Chapter 3

MATRICES AND DETERMINANTS

Matrix

A rectangular array of numbers enclosed by a pair of brackets such as:

$$\begin{bmatrix} 2 & -2 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 5 \\ 2 & 7 & 9 \\ 4 & 1 & 2 \end{bmatrix}$$

(i) (ii)

is called a matrix. The horizontal lines of numbers are called rows and the vertical lines of numbers are called columns.

Order of a Matrix

If a matrix has m rows and n columns then its order is m x n.

In above, matrix (i) has order 2 x 3 and matrix (ii) has order 4×3.

Addition of Matrices

Two matrices A and B can be added if A and B have same order.

The sum of A and B, A + B can be obtained by adding their corresponding elements.

For example if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 3 \\ 1 & 2 \end{bmatrix}$$

then $$A + B = \begin{bmatrix} 2 & 1 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+7 & 1+3 \\ 3+1 & 9+2 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 4 & 11 \end{bmatrix}$$

Subtraction of Matrices

A matrix B can be subtracted from a matrix A if A and B have same order. Subtraction of B from A, A – B can be obtained by subtracting each element of matrix B from the corresponding element of matrix A.

If

$$A = \begin{bmatrix} 3 & 1 \\ 9 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 9 & 8 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 3 & 1 \\ 9 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 9 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 1-4 \\ 9-9 & 7-8 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$$

Scalar Multiplication

If A is a matrix of order m x n and k is a scalar, then the product k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, and this process is called scalar multiplication.

If $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ and $2 \in R$

then

$$2A = 2\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times 4 & 2 \times 2 \\ 2 \times 3 & 2 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 6 & 2 \end{bmatrix}$$

Multiplication of Two Matrices

Two matrices A and B are said to be conformable for the product AB if

The number of columns of A = The number of rows of B.

If A is a matrix of order m x n and B is a matrix of order n x p then we can find AB = C (say) and order of matrix C will be m x p.

For example

If

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(2)+(1)(1) & (2)(1)+(1)(2) & (2)(3)+(1)(4) \\ (3)(2)+(4)(1) & (3)(1)+(4)(2) & (3)(3)+(4)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 2+2 & 6+4 \\ 6+4 & 3+8 & 9+16 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 10 \\ 10 & 11 & 25 \end{bmatrix}$$

Transpose of A Matrix

If A is a matrix of order m x n then an n x m matrix obtained by interchanging the rows and columns of A, is called the transpose of A and it is denoted by A^t.

For example if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Determinant of 2 x 2 Matrix

We can associate a unique number with every square matrix A over R or C, this number is known as the determinant of A.

For example if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Singular and Non–singular Matrices

A matrix A is singular if $|A| = 0$

A matrix A is non–singular if $|A| \neq 0$

Adjoint of 2 x 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adj A and is defined as

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of 2 x 2 Matrix

Let A be a non–singular square matrix of order 2. If there exists a matrix B such that $AB = BA\ I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then B is called the multiplicative inverse of A and is usually denoted by A^{-1}. i.e. $B = A^{-1}$

$$\Rightarrow \quad A A^{-1} = A^{-1} A = I_2$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

EXERCISE 3.1

Q.1 **If** $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$ **then show that**

(i) $4A - 3A = A$ **(ii)** $3B - 3A = 3(B - A)$

Solution:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$$

(i) **To show**

$4A - 3A = A$

L.H.S.

$$4A - 3A = 4\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (4)(2) & (4)(3) \\ (4)(1) & (4)(5) \end{bmatrix} - \begin{bmatrix} (3)(2) & (3)(3) \\ (3)(1) & (3)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6 & 12-9 \\ 4-3 & 20-15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = A = \text{R.H.S.}$$

(ii) **To show** $3B - 3A = 3(B - A)$

L.H.S.

$$3B - 3A = 3\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(1) & (3)(7) \\ (3)(6) & (3)(4) \end{bmatrix} - \begin{bmatrix} (3)(2) & (3)(3) \\ (3)(1) & (3)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 21 \\ 18 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3-6 & 21-9 \\ 18-3 & 12-15 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 12 \\ 15 & -3 \end{bmatrix}$$

R.H.S.

$$3(B - A) = 3\left(\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} 1-2 & 7-3 \\ 6-1 & 4-5 \end{bmatrix}$$

$$= 3\begin{bmatrix} -1 & 4 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} (3)(-1) & (3)(4) \\ (3)(5) & (3)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 12 \\ 15 & -3 \end{bmatrix} = \text{R.H.S.}$$

$\Rightarrow$ $3B - 3A = 3(B - A)$

Q.2 **If** $\mathbf{A} = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$, **show that** $\mathbf{A^4 = I_2}$.

Solution:

To show that $A^4 = I^2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Take

$$A^2 = A.A. = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$= \begin{bmatrix} (i)(i)+(0)(1) & (i)(0)+(0)(-i) \\ (1)(i)+(-i)(1) & (1)(0)+(-i)(-i) \end{bmatrix}$$

$$= \begin{bmatrix} i^2+0 & 0-0 \\ i-i & 0+i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \because \quad i^2 = -1$$

Now

$$A^4 = A^2 . A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-1)+(0)(0) & (-1)(0)+(0)(-1) \\ (0)(-1)+(-1)(0) & (0)(0)+(-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Hence proved.

Equal Matrices

Two matrices of the same order said to be equal if their corresponding elements are equal.

Q.3 **Find x and y if**

(i) $\begin{bmatrix} \mathbf{x+3} & \mathbf{1} \\ \mathbf{-3} & \mathbf{3y-4} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix}$

(ii) $\begin{bmatrix} \mathbf{x+3} & \mathbf{1} \\ \mathbf{-3} & \mathbf{3y-4} \end{bmatrix} = \begin{bmatrix} \mathbf{y} & \mathbf{1} \\ \mathbf{-3} & \mathbf{2x} \end{bmatrix}$

Solution:

(i) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

As the two matrices are equal so their corresponding elements are equal

so

$x + 3 = 2$ and $3y - 4 = 2$

$\Rightarrow$ $x = 2 - 3$ and $3y = 4 + 2$

$\Rightarrow$ $x = -1$ and $3y = 6$

$y = 2$

(ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

As the two matrices are equal so their corresponding elements are equal

so

$\Rightarrow$ $x + 3 = y$ (i) and $3y - 4 = 2x$ (ii)

Put $y = x + 3$ from equation (i) in equation (ii)

$3(x + 3)\ 4 = 2x$

$\Rightarrow$ $3x + 9 - 4 = 2x$

$\Rightarrow$ $3x + 5 = 2x$

$\Rightarrow$ $3x + 5 - 2x = 0$

$\Rightarrow$ $x + 5 = 0$

$\Rightarrow$ $\boxed{x = -5}$

Put this value in equation (i)

$y = -5 + 3 = -2$

$\Rightarrow$ $\boxed{y = -2}$

Q.4 **If** $\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

find the following matrices.

(i) 4A – 3B (ii) A + 3 (B – A)

Solution:

(i) to find $4A - 3B$

$$4A - 3B = 4\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 9 & 6 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-0 & 8-9 & 12-6 \\ 4-3 & 0-(-3) & 8-6 \end{bmatrix} = \begin{bmatrix} -4 & -1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

(ii) $A + 3(B - A)$

$$A + 3(B - A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\left(\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 0-(-1) & 3-2 & 2-3 \\ 1-1 & -1-0 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -3 \\ 0 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3 & 2+3 & 3+(-3) \\ 1+0 & 0+(-3) & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 0 \\ 1 & 3 & 2 \end{bmatrix}$$

Q.5 Find x and y if

$$\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{x} \\ \mathbf{1} & \mathbf{y} & \mathbf{3} \end{bmatrix} + \mathbf{2}\begin{bmatrix} \mathbf{1} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{-2} & \mathbf{3} \\ \mathbf{1} & \mathbf{6} & \mathbf{1} \end{bmatrix}$$

Solution:

Given

$$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2x & 2y \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2+2 & 0+2x & x+2y \\ 1+0 & y+4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2x & x+2y \\ 1 & y+4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

As the two matrices are equal so their corresponding elements are equal.

$\Rightarrow$ $2x = -2$, $y + 4 = 6$

$\Rightarrow$ $x = -1$, $y = 6 - 4 = 2$

$\Rightarrow$ $\boxed{x = -1}$ and $\boxed{y = 2}$

Q.6 If $A = [a_{ij}]_{3x3}$ show that

(i) $\lambda(\mu A) = (\lambda\mu) A$ (ii) $(\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda A - A = (\lambda - 1) A$

Solution:

(i) To show $\lambda(\mu A) = (\lambda\mu) A$

Where, $A = [a_{ij}]_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Take L.H.S.

$$\lambda(\mu A) = \lambda\left(\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$= \lambda\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix}$$

Now take R.H.S.

$$(\lambda\mu) A = (\lambda\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix} = \text{L.H.S.}$$

$\Rightarrow$ $\lambda(\mu A) = (\lambda\mu) A$

(ii) $(\lambda + \mu) A = \lambda A + \mu A$

To show $(\lambda + \mu) A = \lambda A + \mu A$

L.H.S.

$$(\lambda + \mu) A = (\lambda + \mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda + \mu) a_{11} & (\lambda + \mu) a_{12} & (\lambda + \mu) a_{13} \\ (\lambda + \mu) a_{21} & (\lambda + \mu) a_{22} & (\lambda + \mu) a_{23} \\ (\lambda + \mu) a_{31} & (\lambda + \mu) a_{32} & (\lambda + \mu) a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} + \mu a_{11} & \lambda a_{12} + \mu a_{12} & \lambda a_{13} + \mu a_{13} \\ \lambda a_{21} + \mu a_{21} & \lambda a_{22} + \mu a_{22} & \lambda a_{23} + \mu a_{23} \\ \lambda a_{31} + \mu a_{31} & \lambda a_{32} + \mu a_{32} & \lambda a_{33} + \mu a_{33} \end{bmatrix}$$

Now take R.H.S.

$$\lambda A + \mu A = \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \mu \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} + \begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} + \mu a_{11} & \lambda a_{12} + \mu a_{12} & \lambda a_{13} + \mu a_{13} \\ \lambda a_{21} + \mu a_{21} & \lambda a_{22} + \mu a_{22} & \lambda a_{23} + \mu a_{23} \\ \lambda a_{31} + \mu a_{31} & \lambda a_{32} + \mu a_{32} & \lambda a_{33} + \mu a_{33} \end{bmatrix}$$

$$= \text{L.H.S.}$$

$\Rightarrow \quad (\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda A - A = (\lambda - 1) A$

To show $\lambda A - A = (\lambda - 1) A$

$$\text{L.H.S.} = \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} - a_{11} & \lambda a_{12} - a_{12} & \lambda a_{13} - a_{13} \\ \lambda a_{21} - a_{21} & \lambda a_{22} - a_{22} & \lambda a_{23} - a_{23} \\ \lambda a_{31} - a_{31} & \lambda a_{32} - a_{32} & \lambda a_{33} - a_{33} \end{bmatrix}$$

Now take R.H.S.

$$(\lambda - 1) A = (\lambda - 1) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda - 1) a_{11} & (\lambda - 1) a_{12} & (\lambda - 1) a_{13} \\ (\lambda - 1) a_{21} & (\lambda - 1) a_{22} & (\lambda - 1) a_{23} \\ (\lambda - 1) a_{31} & (\lambda - 1) a_{32} & (\lambda - 1) a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} - a_{11} & \lambda a_{12} - a_{12} & \lambda a_{13} - a_{13} \\ \lambda a_{21} - a_{21} & \lambda a_{22} - a_{22} & \lambda a_{23} - a_{23} \\ \lambda a_{31} - a_{31} & \lambda a_{32} - a_{32} & \lambda a_{33} - a_{33} \end{bmatrix}$$

$$= \text{L.H.S.}$$

$\Rightarrow \quad \lambda A - A = (\lambda - 1) A.$

Q.7 If $A = [a_{ij}]_{2x3}$ and $B = [b_{ij}]_{2x3}$ show that $\lambda (A + B) = \lambda A + \lambda B$.

Solution:

Given

$$A = [a_{ij}]_{2x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = [b_{ij}]_{2x3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

To show that

$$\lambda (A + B) = \lambda A + \lambda B$$

Take L.H.S.

$$\lambda (A + B) = \lambda \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \right)$$

$$= \lambda \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda (a_{11} + b_{11}) & \lambda (a_{12} + b_{12}) & \lambda (a_{13} + b_{13}) \\ \lambda (a_{21} + b_{21}) & \lambda (a_{22} + b_{22}) & \lambda (a_{23} + b_{23}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} + \lambda b_{11} & \lambda a_{12} + \lambda b_{12} & \lambda a_{13} + \lambda b_{13} \\ \lambda a_{21} + \lambda b_{21} & \lambda a_{22} + \lambda b_{22} & \lambda a_{23} + \lambda b_{23} \end{bmatrix}$$

Now R.H.S.

$$\lambda A + \lambda B = \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \lambda \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} & \lambda b_{13} \\ \lambda b_{21} & \lambda b_{22} & \lambda b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} + \lambda b_{11} & \lambda a_{12} + \lambda b_{12} & \lambda a_{13} + \lambda b_{13} \\ \lambda a_{21} + \lambda b_{21} & \lambda a_{22} + \lambda b_{22} & \lambda a_{23} + \lambda b_{23} \end{bmatrix}$$

$$= \text{L.H.S.}$$

$$\Rightarrow \quad \lambda (A + B) = \lambda A + \lambda B$$

Q.8 **If** $A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ **and** $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **find the values of b.**

Solution:

Given

$$A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (2)(a) & (1)(2) + (2)(b) \\ (a)(1) + (b)(a) & (2)(2) + (b)(b) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 + 2a & 2 + 2b \\ a + ab & 4 + b^2 \end{bmatrix} \quad \text{but} \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 + 2a & 2 + 2b \\ a + ab & 2a + b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As the two matrices are equal so their corresponding elements are equal.

$$\Rightarrow \quad 1 + 2a = 0 \qquad 2 + 2b = 0$$

$$\Rightarrow \quad a = -\frac{1}{2} \qquad b = -1$$

$$\Rightarrow \quad \text{required values are } \boxed{a = -\frac{1}{2}} \quad , \quad \boxed{b = -1}$$

Q.9 **If** $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ **and** $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**, find values of a and b.**

Solution:

Given $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}\begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (-1)(a) & (1)(-1) + (-1)(b) \\ (a)(1) + (b)(a) & (a)(-1) + (b)(b) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 - a & -1 - b \\ a + ab & -a + b^2 \end{bmatrix}$$

But it is given that

$$\Rightarrow \quad \begin{bmatrix} 1 - a & -1 - b \\ a + ab & -a + b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the two matrices are equal so their corresponding elements are equal.

$$\Rightarrow \quad 1 - a = 1 \quad \text{and} \quad -1 - b = 0$$

$$\Rightarrow \quad a = 0 \qquad b = -1$$

$$\Rightarrow \quad \text{required values are } \boxed{a = 0} \quad \text{and} \quad \boxed{b = -1}$$

Q.10 **If** $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

then show that $(A + B)^t = A^t + B^t$.

