 =$

(a) 1 (b) ω

(c) ω^2 (d) 0

[Lahore Board 2014, Lahore Board 2011]

(34) When $3x^4 + 4x^3 + x - 5$ is divided by $x + 1$ then remainder is:

(a) 7 (b) 6

(c) -6 (d) -7

[Lahore Board 2012]

(35) In a quadratic equation $ax^2 + bx + c = 0$ if $b^2 - 4ac < 0$, then roots are:

(a) Real (b) Equal

(c) Rational (d) Complex / imaginary

[Lahore Board 2012]

(36) $1 + w^4 =$

(a) w (b) $-w$

(c) w^2 (d) $-w^2$

[Lahore Board 2013]

(37) Product of roots of $ax^2 + bx + c = 0$:

(a) $\frac{-C}{a}$ (b) $\frac{C}{a}$

(c) $\frac{-a}{C}$ (d) $\frac{-b}{a}$

[Lahore Board 2013]

(38) The polynomial $3x^2 + 2x + 1$ has degree:

(a) 0 (b) 3

(c) 2 (d) 4

[Lahore Board 2013]

(39) If $4^x = \frac{1}{2}$ then $x =$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$

(c) 2 (d) -2

[Lahore Board 2013]

(40) The equation $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$ is ———— equation:

(a) Reciprocal (b) Redical

(c) Exponential (d) All of these

(41) The equation of the form $\sqrt{x + a} + \sqrt{x + b} = \sqrt{x + c}$ is called ———— equation:

(a) Reciprocal (b) Redical

(c) Exponential (d) All of these

(42) If ω and ω^2 are complex cube root of unity then $1 + \omega + \omega^2 =$ ————

(a) 0 (b) 1

(c) 2 (d) None of these

(43) $\left(\frac{-1 + \sqrt{3}\, i}{2}\right)^2 =$ ————

(a) 0 (b) 1

(c) $\frac{-1 - \sqrt{3}\, i}{2}$ (d) $\frac{1 + 3i}{4}$

(44) Complex fourth roots of unity are ———— of each other:

(a) Inverse (b) Conjugate

(c) Imaginary (d) None of these

(45) If 'x' is at tens place and 'y' is at unit place then the two digit number xy = ————

(a) $10x + 10y$ (b) $10y + x$

(c) $y + 10x$ (d) $10x + y$

(46) Solution set of system of equations $x^2 + y^2 = 5$ and $xy = 2$ is equals:

(a) $\{(1, -2), (-1, 2), (2, 1), (-2, -1)\}$

(b) $\{(1, 2), (-1, -2), (2, 1), (-2, -1)\}$

(c) $\{(1, 2), (-1, -2), (2, -1), (2, 1)\}$

(d) None of these

(47) If a number 'x' is greater than 'y' by 5 then it can be written as:

(a) $x + 5 = y$ (b) $y - 5 = x$

(c) $-x - 5 = y$ (d) $x - 5 = y$

(48) If $x^n + a^n$ is divided by $x + a$, then remainder will be:

(a) 0 (b) a^n

(c) $2a^n$ (d) (a) or (c)

(49) $\sqrt{-\frac{1}{2} - \frac{\sqrt{-3}}{2}} = ?$

(a) $\frac{-1 + \sqrt{-3}}{2}$ (b) $\sqrt{\frac{-1 - \sqrt{-3}}{2}}$

(c) $\sqrt{\frac{1 - \sqrt{-3}}{2}}$ (d) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$

(50) ω^{2-6n} where $n \in w$, is equals:

(a) 0 (b) 1

(c) ω (d) ω^2

(51) Product of squares of four fourth roots of unity is equals:

(a) 0 (b) 1

(c) -1 (d) i

(52) $\frac{\text{Dividend}}{\text{Divisor}} - \text{Quotient} =$ ————

(a) $\frac{\text{Remainder}}{\text{dividend}}$ (b) $\frac{\text{Dividend}}{\text{divisor}}$