Solution:

To show $(A + B)^t = A^t + B^t$

Take

$$(A + B) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -1+3 & 2+0 \\ 0+1 & 3+2 & 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}$$

$$(A + B)^t = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}^t = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix} \quad \ldots\ldots\ldots (1)$$

Now

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

and $$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

Now

$$A^t + B^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 0+1 \\ -1+3 & 3+2 \\ 2+0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix} \quad \ldots\ldots\ldots (2)$$

From (1) and (2), it is clear that

$$(A + B)^t = A^t + B^t$$

Q.11 Find A^3 if $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1)+(1)(5)+(3)(-2) & (1)(1)+(1)(2)+(3)(-1) & (1)(3)+(1)(6)+(3)(-3) \\ (5)(1)+(2)(5)(6)(-2) & (5)(1)+(2)(2)+(6)(-1) & (5)(3)+(2)(6)+(6)(-3) \\ (-2)(1)+(-1)(5)+(-3)(-2) & (-2)(1)+(-1)(2)+(-3)(-1) & (-2)(3)+(-1)(6)+(-3).(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Now

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (0)(1)+(0)(5)+(0)(-2) & (0)(1)+(0)(2)+(0)(-1) & (0)(3)+(0)(6)+(0)(-3) \\ (3)(1)+(3)(5)+(9)(-2) & (3)(1)+(3)(2)+(9)(-1) & (3)(3)+(3)(6)+(9)(-3) \\ -(1)(1)+(-1)(5)+(-3)(-2) & (-1)(1)+(-1)(2)+(-3)(-1) & (-1)(3)+(-1)(6)+(-3).(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0-0 & 0+0-0 & 0+0-0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q.12 Find matrix x if

(i) $\mathbf{X}\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

Solution:

(i) $X\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$

$X A = B$ (say)

$X A A^{-1} = B A^{-1}$

$XI = B A^{-1}$

$X = B A^{-1}$ ..... (1)

To find A^{-1}

Here

$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix} = (5)(1) - (-2)(2) = 5 + 4 = 9 \neq 0$

$adJ = A\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

As

$A^{-1} = \dfrac{adJ\ A}{|A|}$

$= \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

Put this value in equation (1)

$X = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} (-1)(1) + (5)(2) & (-1)(-2) + (5)(5) \\ (12)(1) + (3)(2) & (12)(-2) + (3)(5) \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} -1 + 10 & 2 + 25 \\ 12 + 6 & -24 + 15 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 27 \\ 18 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{9} & \frac{27}{9} \\ \frac{18}{9} & -\frac{9}{9} \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

which is required matrix.

(ii) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{10} \end{bmatrix}$

$$\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$$

$A X = B$ (say)

$A^{-1} A X = A^{-1} B$

$I X = A^{-1} B$

$X = A^{-1} B$ ..... (1)

To find A^{-1}

$$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix} = (5)(1) - (-2)(2) = 5 + 4 = 9 \neq 0$$

$$\text{adj}A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

As

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

Put this value in equation (1)

$$X = \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (1)(2) + (-2)(5) & (1)(1) + (-2)(10) \\ (2)(2) + (5)(5) & (2)(1) + (5)(10) \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 2 - 10 & 1 - 20 \\ 4 + 25 & 2 + 50 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -8 & -19 \\ 29 & 52 \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{8}{9} & -\frac{19}{9} \\ \frac{29}{9} & \frac{52}{9} \end{bmatrix}$$ is required matrix.

Q.13 Find the matrix A if

(i) $\begin{bmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{7} & \mathbf{2} \end{bmatrix}$

(ii) $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{-7} \end{bmatrix}$

Solution:

(i) $\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

Let

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the required matrix.

$\Rightarrow \quad \begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} (5)(a) + (-1)(c) & (5)(b) + (-1)(d) \\ (0)(a) + (0)(c) & (0)(b) + (0)(d) \\ (3)(a) + (1)(c) & (3)(b) + (1)(d) \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 5a - c & 5b - d \\ 0 & 0 \\ 3a + c & 3b + d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

As the matrices are equal so their corresponding elements are equal.

i.e.

$5a - c = 3 \qquad \ldots\ldots\ldots (1)$

$5b - d = -7 \qquad \ldots\ldots\ldots (2)$

$3a + c = 7 \qquad \ldots\ldots\ldots (3)$

$3b + d = 2 \qquad \ldots\ldots\ldots (4)$

Adding equation (1) and (3)

$5a - c = 3$

$3a + c = 7$

$8a = 10 \quad \Rightarrow \quad a = \frac{10}{8} \Rightarrow \boxed{\frac{5}{4}}$

Put $a = \frac{5}{4}$ in equation (1)

$$5\left(\frac{5}{4}\right) - c = 3$$

$$\frac{25}{4} - c = 3 \Rightarrow c = \frac{25}{4} - 3$$

$$c = \frac{25 - 12}{4} = \frac{13}{4}$$

$$\boxed{c = \frac{13}{4}}$$

Now add equation (2) and (4)

$$5b - d = -7$$

$$3b + d = 2$$

$$8b = -5 \Rightarrow \boxed{b = -\frac{5}{8}}$$

Put $b = -\frac{5}{8}$ in equation (2)

$$5\left(-\frac{5}{8}\right) - d = -7$$

$$-\frac{25}{8} - d = -7$$

$$-\frac{25}{8} + 7 = d$$

$$\frac{-25 + 56}{8} = d$$

$$\frac{31}{8} = d \Rightarrow \boxed{d = \frac{31}{8}}$$

$\Rightarrow$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{5}{8} \\ \frac{13}{4} & \frac{31}{8} \end{bmatrix}$ is the required matrix.

(ii) $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{-7} \end{bmatrix}$

$B\,A = C$ (say)

$B^{-1}\,B\,A = B^{-1}\,C$

$I\,A = B^{-1}\,C$

$A = B^{-1}\,C$ ..... (1)

To find B^{-1}

Take $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$|B| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-1)(-1) = 4 - 1 = 3 \neq 0$

$\text{adj } B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

As $B^{-1} = \dfrac{\text{adj } B}{|B|}$

$\Rightarrow$ $B^{-1} = \dfrac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Put this value in (1)

$$A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} (2)(0)+(1)(3) & (2)(-3)+(1)(3) & (2)(8)+(1)(-7) \\ (1)(0)+(2)(3) & (1)(-3)+(2)(3) & (1)(8)+(2)(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 0+3 & -6+3 & 16-7 \\ 0+6 & -3+6 & 8-14 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 9 \\ 6 & 3 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3} & -\frac{3}{3} & \frac{9}{3} \\ \frac{6}{3} & \frac{3}{3} & -\frac{6}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

Q.14 Show that

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = r\,I_3$$

Solution:

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\phi\,.\cos\phi + 0.0 + (-\sin\phi)(-r\sin\phi) & r\cos\phi\,.0 + 0.1 + (-\sin\phi) & r\cos\phi\,.\sin\phi + 0.0 + (-\sin\phi)(r\cos\phi) \\ 0\,.\cos\phi + r\,.0 + 0\,.(-r\sin\phi) & 0.0 + r.1 + 0.0 & 0\,.\sin\phi + r.0 + 0.r\cos\phi \\ r\sin\phi\,.\cos\phi + 0.0 + \cos\phi\,(-r\sin\phi) & r\sin\phi\,.0 + 0.1 + \cos\phi\,.0 & r\sin\phi\,.\sin\phi + 0.0 + \cos\phi\,.r\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\cos^2\phi + 0 + r\sin^2\phi & 0+0+0 & r\cos\phi\sin\phi + 0 - r\cos\phi\sin\phi \\ 0+0-0 & 0+r+0 & 0+0+0 \\ r\sin\phi\cos\phi + 0 - r\sin\phi\cos\phi & 0+0+0 & r\sin^2\phi + 0 + r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos^2\phi + \sin^2\phi) & 0 & -0 \\ 0 & r & 0 \\ 0 & 0 & r(\sin^2\phi + \cos^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$= r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = r\,I_3 = \text{R.H.S.}$$

Hence proved.

EXERCISE 3.2

Q.1 If $A = [a_{ij}]_{3x4}$ then show that
(i) $I_3 A = A$ (ii) $AI_4 = A$

Solution:

Given

$$A = [a_{ij}]_{3x4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

(i) To show $I_3 A = A$ where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Take L.H.S.

$$I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= = = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

$$= \begin{bmatrix} 1.a_{11} + 0.a_{21} + 0.a_{31} & 1.a_{12} + 0.a_{22} + 0.a_{32} & 1.a_{13} + 0.a_{23} + 0.a_{33} & 1.a_{14} + 0.a_{24} + 0.a_{34} \\ 0.a_{11} + 1.a_{21} + 0.a_{31} & 0.a_{12} + 1.a_{22} + 0.a_{32} & 0.a_{13} + 1.a_{23} + 0.a_{33} & 0.a_{14} + 1.a_{24} + 0.a_{34} \\ 0.a_{11} + 0.a_{21} + 1.a_{31} & 0.a_{12} + 0.a_{22} + 1.a_{32} & 0.a_{13} + 0.a_{23} + 1.a_{33} & 0.a_{14} + 0.a_{24} + 1.a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + 0 + 0 & a_{12} + 0 + 0 & a_{13} + 0 + 0 & a_{14} + 0 + 0 \\ 0 + a_{21} + 0 & 0 + a_{22} + 0 & 0 + a_{23} + 0 & 0 + a_{24} + 0 \\ 0 + 0 + a_{31} & 0 + 0 + a_{32} & 0 + 0 + a_{33} & 0 + 0 + a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A = \text{R.H.S.}$$

(ii) $\mathbf{AI_4 = A}$

To show that

$$AI_4 = A \quad \text{where} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

L.H.S.

$$A\,I_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + 0 + 0 + 0 & 0 + a_{12} + 0 + 0 & 0 + 0 + a_{13} + 0 & 0 + 0 + 0 + a_{14} \\ a_{21} + 0 + 0 + 0 & 0 + a_{22} + 0 + 0 & 0 + 0 + a_{23} + 0 & 0 + 0 + 0 + a_{24} \\ a_{31} + 0 + 0 + 0 & 0 + a_{32} + 0 + 0 & 0 + 0 + a_{33} + 0 & 0 + 0 + 0 + a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A \text{ R.H.S.}$$

Q.2 Find inverse of the following matrices

(i) $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}$

(iii) $\begin{bmatrix} \mathbf{2\,i} & \mathbf{i} \\ \mathbf{i} & \mathbf{-i} \end{bmatrix}$ **(iv)** $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}$

Solution:

(i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-1)(2) = 3 + 2 = 5 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

(ii) $\begin{bmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = (-2)(5) - (-4)(3) = -10 + 12 = 2 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ \frac{4}{2} & -\frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

(iii) $\begin{bmatrix} \mathbf{2\,i} & \mathbf{i} \\ \mathbf{i} & \mathbf{-i} \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix} = (2i)(-i) - (i)(i) = -2i^2 - i^2 = -2(-1) - (-1) = 2 + 1 = 3$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$

As $$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{3}\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3}i & -\frac{1}{3}i \\ -\frac{1}{3}i & \frac{2}{3}i \end{bmatrix}$$

(iv) $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$|A| = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = (2)(3) - (6)(1) = 6 - 6 = 0$

As $|A| = 0 \Rightarrow$ inverse does not exists.

Q.3 Solve the following system of linear equations.

(i) $\left.\begin{matrix} \mathbf{2x_1 - 3x_2 = 5} \\ \mathbf{5x_1 + x_2 = 4} \end{matrix}\right\}$ **(ii)** $\left.\begin{matrix} \mathbf{4x_1 + 3x_2 = 5} \\ \mathbf{3x_1 - x_2 = 7} \end{matrix}\right\}$ **(iii)** $\left.\begin{matrix} \mathbf{3x - 5x = 1} \\ \mathbf{-2x + y = -3} \end{matrix}\right\}$

Solution:

(i) Given

$2x_1 - 3x_2 = 5$

$5x_1 + x_2 = 4$

In matrix form

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ ..... (1)

where

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = (2)(1) - (5)(-3) = 2 + 15 = 17 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists

$$\text{Adj } A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

As

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} (1)(5)+(3)(4) \\ (-5)(5)+(2)(4) \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 5+12 \\ -25+8 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 17 \\ -17 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{17} \\ -\frac{17}{17} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x_1 = 1, \quad x_2 = -1$$

(ii) Given

$4x_1 + 3x_2 = 5$

$3x_1 - x_2 = 7$

In matrix form

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$AX = B$ (say)

$X = A^{-1}B$ (1)

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; A = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = (4)(-1) - (3)(3) = -4 - 9 = -13 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

As, $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$$X = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} (-1)(5)+(-3)(7) \\ (-3)(5)+(4)(7) \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} -5-21 \\ -15+28 \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} -26 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-26}{-13} \\ -\frac{13}{-13} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \boxed{x_1 = 2} \quad , \quad \boxed{x_2 = -1}$$

(iii) Given

$3x - 5y = 1$

$-2x + y = -3$

In matrix form

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ ..... (1)

where

$$A = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \; x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = (3)(1) - (-2)(-5) = 3 - 10 = -7 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

Now

$$\text{Adj } A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

As

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= -\frac{1}{7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

Putting values in equation (1)

$$X = -\frac{1}{7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} (1)(1)+(5)(-3) \\ (2)(1)+(3)(-3) \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} 1-15 \\ 2-9 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-14}{-7} \\ \frac{-7}{-7} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \boxed{x = 2} \quad , \quad \boxed{y = 1}$$

Q.4 **If** $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ **then find**

(i) A – B (ii) B – A (iii) (A – B) – C (iv) A – (B – C)

Solution:

(i) Given

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

Then $A - B = \begin{bmatrix} 1-2 & -1-1 & 2-(-1) \\ 3-1 & 2-3 & 5-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix}$

$$A - B = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

(ii) B – A

$$B - A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1-(-1) & -1-2 \\ 1-3 & 3-2 & 4-5 \\ -1-(-1) & 2-0 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

(iii) **(A – B) – C**

$$(A - B) - C = \left(\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} \right) - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1-1 & 2-(-1) \\ 3-1 & 2-3 & 5-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-1 & -2-3 & 3-(-2) \\ 2-(-1) & -1-2 & 1-0 \\ 0-3 & -2-4 & 3-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 & 5 \\ 3 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$$

(iv) **A – (B – C)**

$$A - (B - C) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \left(\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2-1 & 1-3 & -1-(-2) \\ 1-(-1) & 3-2 & 4-0 \\ -1-3 & 2-4 & 1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1-(-2) & 2-1 \\ 3-2 & 2-1 & 5-4 \\ -1-(-4) & 0-(-2) & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

Q.5 If $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$ then show that

(i) $(AB)\,C = A\,(BC)$ **(ii)** $(A + B)\,C = AC + BC$.

Solution:

(i) **To show** $(AB)\,C = A\,(BC)$

L.H.S. $= (AB)\,C$

$$(AB)\,C = \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}-\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right)\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (i)(-i) + (2i)(2i) & (i)(1) + (2i)(i) \\ (1)(-i) + (-i)(2i) & (1)(1) + (-i)(i) \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 + 4i^2 & i + 2i^2 \\ -i - 2i^2 & 1 - i^2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -(-1) + 4(-1) & i + 2(-1) \\ i - 2(-1) & 1 - (-1) \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -3 & i - 2 \\ 2 - i & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} -3 & i - 2 \\ -i + 2 & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (-3)(2i) + (i - 2)(-1) & -(3)(-1) + i(i - 2)(i) \\ (2 - i)(2i) + 2(-i) & (2 - i)(-1) + (2)(i) \end{bmatrix}$$

$$= \begin{bmatrix} -6i - i^2 + 2i & 3 + i^2 - 2i \\ 4i + 2i^2 - 2i & -2 + i + 2i \end{bmatrix}$$

$$= \begin{bmatrix} -4i - (-1) & 3 + (-1) - 2i \\ 2i - 2(-1) & -2 + 3i \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 4i & 2 - 2i \\ 2 + 2i & 3i - 2 \end{bmatrix}$$

R.H.S.

$$A\,(BC) = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\left(\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}-\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}\right)$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} (-i)(2i) + (1)(-i) & (-i)(-1) + (1)(i) \\ (2i)(2i) + (i)(-i) & (2i)(-1) + (i)(i) \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2(-1) - i & 2i \\ 4(-1) - (-1) & -2i + (-1) \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

$$= \begin{bmatrix} (i)(2-i)+(2i)(-3) & (i)(2i)+(2i)(-2i-1) \\ (1)(2-i)+(-3)(-i) & (1)(2i)+(-i)(-2i-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2i-i^2-6i & 2i^2-4i^2-2i \\ 2-i+3i & 2i+2i^2+i \end{bmatrix}$$

$$= \begin{bmatrix} -4i-(-1) & 2(-1)-4(-1)-2i \\ 2+2i & 2i+2(-1)+i \end{bmatrix}$$

$$= \begin{bmatrix} -4i+1 & 2-2i \\ 2+2i & 3i-2 \end{bmatrix}$$

$\Rightarrow$ $(AB)C = A(BC)$

(ii) To show $(A + B)C = AC + BC$

L.H.S.

$$(A+B)C = \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right)\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 2i+1 \\ 1+2i & -i+i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2i+1 \\ 2i+1 & 0 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (0)(2i)+(1+2i)(-i) & (0)(-1)+(1+2i)i \\ (1+2i)(2i)+(0)(-i) & (1+2i)(-1)+(0)(i) \end{bmatrix}$$

$$= \begin{bmatrix} 0-2i^2-i & 0+2i^2+i \\ 4i^2+2i+0 & -2i-1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2(-1)-i & 2(-1)+i \\ 4(-1)+2i & 2i-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-i & -2+i \\ -4+2i & -2i-1 \end{bmatrix}$$

Now R.H.S

$$(AC)+BC = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (i)(2i)+(2i)(-i) & (i)(-1)+(2i)(i) \\ (1)(2i)+(-i)(-i) & (1)(-1)(-1)+(-i)(i) \end{bmatrix}$$

$$+ \begin{bmatrix} (-i)(2i)+(1)(-i) & (-i)(-1)+(1)(i) \\ (2i)(2i)+(i)(-i) & (2)(i)(-1)+(i)(i) \end{bmatrix}$$

$$= \begin{bmatrix} +2i^2 - 2i^2 & -i + 2i^2 \\ 2i + i^2 & -1 + i^2 \end{bmatrix} + \begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i + 2(-1) \\ 2i + (-1) & -1 - (-1) \end{bmatrix} + \begin{bmatrix} -2(-1) - i & 2i \\ 4(-1) - (-1) & -2i + (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i - 2 \\ 2i - 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 - i & 2i \\ -3 & -2i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 2 - i & -i - 2 + 2i \\ 2i - 1 - 3 & 0 - 2i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - i & i - 2 \\ -4 + 2i & -2i - 1 \end{bmatrix}$$

$\Rightarrow$ $(A + B)C = AC + BC$

Hence proved.

Q.6 If A and B are square matrices of same order, then explain why in general.

(i) $\mathbf{(A + B)^2 \neq A^2 + 2AB + B^2}$

(ii) $\mathbf{(A - B)^2 \neq A^2 - 2AB + B^2}$

(iii) $\mathbf{(A + B)(A - B) \neq A^2 - B^2}$

Solution:

Let A and B are square matrices of the same order.

(i) As $(A + B)^2 = (A + B) \cdot (A + B)$

$= A.A + A.B + B.A + B.B$

$= A^2 + A.B + B.A + B^2$

but in general $AB \neq BA$

$\Rightarrow$ $A.B + B.A \neq 2AB$

$\Rightarrow$ $(A + B)^2 \neq A^2 + 2AB + B^2$

(ii) $(A - B)^2 = (A - B)(A - B)$

$= A.A - A.B - B.A + B.B$

$= A^2 - AB - B.A + B^2$

but in general $AB \neq BA$

$\Rightarrow$ $-BA - BA \neq 2AB$

$\Rightarrow$ $(A - B)^2 \neq A^2 - 2AB + B^2$

(iii) $(A + B)(A - B) = A.A - A.B + B.A - B^2$

$= A^2 - AB + B.A - B^2$

but in general $AB \neq BA \Rightarrow -AB + BA \neq 0$

$\Rightarrow$ $(A + B)(A - B) \neq A^2 - B^2$

Q.7 If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$, then find AA^t and A^tA.

Solution:

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

Then $$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$A.A^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(2)+(-1)(-1)+(3)(3)+(0)(0) & (2)(1)+(-1)(0)+(3)(4)+(0)(-2) \\ (1)(2)+(0)(-1)+(4)(3)+(-2)(0) & (1)(1)+(0)(0)+(4)(4)+(-2)(-2) \\ (-3)(2)+(5)(-1)+(2)(3)+(-1)(0) & (-3)(1)+(5)(0)+(2)(4)+(-1)(-2) \end{bmatrix}$$

$$\begin{matrix} 2(-3)+(-1)(5)+(3)(2)+(0)(-1) \\ (1)(-3)+(0)(5)+(4)(2)+(-2)(-1) \\ (-3)(-3)+(5)(5)+(2)(2)+(-1)(-1) \end{matrix} \Bigg]$$

$$= \begin{bmatrix} 4+1+9+0 & 2+0+12-0 & -6-5+6+0 \\ 2-0+12-0 & 1+0+16+4 & -3+0+8+2 \\ 6-5+6-0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9 & -2+0-15 & 6+4-6 & 0-2+3 \\ -2+0-15 & 1+0+25 & -3+0+10 & 0-0-5 \\ 6+4-6 & -3+0+10 & 9+16+4 & -0-8-2 \\ 0-2+3 & 0+0-5 & 0-8-2 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -17 & 4 & 1 \\ -17 & 26 & 7 & -5 \\ 4 & 7 & 29 & -10 \\ 1 & -5 & -10 & 5 \end{bmatrix}$$

Q.8 Solve the following matrix equations for X:

(i) $\mathbf{3x - 2A = B}$ **if** $\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

(ii) $\mathbf{2x - 3A = B}$ **if** $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

Solution:

(i) Given

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$3X - 2A = B$$

$$3X = B + 2A$$

$$X = \frac{1}{3}(B + 2A)$$

Put values of A and B

$$X = \frac{1}{3}\left(\begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 2+4 & -3+6 & 1-4 \\ 5-2 & 4+2 & -1+10 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{3} & \frac{3}{3} & -\frac{3}{3} \\ \frac{3}{3} & \frac{6}{3} & \frac{9}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

(ii) Given $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

and

$$2X - 3A = B$$

$$2X = B + 3A$$

$$X = \frac{1}{2}(B + 3A)$$

Put values of A and B

$$X = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & -1-3 & 0+6 \\ 4-6 & 2+12 & 1+15 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{2} & -\frac{4}{2} & \frac{6}{2} \\ -\frac{2}{2} & \frac{14}{2} & \frac{16}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

Q.9 Solve the following matrix equation for A.

(i) $\begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{2} \end{bmatrix}\mathbf{A} - \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{-4} \\ \mathbf{3} & \mathbf{6} \end{bmatrix}$

(ii) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix} - \begin{bmatrix} \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{5} \end{bmatrix}$

Solution:

(i) Given

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1+2 & -4+3 \\ 3-1 & 6-2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$BA = C$ (say)

$B^{-1}B\,A = B^{-1}C$

$IA = B^{-1}C$

$A = B^{-1}C$ ..... (1)

Where

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = (4)(2) - (3)(2) = 8 - 6 = 2$$

As $|B| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } B = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{\text{Adj } B}{|B|} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

Putting this values in (1)

$$A = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (2)(1) + (-3)(2) & (2)(-1) + (-3)(4) \\ (-2)(1) + (4)(2) & (-2)(-1) + (4)(4) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 - 6 & -2 - 12 \\ -2 + 8 & 2 + 16 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & -14 \\ 6 & 18 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{4}{2} & -\frac{14}{2} \\ \frac{6}{3} & \frac{18}{2} \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}$$

is the required matrix.

(ii) Given

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 - 1 & 0 + 2 \\ -1 + 3 & 5 + 1 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$AB = C$ (say)

$ABB^{-1} = CB^{-1}$

$AI = CB^{-1}$

$A = CB^{-1}$ ..... (1)

where

$$B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = (3)(2) - (4)(1) = 6 - 4 = 2$$

$|B| \neq 0 \Rightarrow$ inverse of B exists.

$$\text{Adj } B = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

As

$$B^{-1} = \frac{\text{Adj } B}{|B|} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Putting values in equation (1)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (1)(2) + (2)(-4) & (1)(-1) + (2)(3) \\ (2)(2) + (6)(-4) & (2)(-1) + (6)(3) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 - 8 & -1 + 6 \\ 4 - 24 & -2 + 18 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -6 & 5 \\ -20 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{2} & \frac{5}{2} \\ -\frac{20}{2} & \frac{16}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & \frac{5}{2} \\ -10 & 8 \end{bmatrix}$$

is the required matrix

DETERMINANT OF ORDER $n \geq 3$

Consider a square matrix of order 3.

such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then}$$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

we can expand this determinant by R_1

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

we can expand $|A|$ by C_1 such that

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

Minor and Cofactor of an Element of A Matrix or Its Determinant

Consider a square matrix A of order 3. then the minor of an element a_{ij}, denoted by M_{ij} is the determinant of the $(3-1) \times (3-1)$ matrix formed by deleting the ith row and jth column of A or $|A|$.

For example if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$\text{Minor of } a_{11} = M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\text{Minor of } a_{12} = M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \text{ and so on.}$$

Cofactor of an element a_{ij} denoted by A_{ij} is defined by

$A_{ij} = (-1)^{i+j} . M_{ij}$ where M_{ij} is the minor of a_{ij}.

So in above matrix

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} M_{11}$

Cofactor of $a_{13} = A_{13} = (-1)^{1+3} M_{13}$ and so on.

Note: In above matrix A

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

Properties of Determinants

(1) For a square matrix A , $|A| = |A^t|$.

(2) If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

(3) If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.

(4) If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.

(5) If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in R$, then the determinant of the resulting matrix is $k|A|$.

(6) If each entry of a row (or a column) of a square matrix consists of two terms then its determinant can be written as the sum of two determinants i.e. if

$$A = \begin{bmatrix} a_{11} + b_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} \end{bmatrix} \text{ then}$$

$$|A| = \begin{vmatrix} a_{11} + b_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}.$$

(7) If to each entry of a row (or a column) of a square matrix A is added a non–zero multiple of the corresponding entry of another row (or column) then the determinant of the resulting matrix is $|A|$.

(8) If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the matrix of cofactors of A is

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and $$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Inverse of Square Matrix of Order $n \geq 3$

Let A be a non singular matrix of order n. If there exists a matrix B such that

$AB = BA = I_n$ then B is called the multiplicative inverse of A and is denoted by A^{-1}, and order of A^{-1} is n x n. Thus $A\,A^{-1} = I_n$ and $A^{-1}A = I_n$. If A is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

EXERCISE 3.3

Q.1 Evaluate the following determinants.

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

(iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

(vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Expanding the determinant by R_1.

$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\,(2 \times (-1) - 1 \times (-3)) + 2\,(3 \times 2 - (-3) \times (-2)) - 4\,(3 \times 1 - (-2) \times (-1))$

$= 5\,(-2 + 3) + 2\,(6 - 6) - 4\,(3 - 2)$

$= 5\,(1) + 2\,(0) - 4\,(1)$

$= 5 + 0 - 4 = 1$

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding the determinant by R_1.