(c) $\frac{\text{Remainder}}{\text{divisor}}$ (d) None of these

(53) Nature of roots of quadratic equation depends on ————

(a) Product of roots (b) Sum of roots

(c) Discriminant (d) All of these

(54) The real fourth roots of unity are ———— of each other:

(a) Multiplicative inverse (b) Additive inverse

(c) Conjugate (d) None of these

(55) If ω is a complex cube root of unity then $\omega^3 = ?$

(a) 0 (b) -1

(c) -2 (d) None of these

(56) $ax^2 + bx + c = 0$ where $a \neq 0$, is a ———— equation:

(a) Linear (b) Quadratic

(c) Cubic (d) None of these

(57) The roots of the quadratic equation $x^2 - 4x + 3 = 0$ are:

(a) Rational (b) Irrational

(c) Unknown (d) Cannot find

(58) If α, β are roots of a quadratic equation then the equation can be written as:

(a) $x^2 + (\alpha + \beta)x + \alpha\beta = 0$ (b) $x^2 - (\alpha + \beta)x - \alpha\beta = 0$

(c) $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ (d) None of these

(59) For the equation $2x^2 + 3x + 7 = 0$ sum and product of roots is:

(a) $\frac{3}{2}, \frac{7}{2}$ (b) $-\frac{3}{2}, \frac{7}{2}$

(c) $\frac{3}{2}, -\frac{7}{2}$ (d) $-\frac{3}{2}, -\frac{7}{2}$

(60) The roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary and unequal if:

(a) Discriminant = 0 (b) Discriminant < 0

(c) Discriminant > 0 (d) None of these

(61) The roots of the quadratic equation $ax^2 + bx + c = 0$ are distinct if:

(a) $b^2 - 4ac = 0$ (b) $b^2 - 4ac \neq 0$

(c) Both (a) and (b) (d) None of these

(62) The three cube roots of unity are:

(a) $1, -1, i$ (b) $i, -i, 0$

(c) $1, \omega, 0$ (d) $1, \omega^2, \omega$

(63) Sum of four fourth roots of unity is:

(a) 0 (b) 1

(c) -1 (d) 2

(64) A radical free equation has some roots which do not satisfy the given radical equation, these roots are called:

(a) Extraneous roots (b) Exact roots

(c) Equal roots (d) None of these

(65) Solution set of equation $2x^4 - 32 = 0$ is:

(a) $\{+2, 2i\}$ (b) $\{+i, +1\}$

(c) $\{+2i, +2\}$ (d) $\{2, +2i\}$

(66) If a polynomial f(x) is divided by $2x + 1$ then remainder is equals:

(a) $f(2)$ (b) $f(a)$

(c) $f\left(\frac{1}{2}\right)$ (d) $f\left(-\frac{1}{2}\right)$

(67) If $\omega^2 = \frac{-1 + \sqrt{3}\, i}{2}$ then $\omega =$ _____

(a) $-\frac{1}{2} + \frac{\sqrt{3}\, i}{2}$ (b) $-\frac{1}{2} - \frac{\sqrt{-3}}{2}$

(c) $\frac{1}{2} - \frac{\sqrt{3}\, i}{2}$ (d) None of these

(68) If α, β are roots of $ax^2 + bx + c = 0$ then $\alpha^2 + \beta^2 =$ _____

(a) $\frac{b^2 + 2ac}{a^2}$ (b) $\frac{b^2 - 2ac}{a^2}$

(c) $\frac{b^2 - 2ac}{b^2}$ (d) $\frac{b^2 - 2ac}{a^2}$

(69) A polynomial f(x) has factor $x - a$ if:

(a) $f(a) \neq 0$ (b) $f(a) = 0$

(c) $f(-a) = 0$ (d) None of these

(70) If 'n' is even positive integer then $\frac{x^{n-1} + a^{n-1}}{x + a}$ has remainder:

(a) a^{n-1} (b) 0

(c) 1 (d) a^{2n-1}

(71) If $f(x - a)$ is divided by $x - a$, then remainder is:

(a) $f(a)$ (b) $f(-a)$

(c) $-f(a)$ (d) $f(0)$

(72) If $b^2 - 4ac > 0$ and perfect square then roots are:

(a) Rational (b) Irrational

(c) Equal (d) Complex

(73) The roots of $x^2 + x - 6 = 0$ are:

(a) Real (b) Equal

(c) Complex (d) Tribal

(74) The sum of roots of equation $ax^2 + bx + c = 0$, $a \neq 0$ is:

(a) $\frac{b}{a}$ (b) $\frac{c}{a}$

(c) $\frac{-b}{a}$ (d) $\frac{-c}{a}$

(75) If w is complex cube root of unity than conjugate of w is:

(a) w^2 (b) $-w^2$

(c) $-w$ (d) 1

[Lahore Board 2015]

Answers

(1)	d	*(2)*	b	*(3)*	a	*(4)*	a	*(5)*	b
(6)	b	*(7)*	d	*(8)*	c	*(9)*	a	*(10)*	a
(11)	d	*(12)*	a	*(13)*	b	*(14)*	b	*(15)*	b
(16)	c	*(17)*	d	*(18)*	b	*(19)*	c	*(20)*	d
(21)	b	*(22)*	c	*(23)*	d	*(24)*	b	*(25)*	b
(26)	b	*(27)*	a	*(28)*	a	*(29)*	c	*(30)*	c
(31)	d	*(32)*	c	*(33)*	d	*(34)*	d	*(35)*	d
(36)	d	*(37)*	b	*(38)*	c	*(39)*	b	*(40)*	a
(41)	b	*(42)*	a	*(43)*	c	*(44)*	b	*(45)*	d
(46)	b	*(47)*	d	*(48)*	b	*(49)*	a	*(50)*	d
(51)	b	*(52)*	c	*(53)*	c	*(54)*	b	*(55)*	d
(56)	b	*(57)*	a	*(58)*	c	*(59)*	b	*(60)*	b
(61)	b	*(62)*	d	*(63)*	a	*(64)*	a	*(65)*	c
(66)	d	*(67)*	b	*(68)*	d	*(69)*	b	*(70)*	b
(71)	d	*(72)*	a	*(73)*	a	*(74)*	c	*(75)*	a

SOLUTION

Q.1 $x + 2 = 0 \Rightarrow x = -2$

Put in given polynomial.

$f(x) = x^3 + 3x^2 - 6x + 2$

$f(-2) = (-2)^3 + 3(-2)^2 - 6(-2) + 2$

$= -8 + 12 + 12 + 2$

$f(-2) = 18$

$\Rightarrow$ Remainder = 18

Q.2 Given $x^2 - kx + k^2 = 0$

Let, α, β are its roots then

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} = k^2$$

Q.5 A quadratic equation is of the form $ax^2 + bx + c = 0$ in which highest power of variable 'x' is 2.

Q.8 Let α, β are the roots of $5x^2 - x + 2 = 0$.

Then,

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} = \frac{2}{5}$$

Q.9 As variable is in exponent (power)

So it is an exponential equation.

Q.10 $f(x) = x^3 + 4x^2 - 2x + 5$, $x - 1 = 0$

$f(1) = (1)^3 + 4(1)^2 - 2(1) + 5$ $x = 1$

$R = 1 + 4 - 2 + 5$

$R = 8$

Q.12 $\omega^{-5} = \frac{1}{\omega^5} = \frac{1}{\omega^3.\omega^2} = \frac{1}{1.\omega^2} = \frac{1}{\omega^2} \times \frac{\omega}{\omega} = \frac{\omega}{\omega^3} = \omega$

Q.14 $x^2 - 3x + a = 0$

Let, α, β are its roots.

$\Rightarrow$ $\alpha + \beta = 3$

Let, $\boxed{\alpha = 2}$ (given)

$2 + \beta = 3$

$\boxed{\beta = 1}$ ..... (1)

Also $\alpha\beta = a$

$2(1) = a \Rightarrow \boxed{a = 2}$

Q.16 $25x^2 - 30x + 9 = 0$

Discriminant $D = b^2 - 4ac = (-30)^2 - 4(25)(9)$

$= 900 - 900 = 0$

As $D = 0$ and perfect square so roots are rational and equal.