$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\,(2 \times (-1) - 1 \times (1)) - 2\,((3) \times (-2) - (-2) \times (1)) - 3\,((3) \times (1) - (-2) \times (-1))$

$= 5\,(2 - 1) - 2\,(-6 + 2) - 3\,(3 - 2)$

$= 5\,(1) - 2\,(-4) - 3\,(1)$

$= 5 + 8 - 3 = 10$

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{-2} & \mathbf{5} & \mathbf{6} \end{vmatrix}$

Expanding the given determinant by R_1

$= 1 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} -2 \begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$= 1\,((3) \times (6) - (5) \times (4)) - 2\,((6) \times (-1) - (-2) \times (4)) - 3\,((-1) \times (5) - (-2) \times (3))$

$= 1\,(18 - 20) - 2\,(-6 + 8) - 3\,(-5 + 6)$

$= -2 - 2\,(2) - 3\,(1)$

$= -2 - 4 - 3 = -9$

(iv) $\begin{vmatrix} \mathbf{a+l} & \mathbf{a-l} & \mathbf{a} \\ \mathbf{a} & \mathbf{a+l} & \mathbf{a-l} \\ \mathbf{a-l} & \mathbf{a} & \mathbf{a+l} \end{vmatrix}$

expanding the given determinant by R_1

$= (a + l) \begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix} - (a + l) \begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix} + a \begin{vmatrix} a & a+l \\ a-l & a \end{vmatrix}$

$= (a + l)\,[(a + l)\,(a + l) - a(a - l)] - (a - l)\,[a\,(a + l) - (a - l)\,(a - l)]$

$+ a\,[(a)\,(a) - (a - l)\,(a + l)]$

$= (a + l)\,[a^2 + al + la + l^2 - a^2 + al] - (a - l)\,[a^2 + al - (a^2 - al - al + l^2)]$

$+ a\,[a^2 - (a^2 + al - al - l^2)]$

$= (a + l)\,[3al + l^2] - (a - l)\,[a^2 + al - (a^2 - 2al + l^2)] + a\,[a^2 - (a^2 - l^2)]$

$= (a + l)\,[3al + l^2] - (a - l)\,[a^2 + al - a^2 + 2al - l^2)] + a\,[a^2 - a^2 + l^2)]$

$= (a + l)\,[3al + l^2] - (a - l)\,[3al - l^2)] + a\,[l^2)]$

$= 3a^2l + al^2 + 3al^2 + l^3 - (3a^2l - al^2 - 3al^2 + l^3) + al^2$

$= 3a^2l + 4al^2 + l^3 - 3a^2l + al^2 + 3al^2 - l^3 + al^2$

$= 9al^2$

2nd Method

$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

$= \begin{vmatrix} a+l+a-l+a & a-l & a \\ a+a+l+a-l & a+l & a-l \\ a-l+a+a+l & a & a+l \end{vmatrix}$ by $C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix}$$

Take common 3a from C_1

$$= 3a \begin{vmatrix} 1 & a-l & a \\ 1 & a+l & a-l \\ 1 & a & a+l \end{vmatrix}$$

$$= 3a \begin{vmatrix} 1 & a-l & a \\ 0 & a+l-a+l & a-l-a \\ 0 & a-a+l & a+l-a \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 3a \begin{vmatrix} 1 & a-l & a \\ 0 & 2l & -l \\ 0 & l & l \end{vmatrix}$$

expanding by C_1

$$= 3a\left[1 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 0 \begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + 0 \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} \right]$$

$$= 3a\,[(2l)\,(l) - (l)\,(-l)] - 0 + 0$$

$$= 3a\,[2l^2 + l^2]$$

$$= 3a\,(3l^2)$$

$$= 9al^2$$

(v) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{vmatrix}$

Expanding the given determinant by R_1

$$= 1 \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= \big((1)\,(-1) - (4)\,(-3)\big) - 2\,\big((-1)\,(-1) - (2)\,(-3)\big) - 2\,\big((-1)\,(4) - (2)\,(1)\big)$$

$$= (-1 + 12) - 2\,(1 + 6) - 2\,(-4 - 2)$$

$$= 11 - 2\,(7) - 2\,(-6)$$

$$= 11 - 14 + 12 = 9$$

(vi) $\begin{vmatrix} \mathbf{2a} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{2b} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{2c} \end{vmatrix}$

Expanding the given determinant by R_1

$$= 2a \begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a \begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a \begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$$

$$= 2a\,\big((2b)\,(2c) - (b)\,(c)\big) - a\,\big((b)\,(2c) - (b)\,(c)\big) + a\,\big((b)\,(c) - (2b)\,(c)\big)$$

$$= 2a\,(4bc - bc) - a\,(2bc - bc) + a\,(bc - 2bc)$$

$$= 6abc - abc - abc = 4abc$$

Q.2 Without expansion show that.

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ **(ii)** $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

Solution:

L.H.S.

$$\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 7-6 & 8 \\ 3 & 4-3 & 5 \\ 2 & 3-2 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 8 \\ 3 & 1 & 5 \\ 2 & 1 & 4 \end{vmatrix} \quad C_2 - C_1$$

$$\begin{vmatrix} 6 & 1 & 8-6 \\ 3 & 1 & 5-3 \\ 2 & 1 & 4-2 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} \quad C_3 - C_1$$

Take common (2) from c_3

$$= 2 \begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

As C_2 and C_3 are same so determinant will be zero.

$$= 2\,(0) = 0 = \text{R.H.S.}$$

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

$$\text{L.H.S.} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3-1 & -1 \\ 1 & 1+0 & 0 \\ 2 & -3+5 & 5 \end{vmatrix} \quad C_2 + C_3$$

$$= \begin{vmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 5 \end{vmatrix} = 0 = \text{R.H.S.}$$

As C_1 and C_2 are same so determinant will be zero.

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

L.H.S.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2-1 & 3-1 \\ 4 & 5-4 & 6-4 \\ 7 & 8-7 & 9-7 \end{vmatrix} \qquad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ 7 & 1 & 2 \end{vmatrix}$$

Take (2) common from C_3

$$= (2) \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 0 = \text{R.H.S.}$$

As C_2 and C_3 are same so determinant will be zero.

Q.3 Show that

(i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

(v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$

(vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

Solution:

(i) $$\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

L.H.S. $= \begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$

Expanding by R_1

$$= a_{11}\begin{vmatrix} a_{22} & a_{23}+\alpha_{23} \\ a_{32} & a_{33}+\alpha_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23}+\alpha_{23} \\ a_{31} & a_{33}+\alpha_{33} \end{vmatrix} + (a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}(a_{33}+\alpha_{33}) - a_{32}(a_{23}+\alpha_{23})] - a_{12}[a_{21}(a_{33}+\alpha_{33}) - a_{31}(a_{23}+\alpha_{23})]$$

$$+ (a_{13}+\alpha_{13})[a_{21}a_{32} - a_{31}a_{22}]$$

$$= a_{11}[a_{22}a_{33} + a_{22}\alpha_{33} - a_{32}a_{23} - a_{32}\alpha_{23}]$$

$$- a_{12}[a_{21}a_{33} + a_{21}\alpha_{33} - a_{31}a_{23} - a_{31}\alpha_{23}]$$

$$+ a_{13}[a_{21}a_{32} - a_{31}a_{22}] + \alpha_{13}[a_{21}a_{32} - a_{31}a_{22}] \quad \ldots\ldots\ldots (1)$$

Now take R.H.S.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

Expanding both by R_1

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$+ a_{11}\begin{vmatrix} a_{22} & \alpha_{23} \\ a_{32} & \alpha_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & \alpha_{23} \\ a_{31} & \alpha_{33} \end{vmatrix} + \alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}a_{33} - a_{32}a_{23}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

$$+ a_{11}[a_{22}\alpha_{33} - a_{32}\alpha_{23}] - a_{12}[a_{21}\alpha_{33} - \alpha_{23}a_{31}] + \alpha_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

$$= a_{11}[a_{22}a_{33} + a_{22}\alpha_{33} - a_{32}a_{23} - a_{32}\alpha_{23}] - a_{12}[a_{21}a_{33} + a_{21}\alpha_{33} - a_{23}a_{31} - \alpha_{23}a_{31}]$$

$$+ a_{13}[a_{21}a_{32} - a_{22}a_{31}] + \alpha_{13}[a_{21}a_{32} - a_{22}a_{31}] \quad \ldots\ldots\ldots (2)$$

from equation (1) and (2) L.H.S. = R.H.S.

(ii) $\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{3} & \mathbf{9} & \mathbf{6} \\ \mathbf{2} & \mathbf{15} & \mathbf{1} \end{vmatrix} = \mathbf{9} \begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} & \mathbf{1} \end{vmatrix}$

L.H.S. $= \begin{bmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{bmatrix}$

Take common (3) from C_2

$= 3\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 6 \\ 2 & 5 & 1 \end{bmatrix}$

Take common (3) from R_2

$= 3 \times 3 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$

$= 9\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$

$=$ R.H.S.

(iii) $\begin{vmatrix} \mathbf{a+l} & \mathbf{a} & \mathbf{a} \\ \mathbf{a} & \mathbf{a+l} & \mathbf{a} \\ \mathbf{a} & \mathbf{a} & \mathbf{a+l} \end{vmatrix} = \mathbf{l^2\,(3a+l)}$

L.H.S. $= \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$

$= \begin{vmatrix} a+l+a+a & a+a+l+a & a+a+a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$ $R_1 + R_2 + R_3$

$= \begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$

$(3a + l)$ common from R_1

$= (3a+l) \begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 1-1 \\ a & a+l-a & a-a \\ a & a-a & a+l-a \end{vmatrix}$ $C_2 - C_1$, $C_3 - C_1$

$$= (3a + l)\begin{vmatrix} 1 & 0 & 0 \\ a & l & 0 \\ a & 0 & l \end{vmatrix}$$

Expanding by R_1

$$= (3a + l)\left[1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0\begin{vmatrix} a & 0 \\ a & l \end{vmatrix} + 0\begin{vmatrix} a & l \\ a & 0 \end{vmatrix}\right]$$

$$= (3a + l)\,[1\,(l^2 - 0) - 0 + 0]$$

$$= (3a + l)\,(l^2)$$

$$= l^2\,(3a + l) = \text{R.H.S.}$$

(iv) $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{yz} & \mathbf{zx} & \mathbf{xy} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \end{vmatrix}$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

Multiply C_1 by x, C_2 by y and C_3 by z

$$= \frac{l}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Take (xyz) common from R_3.

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Interchange R_1 and R_3.

$$= -\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \text{R.H.S.}$$

(v) $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{a+b} \end{vmatrix} = \mathbf{4abc}$

L.H.S. $= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$= \begin{vmatrix} b+c & a-a & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix} C_2 - C_3$

$= \begin{vmatrix} b+c & 0 & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix}$

expanding by R_1

$= (b+c)\begin{vmatrix} c+a-b & b \\ c-a-b & a+b \end{vmatrix} - 0 + a\begin{vmatrix} b & c+a-b \\ c & c-a-b \end{vmatrix}$

$= (b+c)[(a+b)(c+a-b) - b(c-a-b)] + a[b(c-a-b) - c(c+a-b)]$

$= (b+c)[ca + a^2 - ab + bc + ba - b^2 - bc + ba + b^2] + a[bc - ba - b^2 - c^2 - ac + bc]$

$= (b+c)[ac + a^2 + ab] + a[2bc - ba - b^2 - c^2 - ac]$

$= bac + ba^2 + ab^2 + ac^2 + a^2c + abc + 2abc - ba^2 - ab^2 - ac^2 - a^2c$

$= abc + abc + 2abc$

$= 4abc$ = R.H.S.

(vi) $\begin{vmatrix} \mathbf{b} & \mathbf{-1} & \mathbf{a} \\ \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{1} & \mathbf{a} & \mathbf{b} \end{vmatrix} = \mathbf{a^3 + b^3}$

expanding this determinant by R_1

$= b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1)\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$

$= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b)$

$= b(b^2) + ab + a(a^2 - b)$

$= b^3 + ab + a^3 - ab$

$= a^3 + b^3$

$=$ R.H.S.

(vii) $\begin{vmatrix} \mathbf{r\cos\phi} & \mathbf{1} & \mathbf{-\sin\phi} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{r\sin\phi} & \mathbf{0} & \mathbf{\cos\phi} \end{vmatrix} = \mathbf{r}$

expanding this determinant by R_1

$= r\cos\phi \begin{bmatrix} 1 & 0 \\ 0 & \cos\phi \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 \\ r\sin\phi & \cos\phi \end{bmatrix} - \sin\phi \begin{bmatrix} 0 & 1 \\ r\sin\phi & 0 \end{bmatrix}$

$= r\cos\phi\,(\cos\phi - 0) - 1\,(0 - 0) - \sin\phi\,(0 - r\sin\phi)$

$= r\cos^2\phi + r\sin^2\phi$

$= r\,(\cos^2\phi + \sin^2\phi)$

$= r\,(1)$

$= r = \text{R.H.S.}$

(viii) $\begin{vmatrix} \mathbf{a} & \mathbf{b+c} & \mathbf{a+b} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b+c} \\ \mathbf{c} & \mathbf{a+b} & \mathbf{c+a} \end{vmatrix} = \mathbf{a^3 + b^3 + c^3 - 3abc}$

$$\text{L.H.S.} = \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b+c & a+b \\ b+c+a & c+a & b+c \\ c+a+b & a+b & c+a \end{vmatrix} \; C_1 + C_2$$

Take $(a + b + c)$ common from c_1

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 0 & c+a-b-c & b+c-a-b \\ 0 & a+b-b-c & c+a-a-b \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix}$$

Expanding by C_1

$$= (a+b+c) \,.\, 1 \,.\, \begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0$$

$= (a + b + c)\,[(a - b)\,(c - b) - (c - a)\,(a - c)]$

$= (a + b + c)\,[ac - ab - bc + b^2 - ac + c^2 + a^2 - ac]$

$= (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ac)$

$= a^3 + b^3 + c^3 - 3abc$

$= \text{R.H.S.}$

(ix) $\begin{vmatrix} \mathbf{a+\lambda} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b+\lambda} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{c+\lambda} \end{vmatrix} = \mathbf{\lambda^2 (a+b+c+\lambda)}$

L.H.S. $= \begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix}$

$= \begin{vmatrix} a+b+\lambda+c & b & c \\ a+b+\lambda+c & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix}$ $C_1 + (C_2 + C_3)$

Taking $(a+b+c+\lambda)$ common from C_1

$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix}$

$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 0 & b+\lambda-b & c-c \\ 0 & b-b & c+\lambda-c \end{vmatrix}$ $R_2 - R_1$, $R_3 - R_1$

$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$

Expanding by C_1

$= (a+b+c+\lambda) \left[1 \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0 \right]$

$= (a+b+c+\lambda)(\lambda^2 - 0)$

$= \lambda^2 (a+b+c+\lambda)$

$=$ R.H.S.

(x) $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)}$

L.H.S. $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ $C_2 - C_1$, $C_3 - C_1$

Expanding by R_1

$= 1 \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} - 0 + 0$

$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$

$= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$

$= (b-a)(c-a)[c+a-(b+a)]$

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b)$

$= -(a-b)(c-a)(-1)(b-c)$

$= (a-b)(b-c)(c-a)$

$=$ R.H.S.

(xi) $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{c+a} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{a+b} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a+b+c)(a-b)(b-c)(c-a)}$

L.H.S. $= \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} b+c+a & a & a^2 \\ c+a+b & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix}$ $\quad C_1 + C_2$

Take $(a+b+c)$ common from c_1

$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1-1 & b-a & b^2-a^2 \\ 1-1 & c-a & c^2-a^2 \end{vmatrix}$ $\quad R_2 - R_1$, $R_3 - R_1$

$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$

Take common $(b-a)$ from R_2, and $(c-a)$ from R_3

$= (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

Expanding by C_1

$= (a+b+c)(b-a)(c-a)\left[1\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} - 0\begin{vmatrix} a & a^2 \\ 1 & c+a \end{vmatrix} + 0\begin{vmatrix} a & a^2 \\ 1 & b+a \end{vmatrix}\right]$

$= (a+b+c)(b-a)(c-a)[c+a-(b+a)]$

$= (a+b+c)(b-a)(c-a)(c+a-b-a)$

$= (a + b + c)(b - a)(c - a)(c - b)$

$= -(a + b + c)(a - b)(c - a)(c - b)$

$= (-1)(-1)(a + b + c)(a - b)(b - c)(c - a)$

$= (a + b + c)(a - b)(b - c)(c - a)$

$=$ R.H.S.

Q.4 (i) **If** $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

Find A_{12}, A_{22}, A_{32}, **and** $|A|$

(ii) $\mathbf{B} = \begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$ **then**

Find B_{21}, B_{22}, B_{23} **and** $|B|$

Solution:

(i) $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

$A_{12} = (-1)^{1+2} M_{12}$ where $M_{12} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

$= (-1)^3 (0)$ $= (0)(1) - (-2)(0)$

$= 0$ $= 0$

$A_{22} = (-1)^{2+2} M_{22}$ where $M_{22} = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$= (-1)^4 (-5)$ $= (1)(1) - (-2)(-3)$

$= (1)(-5)$ $= 1 - 6 = -5$

$= -5$ $= -5$

$A_{32} = (-1)^{3+2} M_{32}$ where $M_{32} = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$= (-1)^5 (0)$ $= (1)(0) - (-3)(0)$

$= 0$ $= 0$

Now

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{vmatrix}$$

Expanding by R_1

$$= 1\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1\big((-2)(1) - (-2)(0)\big) - 2\big((0)(1) - (-2)(0)\big) - 3\big((0)(-2) - (-2)(-2)\big)$$

$$= (-2 - 0) - 2(0 - 0) - 3(0 - 4)$$

$$= -2 - 0 - 3(-4)$$

$$= -2 + 12 = 10$$

(ii) $\mathbf{B} = \begin{bmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{5} \\ \mathbf{3} & \mathbf{-1} & \mathbf{4} \\ \mathbf{-2} & \mathbf{1} & \mathbf{-2} \end{bmatrix}$

$B_{21} = (-1)^{2+1} M_{21}$ where $M_{21} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$

$= (-1)^3 (-1)$ $= (-2)(-2) - (1)(5)$

$= (-1)(-1) = 1$ $= 4 - 5 = -1$

$B_{22} = (-1)^{2+2} M_{22}$ where $M_{22} = \begin{bmatrix} 5 & 5 \\ -2 & -2 \end{bmatrix}$

$= (-1)^4 (0)$ $= (5)(-2) - (5)(-2)$

$= 0$ $= -10 + 10 = 0$

$B_{23} = (-1)^{2+3} M_{23}$ where $M_{23} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$= (-1)^5 (1)$ $= (5)(1) - (-2)(2)$

$= (-1)(1)$ $= 5 - 4$

$= -1$ $= 1$

Now $|B| = \begin{vmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding by R_1

$$= 5\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 4 \\ -2 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= 5\big((-1)(-2) - (1)(4)\big) + 2\big((3)(-2) - (-2)(4)\big) + 5\big((3)(1) - (-1)(-2)\big)$$

$$= 5(2 - 4) + 2(-6 + 8) + 5(3 - 2)$$

$$= 5(-2) + 2(2) + 5(1)$$

$$= -10 + 4 + 5$$

$$= -1$$

Q.5 Without expansion verify that

(i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ac} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

(iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

(v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

(vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

(vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

(vii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \beta+\gamma+\alpha & \gamma+\alpha & 1 \\ \gamma+\alpha+\beta & \alpha+\beta & 1 \end{vmatrix} \quad C_1 + C_2$$

Take $(\alpha+\beta+\gamma)$ common from C_1

$$= (\alpha+\beta+\gamma) \begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix} \quad \because \ C_1 \text{ and } C_3 \text{ are same}$$

$$= (\alpha+\beta+\gamma) \cdot 0$$

$$= 0$$

(ii) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3x} \\ \mathbf{2} & \mathbf{3} & \mathbf{6x} \\ \mathbf{3} & \mathbf{5} & \mathbf{9x} \end{vmatrix} = \mathbf{0}$

L.H.S. $= \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$

Take $3x$ common from C_3

$= 3x \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$

$= 3x\,(0) = 0$

As C_1 and C_3 is same so determinant will be zero.

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{a^2} & \mathbf{\frac{a}{bc}} \\ \mathbf{1} & \mathbf{b^2} & \mathbf{\frac{b}{ac}} \\ \mathbf{1} & \mathbf{c^2} & \mathbf{\frac{c}{ab}} \end{vmatrix} = \mathbf{0}$

L.H.S. $= \begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ac} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix}$

multiplying C_3 by abc

$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & \frac{a \,.\, abc}{bc} \\ 1 & b^2 & \frac{b \,.\, abc}{ac} \\ 1 & c^2 & \frac{c \,.\, abc}{ab} \end{vmatrix}$

$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$

$= 0$

As C_2 and C_3 are same so determinant will be zero.

(iv) $\begin{vmatrix} \mathbf{a-b} & \mathbf{b-c} & \mathbf{c-a} \\ \mathbf{b-c} & \mathbf{c-a} & \mathbf{a-b} \\ \mathbf{c-a} & \mathbf{a-b} & \mathbf{b-c} \end{vmatrix} = \mathbf{0}$

L.H.S. $= \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$ $\quad C_1 + C_2 + C_3$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$ $\quad \because$ C_1 is zero.

$= 0$

(v) $\begin{vmatrix} \mathbf{bc} & \mathbf{ca} & \mathbf{ab} \\ \mathbf{\frac{1}{a}} & \mathbf{\frac{1}{b}} & \mathbf{\frac{1}{c}} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{vmatrix} = \mathbf{0}$

L.H.S. $= \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$

multiplying R_2 by abc

$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ a & b & c \end{vmatrix}$

$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix}$

$= 0$

As R_1 and R_2 are same so det will be zero.

(vi) $\begin{vmatrix} \mathbf{mn} & \boldsymbol{l} & \boldsymbol{l}^{\mathbf{2}} \\ \mathbf{n}\boldsymbol{l} & \mathbf{m} & \mathbf{m^2} \\ \boldsymbol{l}\mathbf{m} & \mathbf{n} & \mathbf{n^2} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \boldsymbol{l}^{\mathbf{2}} & \boldsymbol{l}^{\mathbf{3}} \\ \mathbf{1} & \mathbf{m^2} & \mathbf{m^3} \\ \mathbf{1} & \mathbf{n^2} & \mathbf{n^3} \end{vmatrix}$

L.H.S. $= \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$

multiplying R_1 by l R_2 by m, r_3 by n.

$= \frac{1}{lmn} \begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$

Taking lmn common from C_1

$$= \frac{lmn}{lmn}\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$$= \text{R.H.S.}$$

(vii) $\begin{vmatrix} \mathbf{2a} & \mathbf{2b} & \mathbf{2c} \\ \mathbf{a+b} & \mathbf{2b} & \mathbf{b+c} \\ \mathbf{a+c} & \mathbf{b+c} & \mathbf{2c} \end{vmatrix} = \mathbf{0}$

$$\text{L.H.S.} = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2b-2a & 2c-2a \\ a+b & 2b-(a+b) & b+c-(a+b) \\ a+c & b+c-(a+c) & 2c-(a+c) \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 2a & 2(b-a) & 2(c-a) \\ a+b & 2b-a-b & b+c-a-b \\ a+c & b+c-a-c & 2c-a-c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2(b-a) & 2(c-a) \\ a+b & b-a & c-a \\ a+c & b-a & c-a \end{vmatrix}$$

Taking $(b-a)$ common from C_2, and $(c-a)$ from C_3.

$$= (b-a)(c-a)\begin{bmatrix} 2a & 2 & 2 \\ a+b & 1 & 1 \\ a+c & 1 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(0) \qquad C_2 \text{ and } C_3 \text{ is same.}$$

$$= 0 = \text{R.H.S.}$$

(viii) $\begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{6} \\ \mathbf{6} & \mathbf{3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{5} & \mathbf{1} \end{vmatrix} = \begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{7} \\ \mathbf{6} & \mathbf{3} & \mathbf{5} \\ \mathbf{-3} & \mathbf{5} & \mathbf{-3} \end{vmatrix} + \begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{-1} \\ \mathbf{6} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-3} & \mathbf{5} & \mathbf{4} \end{vmatrix}$

$$\text{L.H.S.} = \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix}$$

by using property of determinant

$$= \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

$$= \text{R.H.S.}$$

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} -ab & 0 & bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \begin{matrix} bR_1 \\ cR_2 \\ aR_3 \end{matrix}$$

Taking ab common from C_1.
Taking ac common from C_2.
Taking bc common from C_3.

$$= \frac{1}{abc}(ab)(ac)(bc) \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= abc \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} R_1 + R_2$$

$$= abc \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= abc \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} R_1 + R_3$$

$= abc\,(0)$ $\quad \therefore$ all elements of R_1 are zero.
$= 0$
$=$ R.H.S

Q.6 Find value of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:

(i) $\begin{vmatrix} \mathbf{3} & \mathbf{1} & \mathbf{x} \\ \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{x} & \mathbf{1} & \mathbf{0} \end{vmatrix} = \mathbf{-30}$

Expanding this determinant by R_1

$\Rightarrow \quad 3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$

$\Rightarrow \quad 3\,(3\,(0) - (4)\,(1)) - 1\,((-1)\,(0) - (x)\,(4)) + x\,((-1)\,(1) - (x)\,(3)) = -30$

$\Rightarrow \quad 3(-4) - 1(-4x) + x(-1 - 3x) = -30$

$\Rightarrow \quad -12 + 4x - x - 3x^2 = -30$

$\Rightarrow \quad -3x^2 + 3x - 12 = -30$

$\Rightarrow \quad -3\,(x^2 - x + 4) = -30$

$\Rightarrow \quad x^2 - x + 4 = \dfrac{-30}{-3} = 10$

$\Rightarrow \quad x^2 - x + 4 - 10 = 0$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad x^2 - 3x + 2x - 6 = 0$

$\Rightarrow \quad x\,(x - 3) + 2\,(x - 3) = 0$

$\Rightarrow \quad (x + 2)\,(x - 3) = 0$

$\Rightarrow \quad \boxed{x = -2, 3}$

(ii) $\begin{vmatrix} \mathbf{1} & \mathbf{x-1} & \mathbf{3} \\ \mathbf{-1} & \mathbf{x+1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{x} \end{vmatrix} = \mathbf{0}$

Expanding this determinant by R_1

$1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad 1\,(x\,(x + 1) - (-2)\,(2)) - (x - 1)\,((-1)\,(x) - (2)\,(2)) + 3\,((-1)\,(-2) - 2\,(x + 1)) = 0$

$\Rightarrow \quad (x^2 + x + 4) - (x - 1)\,(-x - 4) + 3\,(2 - 2x - 2) = 0$

$\Rightarrow \quad x^2 + x + 4 - (-x^2 - 4x + x + 4) + 3\,(-2x) = 0$

$\Rightarrow \quad x^2 + x + 4 + x^2 + 4x - x - 4 - 6x = 0$

$\Rightarrow \quad 2x^2 - 2x = 0$

$\Rightarrow \quad 2x\,(x - 1) = 0$

$\Rightarrow \quad 2x = 0 \qquad\qquad x - 1 = 0$

$\Rightarrow \quad \boxed{x = 0} \qquad\qquad \boxed{x = 1}$

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{x} & \mathbf{2} \\ \mathbf{3} & \mathbf{6} & \mathbf{x} \end{vmatrix} = \mathbf{0}$

Expanding this determinant by R_1

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad (x^2 - 12) - 2(2x - 6) + (12 - 3x) = 0$

$\Rightarrow \quad x^2 - 12 - 4x + 12 + 12 - 3x = 0$

$\Rightarrow \quad x^2 - 7x + 12 = 0$

$\Rightarrow \quad x^2 - 4x - 3x + 12 = 0$

$\Rightarrow \quad x(x - 4) - 3(x - 4) = 0$

$\Rightarrow \quad (x - 3)(x - 4) = 0$

$\Rightarrow \quad x - 3 = 0 \qquad x - 4 = 0$

$\Rightarrow \quad \boxed{x = 3} \qquad \boxed{x = 4}$

Q.7 Evaluate the following determinants:

(i) $\begin{vmatrix} \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{7} \\ \mathbf{2} & \mathbf{5} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{-3} & \mathbf{5} \\ \mathbf{4} & \mathbf{1} & \mathbf{-2} & \mathbf{6} \end{vmatrix}$

(ii) $\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{4} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{2} & \mathbf{-1} & \mathbf{6} \\ \mathbf{3} & \mathbf{-7} & \mathbf{2} & \mathbf{-2} \end{vmatrix}$

(iii) $\begin{vmatrix} \mathbf{-3} & \mathbf{9} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{9} & \mathbf{7} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{7} \\ \mathbf{2} & \mathbf{5} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{-3} & \mathbf{5} \\ \mathbf{4} & \mathbf{1} & \mathbf{-2} & \mathbf{6} \end{vmatrix}$

Interchange R_1 and R_3

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 2 & 5 & 0 & 3 \\ 3 & 4 & 2 & 7 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 2-2(1) & 5-2(2) & 0-2(-3) & 3-2(5) \\ 3-3(1) & 4-3(2) & 2-3(-3) & 7-3(5) \\ 4-4(1) & 1-4(2) & -2-4(-3) & 6-4(5) \end{vmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 0 & 5-4 & 0+6 & 3-10 \\ 0 & 4-6 & 2+9 & 7-15 \\ 0 & 1-8 & -2+12 & 6-20 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 6 & -7 \\ 0 & -2 & 11 & -8 \\ 0 & -7 & 10 & -14 \end{vmatrix}$$

Expanding by C_1

$$= -\left[1 \begin{vmatrix} 1 & 6 & -7 \\ -2 & 11 & -8 \\ -7 & 10 & -14 \end{vmatrix} - 0 + 0 - 0 \right]$$

$$= -\begin{vmatrix} 1 & 6 & -7 \\ -2 & 11 & -8 \\ -7 & 10 & -14 \end{vmatrix}$$

Expanding by R_1

$$= -\left[1 \begin{vmatrix} 11 & -8 \\ 10 & -14 \end{vmatrix} - 6 \begin{vmatrix} -2 & -8 \\ -7 & -14 \end{vmatrix} + (-7) \begin{vmatrix} -2 & 11 \\ -7 & 10 \end{vmatrix} \right]$$

$$= -[((11)(-14) - (10)(-8)) - 6((-2)(-14) - (-7)(-8)) - 7[((-2)(10) - (11)(-7)]$$

$$= -[-154 + 80 - 6(28 - 56) - 7(-20 + 77)]$$

$$= -[-74 - 6(-28) - 7(57)]$$

$$= 305$$

(ii) $\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{4} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{2} & \mathbf{-1} & \mathbf{6} \\ \mathbf{3} & \mathbf{-7} & \mathbf{2} & \mathbf{-2} \end{vmatrix}$

Interchange C_1 and C_3

$$= -\begin{vmatrix} 1 & 3 & 2 & -1 \\ 2 & 0 & 4 & 1 \\ -1 & 2 & 5 & 6 \\ 2 & -7 & 3 & -2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 3 & 2 & -1 \\ 2-2(1) & 0-2(3) & 4-2(2) & 1-2(-1) \\ -1+1 & 2+3 & 5+2 & 6-1 \\ 2-2(1) & -7-2(3) & 3-2(2) & -2-2(-1) \end{vmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & -6 & 0 & 3 \\ 0 & 5 & 7 & 5 \\ 0 & -13 & -1 & 0 \end{vmatrix}$$