Q.17 $x^2 - kx + 4 = 0$

Let, α, β are its roots.

Then, $\alpha + \beta = \dfrac{-b}{a} = k$

And $\alpha\beta = 4$

As sum of roots = product of roots

$\boxed{k = 4}$

Q.19 $1 + (-1) + i + (-i) = 1 - 1 + i - i = 0$

Q.21 $4^x = \dfrac{1}{2}$

$(2^2)^x = 2^{-1}$

$2^{2x} = 2^{-1} \Rightarrow 2x = -1 \Rightarrow \boxed{x = -\dfrac{1}{2}}$

Q.22 Given polynomial is:

$f(x) = x^2 + 3x + 7$ $\qquad x + 1 = 0$

$f(-1) = (-1)^2 + 3(-1) + 7$ $\qquad x = -1$

$= 1 - 3 + 7 = 5$

$\boxed{R = 5}$

Q.23 $4^x = 2$

$(2^2)^x = 2$

$2^{2x} = 2^1$

$2x = 1 \Rightarrow x = \dfrac{1}{2}$

Q.25 Let, $\alpha = 2 - \sqrt{3}$, $\beta = 2 + \sqrt{3}$

$S = \alpha + \beta = 2 - \sqrt{3} + 2 + \sqrt{3} = 4$

$P = \alpha\beta = (2 - \sqrt{3})(2 + \sqrt{3})$

$= (2)^2 - (\sqrt{3})^2$

$= 4 - 3 = 1$

Equ will be $x^2 - Sx + P = 0$

$\Rightarrow x^2 - 4x + 1 = 0$

Q.26 $4x^2 + 7x - 3 = 0$

As, Sum of roots $= \frac{-b}{a} = \frac{-7}{4}$

Product of roots $= \frac{c}{a} = \frac{-3}{4}$

Q.27 $x^2 - x - 2 = 0$

$$x = \frac{-(b) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$x = \frac{1+3}{2}$ or $x = \frac{1-3}{2}$

$= \frac{4}{2}$ $= \frac{-2}{2}$

$x = 2$ $x = -1$

Q.28 Three cube roots of unity are $1, \omega, \omega^2$.

$\Rightarrow \quad 1.\omega.\omega^2 = w^3 = 1$

Q.29 Four fourth roots of unity are $1, -1, i, -i$

$(1)(-1)(i)(-i)$

$= (-1)(-i^2) \quad \therefore \ i^2 = -1$

$= (-1)(+1) = -1$

Q.30 As α, β are roots of $3x^2 - 2x + 4 = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} = \frac{2}{3}$

$\alpha\beta = \frac{c}{a} = \frac{4}{3}$

Now, $S = 2\alpha + 2\beta$ (Sum of roots of required equation)

$= 2(\alpha + \beta)$

$= 2\left(\frac{2}{3}\right)$ from (1)

$= \frac{4}{3}$

$$P = (2\alpha)\,.\,(2\beta) \qquad \text{(Product of roots of required equation)}$$

$$= 4\alpha\beta = 4\left(\frac{4}{3}\right) \text{ from (2)}$$

$$= \frac{16}{3}$$

Required equation will be:

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{4}{3}x + \frac{16}{3} = 0$$

$$3x^2 - 4x + 16 = 0$$

Q.36 Given equation $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$ replace x by $\frac{1}{x}$.

$$\left(\frac{1}{x}\right)^4 - 3\left(\frac{1}{x}\right)^3 + 4\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 1 = 0$$

$$\frac{1}{x^4} - 3\,.\,\frac{1}{x^3} + 4\,.\,\frac{1}{x^2} - 3\,.\,\frac{1}{x} + 1 = 0$$

Multiplying by x^4.

$$1 - 3x + 4x^2 - 3x^3 + x^4 = 0$$

or $$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$$

As equation remains unchanged if we replace x by $\frac{1}{x}$ so it is reciprocal equation.