Expanding by C_1

$$= -\left[1 \begin{vmatrix} -6 & 0 & 3 \\ 5 & 7 & 5 \\ -13 & -1 & 0 \end{vmatrix} - 0 + 0 - 0\right]$$

$$= -\begin{vmatrix} -6 & 0 & 3 \\ 5 & 7 & 5 \\ -13 & -1 & 0 \end{vmatrix}$$

Expanding by R_1

$$= -\left[-6 \begin{vmatrix} 7 & 5 \\ -1 & 0 \end{vmatrix} - 0 + 3 \begin{vmatrix} 5 & 7 \\ -13 & -1 \end{vmatrix}\right]$$

$$= -\left[-6\left(0-(-1)(5)\right) + 3\left((5)(-1)-(-13)(7)\right)\right]$$

$$= -\left[-6(5) + 3(-5+91)\right]$$

$$= -\left[-30 + 3(86)\right] = -228$$

(iii) $\begin{vmatrix} \mathbf{-3} & \mathbf{9} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{9} & \mathbf{7} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{vmatrix}$

Interchange C_1 and C_3

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ -1 & 3 & 0 & 2 \\ -1 & 7 & 9 & 1 \\ 1 & 0 & -2 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ -1+1 & 3+9 & 0-3 & 2+1 \\ -1+1 & 7+9 & 9-3 & 1+1 \\ 1-1 & 0-9 & -2+3 & -1-1 \end{vmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ 0 & 12 & -3 & 3 \\ 0 & 16 & 6 & 2 \\ 0 & -9 & 1 & -2 \end{vmatrix}$$

Expanding by C_1

$$= -1 \begin{vmatrix} 12 & -3 & 3 \\ 16 & 6 & 2 \\ -9 & 1 & -2 \end{vmatrix}$$

Expanding by R_1

$$= -\left[12 \begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix} - (-3) \begin{vmatrix} 16 & 2 \\ -9 & -2 \end{vmatrix} + 3 \begin{vmatrix} 16 & 6 \\ -9 & 1 \end{vmatrix} \right]$$

$$= -[12(-12-2) + 3(-32+18) + 3(16+54)]$$

$$= -[12(-14) + 3(-14) + 3(70)]$$

$$= -[-210 + 210] = 0$$

Q.8 Show that $\begin{vmatrix} \mathbf{x} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{x} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{x} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{x} \end{vmatrix} = \mathbf{(x+3)(x-1)^3}$

Solution:

L.H.S.

$$\text{L.H.S.} = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+1+1+1 & 1 & 1 & 1 \\ 1+x+1+1 & x & 1 & 1 \\ 1+1+x+1 & 1 & x & 1 \\ 1+1+1+x & 1 & 1 & x \end{vmatrix} \qquad C_1 + C_2 + C_3 + C_4$$

$$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & x & 1 \\ x+3 & 1 & 1 & x \end{vmatrix}$$

Take $(x+3)$ common from C_1

$$= (x+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

Expanding by C_1

$$= (x+3)\begin{bmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

Expanding by C_1

$$= (x+3)\left[(x-1)\begin{bmatrix} x-1 & 0 \\ 0 & x-1 \end{bmatrix}\right]$$

$$= (x+3)\left[(x-1)\left((x-1)(x-1)-(0)(0)\right)\right]$$

$$= (x+3)\left[(x-1)(x-1)(x-1)\right]$$

$$= (x+3)(x-1)^3$$

$$= \text{R H.S.}$$

Q.9 Find $|A\,A^t|$ and $|A^t\,A|$ if

(i) If $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{bmatrix}$ **(ii)** $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$

Solution:

(i) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{bmatrix}$

$$A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$A\,A^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(3)+(2)(2)+(-1)(-1) & (3)(2)+(2)(1)+(-1)(3) \\ (2)(3)+(1)(2)+(3)(-1) & (2)(2)+(1)(1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

Now $|A\,A^t| = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix} = (14)(14) - 5(5) = 196 - 25 = 171$

Now $A^t\,A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} (3)(3)+(2)(2) & (3)(2)+(2)(1) & (3)(-1)+(2)(3) \\ (2)(3)+(1)(2) & (2)(2)+(1)(1) & (2)(-1)+(1)(3) \\ (-1)(3)+(3)(2) & (-1)(2)+(3)(1) & (-1)(-1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

$$|A^t A| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

Expanding by R_1

$$= 13 \begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8 \begin{vmatrix} 8 & 1 \\ 3 & 10 \end{vmatrix} + 3 \begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 13\,(50-1) - 8\,(80-3) + 3\,(8-15)$$

$$= 13\,(49) - 8\,(77) + 3\,(-7)$$

$$= 637 - 616 - 21$$

$$= 637 - 637 = 0$$

(ii) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$

$$A^t = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$A\,A^t = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{bmatrix}$$

$$|A\,A^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$

$$= \begin{vmatrix} 25-3\,(7) & 10-3\,(3) & 7-3\,(2) & 18-3\,(5) \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18-2\,(7) & 7-2\,(3) & 5-2\,(2) & 13-2\,(5) \end{vmatrix} \begin{matrix} \\ \\ R_1 - 3R_3 \\ R_4 - 2R_3 \end{matrix}$$

$$= \begin{vmatrix} 4 & 1 & 1 & 3 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 4 & 1 & 1 & 3 \end{vmatrix}$$

As R_1 and R_4 are same so det will be zero.

$$A^t A = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(3)+(2)(2)+(1)(1)+(2)(2) & (3)(4)+(2)(1)+(1)(1)+(2)(3) \\ (4)(3)+(1)(2)+(1)(1)+(3)(2) & (4)(4)+(1)(1)+(1)(1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 21 \\ 21 & 27 \end{bmatrix}$$

$$|A^t A| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix}$$

$$= (18)(27) - (21)(21) = 486 - 441 = 45$$

Q.10 If 'A' is a square matrix of order 3, then show that $|KA| = K^3 |A|$

Solution:

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

$$|kA| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Take 'k' common from R_1, R_2 and R_3

$$= k.k.k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$|kA| = k^3 |A| =$ R.H.S.

Hence proved.

Q.11 Find value of 'λ' if A and B are singular

(i) $\mathbf{A} = \begin{bmatrix} \mathbf{4} & \boldsymbol{\lambda} & \mathbf{3} \\ \mathbf{7} & \mathbf{3} & \mathbf{6} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$

(ii) $\mathbf{B} = \begin{bmatrix} \mathbf{5} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{8} & \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \boldsymbol{\lambda} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

Solution:

(i) $\mathbf{A} = \begin{bmatrix} \mathbf{4} & \boldsymbol{\lambda} & \mathbf{3} \\ \mathbf{7} & \mathbf{3} & \mathbf{6} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$

As A is singular so $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Expanding by R_1

$$\Rightarrow \quad 4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad 4(3-18) - \lambda(7-12) + 3(21-6) = 0$$

$$\Rightarrow \quad 4(-15) - \lambda(-5) + 3(15) = 0$$

$$\Rightarrow \quad -60 + 5\lambda + 45 = 0$$

$$\Rightarrow \quad -15 + 5\lambda = 0 \Rightarrow 5(-3+\lambda) = 0$$

$$\Rightarrow \quad -3 + \lambda = 0 \Rightarrow \boxed{\lambda = 3}$$

(ii) $\mathbf{B} = \begin{bmatrix} \mathbf{5} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{8} & \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \boldsymbol{\lambda} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

As 'B' is a singular so $|B| = 0$

$$\Rightarrow \quad \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

Interchange C_1 and C_4

$$\Rightarrow \quad -\begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 1 & 2 & 0 & 3 \\ 3 & \lambda & -1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 1-1 & 2-2 & 0-5 & 3-8 \\ 3-3(1) & \lambda-3(2) & -1-3(5) & 2-3(8) \end{vmatrix} = 0 \quad \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - 3R_2 \end{matrix}$$

$$\Rightarrow \quad -\begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 0 & 0 & -5 & -5 \\ 0 & \lambda-6 & -16 & -22 \end{vmatrix} = 0$$

Expanding by C_1

$$\Rightarrow \quad (-1)(-1)\begin{vmatrix} 1 & 2 & 5 \\ 0 & -5 & -5 \\ \lambda-6 & -16 & -22 \end{vmatrix} = 0$$

Expanding by C_1

$$\Rightarrow \quad \begin{vmatrix} -5 & -5 \\ -16 & -22 \end{vmatrix} - 0 + (\lambda-6)\begin{vmatrix} 2 & 5 \\ -5 & -5 \end{vmatrix} = 0$$

$$\Rightarrow \quad (110-80) + (\lambda-6)(15) = 0$$

$$\Rightarrow \quad 30 + (\lambda-6)(15) = 0$$

$$\Rightarrow \quad 30 + 15\lambda - 90 = 0$$

$$\Rightarrow \quad 15\lambda - 60 = 0$$

$$\Rightarrow \quad 15(\lambda-4) = 0$$

$$\Rightarrow \quad \lambda - 4 = 0 \Rightarrow \boxed{\lambda = 4}$$

Q.12 Which of the following matrices are singular and which of them are non–singular.

(i) $\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \end{vmatrix}$

(ii) $\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{5} \end{vmatrix}$

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{4} \end{vmatrix}$

Solution:

(i) $\mathbf{A} = \begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \end{vmatrix}$

Expanding by R_1

$= \quad 1\,(4 - (-2)) - 0 + 3\,(6 - 0)$

$= \quad 4 + 2 + 3\,(6) = 6 + 18 = 24 \neq 0$

$\Rightarrow$ A is non–singular.

(ii) $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{5} \end{bmatrix}$

$|A| = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

Expanding by R_1

$= 2\,(5 - 0) - 3\,(5 - 0) + (-1)\,(-3 - 2)$

$= 2\,(5) - 3\,(5) - 1\,(-5)$

$= 10 - 15 + 5 = 0 \Rightarrow$ A is singular.

(iii) $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

Let

$$|A| = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1-1 & 2-1 & -1-2 & -3-(-1) \\ 2-2\,(1) & 3-2\,(1) & 1-2\,(2) & 2-2\,(-1) \\ 3-3\,(1) & -1-3\,(1) & 3-3\,(2) & 4-3\,(-1) \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & -4 & -3 & 7 \end{vmatrix}$$

Expanding by C_1

$$= 1 . \begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix}$$

Expanding by R_1

$$= 1 \begin{vmatrix} -3 & 4 \\ -3 & 7 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 4 \\ -4 & 7 \end{vmatrix} - 2 \begin{vmatrix} 1 & -3 \\ -4 & -3 \end{vmatrix}$$

$$= (-21 - (-12)) + 3 (7 - (-16)) - 2 (-3 - 12)$$

$$= -21 + 12 + 3 (7 + 16) - 2 (-15)$$

$$= -21 + 12 + 69 + 30$$

$$= 90 \neq 0$$

$\Rightarrow$ A is non–singular.-

Q.13 Find inverse of $\mathbf{A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}}$ **and show that** $\mathbf{A^{-1} A = I_3}$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

As $A^{-1} = \dfrac{adj}{|A|}$ ..... (1)

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + 0$$

$$= 2 (5 - 0) - 1 (5 - 0)$$

$$= 10 - 5 = 5 \neq 0$$

Now

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t \quad \text{..... (2)}$$

where

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}$$

$$= 5 - 0 = 5$$

Expanding by C_1

$$= 1 . \begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix}$$

Expanding by R_1

$$= 1 \begin{vmatrix} -3 & 4 \\ -3 & 7 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 4 \\ -4 & 7 \end{vmatrix} - 2 \begin{vmatrix} 1 & -3 \\ -4 & -3 \end{vmatrix}$$

$$= (-21 - (-12)) + 3 (7 - (-16)) - 2 (-3 - 12)$$

$$= -21 + 12 + 3 (7 + 16) - 2 (-15)$$

$$= -21 + 12 + 69 + 30$$

$$= 90 \neq 0$$

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Q.13 Find inverse of $\mathbf{A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}}$ **and show that** $\mathbf{A^{-1} A = I_3}$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

As $A^{-1} = \dfrac{adj}{|A|}$ ..... (1)

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + 0$$

$$= 2 (5 - 0) - 1 (5 - 0)$$

$$= 10 - 5 = 5 \neq 0$$

Now

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t \quad \text{..... (2)}$$

where

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}$$

$$= 5 - 0 = 5$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & \frac{8}{5} & \frac{1}{5} \end{bmatrix}$$

To show $A^{-1} A = I_3$

$$A^{-1} A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & \frac{8}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2) + (-1)(1) + (0)(2) & (1)(1) + (-1)(1) + (0)(-3) & (1)(0) + (-1)(0) + (0)(5) \\ (-1)(2) + (2)(1) + (0)(2) & (-1)(1) + (2)(1) + (0)(-3) & (-1)(0) + (2)(0) + (0)(5) \\ (-1)(2) + \left(\frac{8}{5}\right)(1) + \left(\frac{1}{5}\right)(2) & (-1)(1) + \left(\frac{8}{5}\right)(1) + \left(\frac{1}{5}\right)(-3) & (-1)(0) + \left(\frac{8}{5}\right)(0) + \left(\frac{1}{5}\right)(5) \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I_3$$

Q.14 Verify that $(AB)^{-1} = B^{-1} . A^{-1}$ if

(i) $\mathbf{A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}} \quad \mathbf{B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}}$

(ii) $\mathbf{A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}}, \quad \mathbf{B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}}$

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-3) + (2)(4) & (1)(1) + (2)(-1) \\ (-1)(-3) + (0)(4) & (-1)(1) + (0)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 8 & 1 - 2 \\ 3 + 0 & -1 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix}$$

As

$$(AB)^{-1} = \frac{\text{adj } (AB)}{\det (AB)} \qquad \ldots\ldots\ldots (1)$$

$$\det (AB) = |AB| = \begin{vmatrix} -1 & 1 \\ -3 & -5 \end{vmatrix} = -5 - (3) = -5 + 3 = -2$$

$$\text{adj } (AB) = \begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$$

Put values in equation (1)

$$(AB)^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

Now

$$B^{-1} = \frac{\text{adj}B}{|B|} \qquad \ldots\ldots\ldots (2)$$

$$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = (-3)(-1) - (4)(1) = 3 - 4 = -1$$

$$\text{AdJ } B = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$$

Put values in equation (2)

$$B^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

Now

$$A^{-1} = \frac{\text{adj}A}{|A|} \qquad \ldots\ldots\ldots (3)$$

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = (0)(1) - (2)(-1) = 2$$

$$\text{adJ} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

put values in equation (3)

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{0}{2} & \frac{-2}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(0) + (1)\left(\frac{1}{2}\right) & (1)(-1) + (1)\left(\frac{1}{2}\right) \\ (4)(0) + (3)\left(\frac{1}{2}\right) & (4)(-1) + (3)\left(\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -1 + \frac{1}{2} \\ \frac{3}{2} & -4 + \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix} \qquad \ldots\ldots\ldots \text{(II)}$$

From (I) & (II)

$(AB)^{-1} = B^{-1}A^{-1}$ proved.

Q.15 Verify that $(AB)^t = B^t A^t$ and if

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} & \mathbf{1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

Solution:

Given that

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$(AB) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (-1)(3) + (2)(0) & (1)(1) + (-1)(2) + (2)(-1) \\ (0)(1) + (3)(3) + (1)(0) & (0)(1) + (3)(2) + (1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 1-2-2 \\ 9+0 & 6-1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 9 & 5 \end{bmatrix}$$

$$(AB)^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \qquad \ldots\ldots\ldots \text{(a)}$$

Now

$$B^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B^t . A^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{vmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} (1)(1) + (3)(-1) + (0)(2) & (1)(0) + (3)(3) + (0)(1) \\ (1)(1) + (2)(-1) + (-1)(2) & (1)(0) + (2)(3) + (-1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 3 & 9 \\ 1 - 2 - 2 & 6 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \qquad \ldots\ldots\ldots (b)$$

From (a) & (b)

$(AB)^t = B^t A^t$

Q.16 Verify that $(A^{-1})^t = (A^t)^{-1}$ if $A = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} \end{bmatrix}$

Solution:

Given that

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adJ } A}{|A|} \qquad \ldots\ldots\ldots (1)$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 - (-3) = 5$$

$$\text{AdJ } A = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

Put values in (1)

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$(A^{-1})^t = \begin{bmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \qquad \ldots\ldots\ldots (a)$$

Now

$$A^t = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$(A^t)^{-1} = \frac{\text{adJ } (A^t)}{|A^t|} \qquad \ldots\ldots\ldots (2)$$

$$\text{adj } A^t = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$|A^t| = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2 - (-3) = 5$$

$$\text{Adj } A^t = \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}$$

Put values in (2)

$$(A^t)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{vmatrix} \qquad \ldots\ldots\ldots (B)$$

From (a) & (b)

$(A^{-1})^t = (A^t)^{-1}$

Hence proved.

Q.17 If A and B are non–singular matrices, then show that

(i) $(AB)^{-1} = B^{-1} A^{-1}$ (ii) $(A^{-1})^{-1} = A$

Solution:

(i) $(AB)^{-1} = B^{-1} A^{-1}$

We know that.

$AA^{-1} = I$

$BB^{-1} = I$

$(AB)(AB)^{-1} = I$ ..... (1)

We take

$(AB)(B^{-1}A^{-1}) = I$ ..... (2)

L.H.S $= (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1}$

$= I$

$=$ R.H.S

From equations (1) and (2), we have $(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow$

$B^{-1}A^{-1}$ is inverse of AB.

$\Rightarrow$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^{-1})^{-1} = A$

As $AA^{-1} = I$ and $A^{-1}A = I$

$\Rightarrow$

A^{-1} is inverse of A

so $(A^{-1})^{-1} = A$. Hence proved.

Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called symmetric if $A^t = A$.

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called skew symmetric matrix if $A^t = A$

Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ with complex entries is called Hermitian matrix if $(\bar{A})^t = A$

Skew Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ with complex enteries is called skew Hermitian if $(\bar{A})^t = -A$

Echelon Form of a Matrix

A matrix A is called in (row) echelon form if

(i) in each successive non–zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

(ii) the first non–zero entry (or leading entry) in each row is 1.

For example

$$\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in echelon form.

Reduced Echelon Form of a Matrix

A matrix A is in reduced echelon form if it is in echelon form and if the first non–zero entry in R_i lies in C_j then all other entries of C_j are zero.

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in reduced echelon form.

Rank of a Matrix

If a matrix A is in reduced echelon form then the number of non–zero rows of matrix A is called the rank of the matrix A.

EXERCISE 3.4

Q.1 **If** $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$

then show that (A + B) is symmetric.

Solution:

$$A + B = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & -2+1 & 5-2 \\ -2+1 & 3+0 & -1-1 \\ 5-2 & -1-1 & 0+2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

$$(A + B)^t = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix} = A + B$$

$\Rightarrow$ $A + B$ is a symmetric matrix.

Q.2 **If** $\mathbf{A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}}$ **show that**

(i) $\mathbf{A + A^t}$ **is symmetric** **(ii)** $\mathbf{A - A^t}$ **is skew symmetric**

Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3-1 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} = A + A^t$$

$\Rightarrow$ $(A + A^t)^t = (A + A^t)$

$\Rightarrow$ $A + A^t$ is symmetric.

(ii) $\mathbf{A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}}$

$$A^t = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$(A - A^t) = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-3 & 0+1 \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3-(-1) & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$(A - A^t)^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = -(A - A^t)$$

$\Rightarrow$ $(A - A^t)$ is a skew symmetric.

Q.3 If A is any square matrix of order 3, show that

(i) $A + A^t$ is symmetric

(ii) $(A - A^t)$ is skew–symmetric.

Solution:

(i) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is a square matrix of order 3, then

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & a_{31}+a_{13} \\ a_{12}+a_{21} & a_{22}+a_{22} & a_{32}+a_{23} \\ a_{13}+a_{31} & a_{23}+a_{32} & a_{33}+a_{33} \end{bmatrix} = A + A^t$$

$\Rightarrow$ $(A + A^t)^t = A + A^t$

$\Rightarrow$ $A + A^t$ is symmetric.

(ii) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is a square matrix of order 3, then

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & a_{22} - a_{22} & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & a_{33} - a_{33} \end{bmatrix}$$

$$(A - A^t)^t = \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & a_{22} - a_{22} & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & a_{33} - a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11} - a_{11} & a_{21} - a_{12} & a_{31} - a_{13} \\ a_{12} - a_{21} & a_{22} - a_{22} & a_{32} - a_{23} \\ a_{13} - a_{31} & a_{23} - a_{32} & a_{33} - a_{33} \end{bmatrix}$$

$$= -\begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & a_{22} - a_{22} & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & a_{33} - a_{33} \end{bmatrix}$$

$\Rightarrow \quad (A - A^t)^t = -(A - A^t)$

$\Rightarrow$ $A - A^t$ is a skew symmetric matrix.

Q.4 If the matrices A and B are symmetric and AB = BA, show that AB is symmetric.

Solution:

If the matrices A and B are symmetric then

$A^t = A$ ..... (1)

and $B^t = B$ ..... (2)

also $AB = BA$ ..... (3)

To show that AB is symmetric

We will prove that $(AB)^t = AB$

Now take $(AB)^t = B^t A^t$ by definition

$= BA$ from (1) and (2)

$= AB$ from (3)

$\Rightarrow \quad (AB)^t = AB$

$\Rightarrow$ AB is symmetric.

Q.5 Show that AA^t and A^tA are symmetric for any matrix of order 2 x 3.

Solution:

Consider a matrix A of order 2 x 3 such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A\,A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}.a_{11} + a_{12}.a_{12} + a_{13}.a_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{21}.a_{11} + a_{22}.a_{12} + a_{23}.a_{13} & a_{21}.a_{21} + a_{22}.a_{22} + a_{23}.a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a_{21}^2 + a_{22}^2 + a_{23}^2 \end{bmatrix}$$

$$(A\,A^t)^t = \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{21}^2 + a_{22}^2 + a_{23}^2 \end{bmatrix}$$

$$(A\,A^t)^t = A\,A^t$$

$\Rightarrow$ $A\,A^t$ is symmetric.

Now $$A^t\,A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}a_{11}+a_{21}a_{21} & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{12}a_{11}+a_{22}a_{21} & a_{12}a_{12}+a_{22}a_{22} & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}a_{13}+a_{23}a_{23} \end{bmatrix}$$

$$A^t\,A = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{12}a_{11}+a_{22}a_{21} & a_{12}^2 + a_{22}^2 & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2 + a_{23}^2 \end{bmatrix}$$

$$(A^t\,A)^t = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{12}a_{11} + a_{22}a_{21} & a_{13}a_{11} + a_{23}a_{21} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 & a_{13}a_{12} + a_{23}a_{22} \\ a_{11}a_{13}+a_{21}a_{23} & a_{12}a_{13} + a_{22}a_{23} & a_{13}^2 + a_{23}^2 \end{bmatrix}$$

$$(A^t\,A)^t = (A^t\,A)$$

$\Rightarrow$ $A^t\,A)$ is symmetric.

Q.6 If

$$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}, \text{ show that}$$

(i) $\mathbf{A + (\bar{A})^t}$ **is Hermitian** **(ii)** $\mathbf{A - (\bar{A})^t}$ **is skew Hermitian.**

Solution:

(i) $$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & -1 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -1-1 \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} 0 & 2+i \\ 2-i & -2 \end{bmatrix}$$

$$\overline{A + (\bar{A})^t} = \begin{bmatrix} 0 & 2-i \\ 2+i & -2 \end{bmatrix}$$

$$\left(\overline{A + (\bar{A})^t}\right)^t = \begin{bmatrix} 0 & 2+i \\ 2-i & -2 \end{bmatrix}$$

$$\left(\overline{A + (\bar{A})^t}\right)^t = A + (\bar{A})^t$$

$\Rightarrow$ $A + (\bar{A})^t$ is Hermitian.

(ii) $$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & -1 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$A-(\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i-(-i) & 1+i-1 \\ 1-(1-i) & -1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} i+i & i \\ 1-1+i & -1+1 \end{bmatrix} \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$A-(\bar{A})^t = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$\overline{A+(\bar{A})^t} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t = -(A-(\bar{A})^t)$$

$\Rightarrow$ $A-(\bar{A})^t$ is skew Hermitian.

Q.7 If A is symmetric or skew symmetric show that A^2 is symmetric.

Solution:

Let A is symmetric

$\Rightarrow$ $A^t = A$ ..... (1)

To show that A^2 is symmetric, we will show that $(A^2)^t = A^2$

$\Rightarrow$ Take

$(A^2)^t = (A.A)^t = A^t . A^t = A . A = A^2$ from (1)

$(A^2)^t = A^2$

$\Rightarrow$ A^2 is symmetric.

Now let A is skew symmetric

$\Rightarrow$ $A^t = -A$ ..... (2)

To show that $(A^2)^t = A^2$

Take

$(A^2)^t = (A\,A)^t = A^t . A^t = (-A)(-A) = A^2$ from (2)

$\Rightarrow$ $(A^2)^t = A^2$

$\Rightarrow$ A^2 is symmetric.

Q.8 If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$ find $A\left(\bar{A}\right)^t$.

Solution:

$$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

$$\left(\bar{A}\right)^t = [1 \quad 1-i \quad -i]$$

$$A\left(\bar{A}\right)^t = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}[1 \quad 1-i \quad -i]$$

$$= \begin{bmatrix} (1)(1) & (1)(1-i) & (1)(-i) \\ (1+i)(1) & (1+i)(1-i) & (1+i)(-i) \\ (i)(1) & (i)(1-i) & (i)(-i) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-(-1) & -i-(-1) \\ i & i-(-1) & -(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & i+1 & 1 \end{bmatrix}$$

Q.9 **Find the inverse of the following matrices. Also find their inverses by using row and column operations.**

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution:

(i) By Adjoint Method

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1\left((-2)(2) - 0\right) - 2\left((0)(2) - (-2)(0)\right) - 3\left((0)(-2) - (-2)(-2)\right)$$

$$= -4 - 2(0) - 3(0 - 4)$$

$= -4 - 3(-4) = -4 + 12 = 8 \neq 0$

$|A| = \neq 0 \Rightarrow A^{-1}$ exists.

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = (-1)^2\big((-2)(2) - (0)(-2)\big) = (1)(-4+0) = -4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = (-1)^3(0-0) = 0$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} = (-1)^4\big(0-(-2)(-2)\big) = -4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = (-1)^3(4-6) = -1(-2) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = (-1)^4(2-6) = -4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = (-1)^5(-2+4) = -(2) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = (-1)^4(0-6) = 1(-6) - 6$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)^5(0-0) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = (-1)^6(-2-0) = 1(-2) = -2$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 0 & -4 \\ 2 & -4 & -2 \\ -6 & 0 & -2 \end{bmatrix}^t$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{8} & \frac{2}{8} & -\frac{6}{8} \\ \frac{0}{8} & -\frac{4}{8} & \frac{0}{8} \\ -\frac{4}{8} & -\frac{2}{8} & -\frac{2}{8} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

To find A^{-1} by row operations:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2+2(1) & -2+2(2) & -2+2(-3) & 0+2(1) & 0+2(0) & 1+2(0) \end{array}\right] R_3+2R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right] \text{By } -\frac{1}{2} R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1+(-2)(0) & 2+(-2)(1) & -3+(-2)(0) & 1+(-2)(0) & 0+(-2)-\frac{1}{2} & 0+(-2)(0) \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0+(-2)(0) & 2+(-2)(1) & -4+(-2)(0) & 2+(-2)(0) & 0+(-2)\left(-\frac{1}{2}\right) & 1+(-2)(0) \end{array}\right] \begin{matrix} R_1+(-2)R_2 \\ R_3+(-2)R_2 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \text{By } -\frac{1}{4} R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3+3(1) & 1+3\left(-\frac{1}{2}\right) & 1+3\left(\frac{1}{4}\right) & 0+3\left(-\frac{1}{4}\right) \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \text{By } R_1 + 3R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

To find A^{-1} by using column operations.

Taking I_3 as

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 2+(-2)(1) & -3+3(1) \\ 0 & -2+(-2)(0) & 0+3(0) \\ -2 & -2+(-2)(-2) & 2+3(0) \\ -- & -- & -- \\ 1 & 0+(-2)(1) & 0+3(1) \\ 0 & 1+(-2)(0) & 0+3(0) \\ 0 & 0+(-2)(0) & 1+3(0) \end{bmatrix} \quad \begin{matrix} \text{By} \\ C_2 + (-2)\,C_1 \\ C_3 + 3C_1 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ -- & -- & -- \\ 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -4 \\ -- & -- & -- \\ 1 & 1 & 3 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By} \quad \left(-\frac{1}{2}\right) C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \\ -- & -- & -- \\ 1 & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \text{ By } \left(-\frac{1}{4}\right) C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2+2(1) & -1+1 & 1 \\ -- & -- & -- \\ 1+2\left(-\frac{3}{4}\right) & 1-\frac{3}{4} & -\frac{3}{4} \\ 0+2(0) & -\frac{1}{2}+0 & 0 \\ 0+2\left(-\frac{1}{4}\right) & 0-\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \text{ By } \begin{matrix} C_1 + 2C_3 \\ C_2 + C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -- & -- & -- \\ -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \quad |A| = 1(-2-0) + 1(6-1) = -2+5 = 3 \neq 0$

$\Rightarrow \quad |A| = \neq 0 \Rightarrow$ inverse of A exists.