Q.37 As given equation has radical expressions of the variable so it is radical equation.

Q.38 As, $$\omega = \frac{-1 + \sqrt{-3}}{2}, \quad \omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

$$1 + \omega + \omega^2 = 1 + \frac{-1 + \sqrt{-3}}{2} + \frac{-1 - \sqrt{-3}}{2}$$

$$= \frac{2 - 1 + \sqrt{-3} - 1 - \sqrt{-3}}{2} = \frac{2 - 2}{2} = \frac{0}{2} = 0$$

Q.39 $$\left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{(-1 + \sqrt{3}i)^2}{(2)^2}$$

$$= \frac{(-1)^2 + (\sqrt{3}i)^2 + 2(-1)(\sqrt{3}i)}{4}$$

$$= \frac{1 + 3i^2 - 2\sqrt{3}i}{4} = \frac{1 - 3 - 2\sqrt{3}i}{4}$$

$$= \frac{-2 - 2\sqrt{3}\,i}{4} = \frac{2(-1 - \sqrt{3}\,i)}{4} = \frac{-1 - \sqrt{3}\,i}{2}$$

Q.40 As complex fourth roots of unity are $i, -i$ which are conjugate of each other.

Q.42 Given:

$$x^2 + y^2 = 5 \quad \ldots\ldots\ldots (1) \qquad xy = 2$$

$$x = \frac{2}{y} \quad \ldots\ldots\ldots (2)$$

Put (2) in (1).

$$\left(\frac{2}{y}\right)^2 + y^2 = 5$$

$$\frac{4}{y^2} + y^2 = 5$$

$$y^2 \cdot \frac{4}{y^2} + y^2 \cdot y^2 = 5.y^2$$

$$4 + y^4 = 5y^2$$

$$y^4 - 5y^2 + 4 = 0$$

$$y^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm \sqrt{9}}{2}$$

$$= \frac{5 \pm 3}{2}$$

$$y^2 = \frac{5+3}{2} \quad , \quad y^2 = \frac{5-3}{2}$$

$$y^2 = 4 \quad , \quad y^2 = 1$$

$$y = \pm 2 \quad , \quad y = \pm 1$$

When, $y = 2 \quad , \quad x = \frac{2}{2} = 1 \Rightarrow (1, 2)$

When, $y = -2 \quad , \quad x = -1 \Rightarrow (-1, -2)$

When, $y = 1 \Rightarrow x = \frac{2}{1} = 2 \Rightarrow (2, 1)$

When, $y = -1 \Rightarrow x = -2 \Rightarrow (-2, -1)$

$\Rightarrow$ Solution will be $\{(1, 2), (-1, -2), (2, 1), (-2, -1)\}$

Q.43 Given:

$$x > y \text{ by } 5$$

$$\Rightarrow \quad x - 5 = y$$

Q.44 Given polynomial.

$$f(x) = x^n + a^n \quad , \quad x + a = 0$$

$$f(-a) = (-a)^n + a^n \qquad x = -a$$

or $\quad R = (-a)^n + a^n$

If n is even then.

$$R = a^n + a^n = 2a^n$$

If n is odd then

$$R = -a^n + a^n = 0$$

$\Rightarrow$ R is $2a^n$ or 0.

Q.45 $\sqrt{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = \sqrt{\frac{-1 - \sqrt{-3}}{2}} = \sqrt{\omega^2} = \omega$

And $\quad \omega = \frac{-1 + \sqrt{-3}}{2}$

Q.46 $w^{2-6n} = ?$ where $n \in \{0, 1, 2, 3, \ldots\ldots\}$

If, $\quad n = 0$

$$w^{2-6n} = \omega^2$$

If $\quad n = 1 \ , \ \omega^{2-6} = \omega^{-4} = \frac{1}{\omega^4} = \frac{1}{\omega^3.\omega} = \frac{1}{\omega} \cdot \frac{\omega^2}{\omega^2} = \frac{\omega^2}{\omega^3} = \omega^2$

If $\quad n = 2 \ , \ \omega^{2-12} = \omega^{-10} = \frac{1}{\omega^{10}} = \frac{1}{(\omega^3)^3.\omega} = \frac{1}{\omega} \cdot \frac{\omega^2}{\omega^2} = \frac{\omega^2}{\omega^3} = \omega^2$

$\Rightarrow \quad \omega^{2-6n} = \omega^2 \ \forall \ n \in \{0, 1, 2, 3, \ldots\ldots\}$

Q.47 As four fourth roots of unity are $1, -1, i, -i$.

Product of their squares $= (1)^2(-1)^2(i)^2(-i)^2 \qquad \because i^2 = -1$

$$= (1)(1)(-1)(-1) = 1$$

Q.48 As,

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

$$\frac{\text{Dividend}}{\text{Divisor}} - \text{quotient} = \frac{\text{Remainder}}{\text{Divisor}}$$

Q.50 As real fourth roots of unity are 1 and -1 which are additive inverse of each other.

Q.51 $\omega^3 = 1$

Q.52 As degree of $ax^2 + bx + c = 0$ is 2 so this is a quadratic equation.

Q.53 $x^2 - 4x + 3 = 0$

Discriminant $= D = b^2 - 4ac = (-4)^2 - 4(1)(3)$

$= 16 - 12 = 4$

$D = 4$ which is a perfect square

So roots will be rational.

Q.55 $2x^2 + 3x + 7 = 0$

Sum of roots $= \dfrac{-b}{a} = \dfrac{-3}{2}$

Product of roots $= \dfrac{c}{a} = \dfrac{7}{2}$

Q.59 Four fourth roots of unity are:

$1, -1, i, -i$ and their sum $1 - 1 + i - i = 0$

Q.61 $2x^4 - 32 = 0$

$\Rightarrow \quad 2(x^4 - 16) = 0$

$\Rightarrow \quad x^4 - 16 = 0$

$\Rightarrow \quad (x^2)^2 - (4)^2 = 0 \Rightarrow (x^2 + 4)(x^2 - 4) = 0$

$\Rightarrow \quad (x^2 + 4) = 0 \quad$ or $\quad x^2 - 4 = 0$

$\Rightarrow \quad x^2 = -4 \qquad\qquad x^2 = 4$

$\Rightarrow \quad x = \pm\sqrt{-4} \qquad\qquad x = \pm\sqrt{4}$

$\Rightarrow \quad x = \pm 2i \qquad\qquad x = \pm 2$

$\Rightarrow$ S.S $= \{\pm 2i, \pm 2\}$

Q.63 If a polynomial f(x) is divided by $x - a$ then remainder $R = f(a)$ so if a polynomial f(x) is divided by $2x + 1$ then remainder $R = f\left(-\dfrac{1}{2}\right)$.

Q.64 If α, β are roots of $ax^2 + bx + c = 0$.

Then, $\alpha + \beta = \dfrac{-b}{a} \qquad \ldots\ldots\ldots (1)$

$\alpha\beta = \dfrac{c}{a} \qquad \ldots\ldots\ldots (2)$

Then,

$$\begin{aligned}\alpha^2 + \beta^2 &= \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) \\ &= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}\end{aligned}$$

Q.65 A polynomial f(x) has factor x – a if remainder = R = 0.

or $f(a) = 0$ $\quad \therefore \ R = f(a)$

Q.66 If $f(x) = \dfrac{x^{n+1} + a^{n+1}}{x + a}$ is divided by x + a then.

Remainder R = $f(-a)$

$= (-a)^{n+1} + a^{n+1}$ $\quad \therefore$ n is even

$= -a^{n+1} + a^{n+1}$ $\quad \Rightarrow n + 1$ is odd

$= 0$

Q.67 As if f(x) is divided by x – a then remainder = f(a)

Similarly,

If f(x – a) is divided by x – a

Then remainder = f(a – a)

= f(0)

Q.75 $x^2 + x - 6 = 0$, $a = 1$, $b = 1$, $c = -6$

Disc = $b^2 - 4ac$

= $1 - 4(-6) = 25 > 0$ roots are real/unequal