Now $A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = (-1)^2(-2+0) = -2$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^3(0-3) = -(3) = 3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (-1)^4(0+1) = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = (-1)^3(4-0) = -4$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = (-1)^4(2+1) = 3$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^5(0-2) = 2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = (-1)^4(6-1) = 1(5) = 5$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = (-1)^5(3-0) = -3$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = (-1)^6(-1-0) = -1$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 3 & 1 \\ -4 & 3 & 2 \\ 5 & -3 & -1 \end{bmatrix}^t = \frac{1}{3}\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

To find A^{-1} by using row operations. Take I_3 as

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1-1 & 0-2 & 2+1 & 0-1 & 0-0 & 1-0 \end{array}\right] \quad \text{By } R_3 - R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & | & 1 & 0 & 0 \\ 0 & -1 & 3 & | & 0 & 1 & 0 \\ 0 & -2 & 3 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & 0 & -1 & 0 \\ 0 & -2 & 3 & | & -1 & 0 & 1 \end{bmatrix} \quad \text{By } (-1)\,R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2-2(1) & -1-2(-3) & | & 1-2(0) & 0-2(-1) & 0-2(0) \\ 0 & 1 & -3 & | & 0 & -1 & 0 \\ 0 & -2+2(1) & 3+2(-3) & | & -1+2(0) & 0+2(-1) & 1+2(0) \end{bmatrix} \quad \text{By } \begin{matrix} R_1 - 2R_2 \\ R_3 + 2R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 5 & | & 1 & 2 & 0 \\ 0 & 1 & -3 & | & 0 & -1 & 0 \\ 0 & 0 & -3 & | & -1 & -2 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 5 & : & 1 & 2 & 0 \\ 0 & 1 & -3 & : & 0 & -1 & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \text{By } \left(-\frac{1}{3}\right) R_3$$

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 5-5(1) & : & 1-5\left(\frac{1}{3}\right) & 2-5\left(\frac{2}{3}\right) & 0-5\left(-\frac{1}{3}\right) \\ 0 & 1 & -3+3(1) & : & 0+3\left(\frac{1}{3}\right) & -1+3\left(\frac{2}{3}\right) & 0+3\left(-\frac{1}{3}\right) \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \text{By } \begin{matrix} R_1 - 5R_3 \\ R_2 + 3R_3 \end{matrix}$$

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 0 & : & -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 0 & : & 1 & 1 & -1 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

To find A^{-1} by using column operations.

Taking I_3 as

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 2-2(1) & -1+1 \\ 0 & -1-2(0) & 3+0 \\ 1 & 0-2(1) & 2+1 \\ -- & --- & --- \\ 1 & 0-2(1) & 0+1 \\ 0 & 1-2(0) & 0+0 \\ 1 & 0-2(0) & 1+(0) \end{bmatrix} \quad \begin{matrix} \text{By} \\ C_2 + (-2)\,C_1 \\ C_3 + C_1 \end{matrix}$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & -2 & 3 \\ -- & -- & -- \\ 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \\ -- & -- & -- \\ 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By} \quad (-1)\, C_2$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3-3\,(1) \\ 1 & 2 & 3-3\,(2) \\ -- & -- & -- \\ 1 & 2 & 1-3\,(2) \\ 0 & -1 & 0-3\,(-1) \\ 0 & 0 & 1-3\,(0) \end{bmatrix} \quad \text{By} \quad C_3 - 3\,C_2$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \\ -- & -- & -- \\ 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \\ -- & -- & -- \\ 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \quad \text{By} \quad \left(-\frac{1}{3}\right) C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-1 & 2-2(1) & 1 \\ -- & -- & -- \\ 1-\frac{5}{3} & 2-2\left(\frac{5}{3}\right) & \frac{5}{3} \\ 0+1 & -1-2(-1) & -1 \\ 0+\frac{1}{3} & 0-2\left(-\frac{1}{3}\right) & -\frac{1}{3} \end{bmatrix} \text{ By } \begin{matrix} C_1 - C_3 \\ C_2 - 2C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -- & -- & -- \\ -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

(iii) $\mathbf{A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$

To find inverse by adjoint method

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$= 1(1-0) + 3(2-0) + 2(-2-0) = 1 + 6 - 4 = 3 \neq 0$

$\Rightarrow$ $|A| \neq 0 \Rightarrow$ inverse of A exists.

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = (-1)^2(1-0) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = (-1)^3(2-0) = -2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-2-0) = -2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 2 \\ -1 & 1 \end{vmatrix} = (-1)^3(-3+2) = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)^4(1-0) = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} = (-1)^5(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^4(0-2) = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = (-1)^5(0-4) = 4$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = (-1)^6(1+6) = 7$$

As

$$A^{-1} = \frac{\text{AdJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ 1 & 1 & 1 \\ -2 & 4 & 7 \end{bmatrix}^t$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 4 \\ -2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

To find A^{-1} by using row operations.

Take I_3 as

$$\left[\begin{array}{ccc:ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 2-2(1) & 1-2(-3) & 0-2(2) & 0-2(1) & 1-2(0) & 0-2(0) \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ By } R_2 - 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 0 & 7 & -4 & : & -2 & 1 & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & \frac{-4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \text{ By } \left(\frac{1}{7}\right) R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3+3(1) & 2+3\left(-\frac{4}{7}\right) & : & 1+3\left(-\frac{2}{7}\right) & 0+3\left(\frac{1}{7}\right) & 0+3(0) \\ 0 & 1 & \frac{-4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & -1+1 & 1-\frac{4}{7} & : & 0-\frac{2}{7} & 0+\frac{1}{7} & 1+0 \end{bmatrix} \text{ By } \begin{matrix} R_1+3R_2 \\ R_3+R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7} & : & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{3}{7} & : & \frac{-2}{7} & \frac{1}{7} & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7} & : & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & : & \frac{-2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \text{ By } \left(\frac{7}{3}\right) R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7}-\frac{2}{7}(1) & : & \frac{1}{7}-\frac{2}{7}\left(-\frac{2}{3}\right) & \frac{3}{7}-\frac{2}{7}\left(\frac{1}{3}\right) & 0-\frac{2}{7}\left(\frac{7}{3}\right) \\ 0 & 1 & -\frac{4}{7}+\frac{4}{7}(1) & : & -\frac{2}{7}+\frac{4}{7}\left(-\frac{2}{3}\right) & \frac{1}{7}+\frac{4}{7}\left(\frac{1}{3}\right) & 0+\frac{4}{7}\left(\frac{7}{3}\right) \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \text{ By } \begin{matrix} R_1-\left(\frac{2}{7}\right)R_3 \\ R_2+\left(\frac{4}{7}\right)R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & : & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

To find A^{-1} by using column operations:

Take I_3 as

$$\left[\begin{array}{ccc} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{C} \left[\begin{array}{ccc} 1 & -3+3(1) & 2-2(1) \\ 2 & 1+3(2) & 0-2(2) \\ 0 & -1+3(0) & 1-2(0) \\ -- & --- & --- \\ 1 & 0+3(1) & 0-2(1) \\ 0 & 1+3(0) & 0-2(0) \\ 0 & 0+3(0) & 1-2(0) \end{array}\right] \text{ By } \begin{array}{l} C_2 + 3C_1 \\ C_3 - 2C_1 \end{array}$$

$$\underset{\sim}{C} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \\ -- & -- & -- \\ 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{C} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -\frac{1}{7} & 1 \\ -- & -- & -- \\ 1 & \frac{3}{7} & -2 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ By } \left(\frac{1}{7}\right) C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2-2(1) & 1 & -4+4(1) \\ 0-2\left(-\frac{1}{7}\right) & -\frac{1}{7} & 1+4\left(-\frac{1}{7}\right) \\ --- & -- & --- \\ 1-2\left(+\frac{3}{7}\right) & \frac{3}{7} & -2+4\left(\frac{3}{7}\right) \\ 0-2\left(\frac{1}{7}\right) & \frac{1}{7} & 0+4\left(\frac{1}{7}\right) \\ 0-2(0) & 0 & 1+4(0) \end{bmatrix} \text{ By } \begin{matrix} C_1-2C_2 \\ C_3+4C_2 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \\ -- & -- & -- \\ \frac{1}{7} & \frac{3}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} & 1 \\ -- & -- & -- \\ \frac{1}{7} & \frac{3}{7} & -\frac{2}{3} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{3} \\ 0 & 0 & \frac{7}{3} \end{bmatrix} \text{ By } \left(\frac{7}{3}\right)C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7}-\frac{2}{7}(1) & -\frac{1}{7}+\frac{1}{7}(1) & 1 \\ --- & --- & -- \\ \frac{1}{7}-\frac{2}{7}\left(-\frac{2}{3}\right) & \frac{3}{7}+\frac{1}{7}\left(-\frac{2}{3}\right) & -\frac{2}{3} \\ -\frac{2}{7}-\frac{2}{7}\left(\frac{4}{3}\right) & \frac{1}{7}+\frac{1}{7}\left(\frac{4}{3}\right) & \frac{4}{3} \\ 0-\frac{2}{7}\left(\frac{7}{3}\right) & 0+\frac{1}{7}\left(\frac{7}{3}\right) & \frac{7}{3} \end{bmatrix} \text{ By } \begin{matrix} C_1-\left(\frac{2}{7}\right)C_3 \\ C_2+\left(\frac{1}{7}\right)C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ - & - & - \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Q.10 Find the rank of the following matrices

(i) $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{-6} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{-3} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{-7} \\ \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-7} & \mathbf{4} \end{bmatrix}$

(iii) $\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$

Solution:

(i) $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{-6} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{-3} \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2-2(1) & -6-2(-1) & 5-2(2) & 1-2(1) \\ 3-3(1) & 5-3(-6) & 4-3(5) & -3-3(1) \end{bmatrix} \text{By} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8 & -2 & -6 \end{bmatrix} \text{By} \left(\frac{-1}{4}\right) R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8-8(1) & -2-8\left(-\frac{1}{4}\right) & -6-8\left(\frac{1}{4}\right) \end{bmatrix} \text{ By } R_3 - 8R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1+1 & 2-\frac{1}{4} & 1+\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix} \text{ By } R_1 + R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

since There are three non–zero rows

$\Rightarrow$ Rank = 3

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{-7} \\ \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-7} & \mathbf{4} \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 2-2(1) & -5-2(-4) & 1-2(-7) \\ 1-1 & -2+4 & 3+7 \\ 3-3(1) & -7-3(-4) & 4-3(-7) \end{bmatrix} \text{ By } \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \text{ By } \left(\frac{1}{3}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4+4(1) & -7+4(5) \\ 0 & 1 & 5 \\ 0 & 2-2(1) & 10-2(5) \\ 0 & 5-5(1) & 25-5(5) \end{bmatrix} \text{ By } \begin{matrix} R_1+4R_2 \\ R_3-2R_2 \\ R_4-5R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are two non–zero rows

$\Rightarrow$ Rank = 2

(iii) $$\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \text{ interchanging } R_1 \text{ \& } R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3-3(1) & -1-3(2) & 3-3(-1) & 0-3(-3) & -1-3(-2) \\ 2-2(1) & 3-2(2) & 4-2(-1) & 2-2(-3) & 5-2(-2) \\ 2-2(1) & 5-2(2) & -2-2(-1) & -3-2(-3) & 3-2(-2) \end{bmatrix} \text{ By } \begin{matrix} R_2-3R_1 \\ R_3-2R_1 \\ R_4-2R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix} \text{ Interchanging } R_2 \text{ \& } R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2-2(1) & -1-2(0) & -3-2(3) & -2-2(7) \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1+1 & 6+0 & 8+3 & 9+7 \\ 0 & -7+7(1) & 6+7(0) & 9+7(3) & 5+7(7) \end{bmatrix} \text{ By } \begin{matrix} R_1-2R_2 \\ R_3+R_2 \\ R_4+7R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 6 & 30 & 54 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 30 & 54\\ 0 & 0 & 6 & 11 & 16\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & 5 & 9\\ 0 & 0 & 6 & 11 & 16\end{bmatrix} \text{ By } \left(\frac{1}{6}\right)R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1+1 & -9+5 & -16+9\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & 5 & 9\\ 0 & 0 & 6-6(1) & 11-6(5) & 16-6(9)\end{bmatrix} \text{ By } \begin{matrix}R_1+R_3\\ R_4-6R_3\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & -4 & -7\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & 5 & 9\\ 0 & 0 & 0 & -19 & -38\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & -4 & -7\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & 5 & 9\\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \text{ By } \left(-\frac{1}{19}\right)R_4$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & -4+4(1) & -7+4(2)\\ 0 & 1 & 0 & 3-3(1) & 7-3(2)\\ 0 & 0 & 1 & 5-5(1) & 9-5(2)\\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \text{ By } \begin{matrix}R_1+4R_4\\ R_2-3R_4\\ R_3-5R_4\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

There are four non–zero rows

$\Rightarrow$ Rank = 4

EXERCISE 3.5

Q.1 Solve the following systems of linear equations by Cramer rule.

(i) $2x + 2y + z = 13$
$3x - 2y - 2z = 1$
$5x + y - 3z = 2$

(ii) $2x_1 - x_2 + x_3 = 5$
$4x_1 + 2x_2 + 3x_3 = 8$
$3x_1 - 4x_2 - x_3 = 3$

(iii) $2x_1 - x_2 + x_3 = 8$
$x_1 + 2x_2 + 2x_3 = 6$
$x_1 - 2x_2 - x_3 = 1$

Solution:

(i) Given system of linear equations is

$2x + 2y + z = 13$

$3x - 2y - 2z = 1$

$5x + y - 3z = 2$

In matrix form

Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix} = 2(6+2) - 2(-9+10) - 1(3+10)$$

$$= 16 - 2 + 13) = 27 \neq 0$$

As $|A| \neq 0 \Rightarrow$ solution exists.

Now by Cramer's rule.

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27}$$

$$= \frac{3(6+2) - 2(-3+4) + 1(1+4)}{27}$$

$$= \frac{24 - 2 + 5}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27}$$

$$= \frac{2(-3+4) - 3(-9+10) + 1(6-5)}{27}$$

$$= \frac{2 - 3 + 1}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27}$$

$$= \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27}$$

$$= \frac{-10 - 2 + 39}{27} = \frac{27}{27} = 1$$

Thus $x = 1 \quad , \quad y = 0 \quad , \quad z = 1$

(ii) Given system of linear equations is

$2x_1 - x_2 + x_3 = 5$

$4x_1 + 2x_2 + 3x_3 = 8$

$3x_1 - 4x_2 - x_3 = 3$

In matrix form

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 2(-2+12) + 1(-4-9) + 1(16-6)$$

$= 20 - 13 - 22 = -15 \neq 0$

As $|A| \neq 0$

$\Rightarrow$ solution exists.

Now by Cramer's rule.

$$x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{-15}$$

$$= \frac{5(-2+12) + 1(-8-9) + 1(-32-6)}{-15}$$

$$= \frac{50 - 17 - 38}{-15} = \frac{-5}{-15} = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{-15}$$

$$= \frac{2(-8-9) - 5(-4-9) + 1(12-24)}{-15}$$

$$= \frac{-34 + 65 - 12}{-15} = \frac{19}{-15} = -\frac{19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{-15}$$

$$= \frac{2(6+32) + 1(12-24) + 5(-16-6)}{-15}$$

$$= \frac{76 - 12 - 110}{-15} = \frac{46}{15}$$

So $x_1 = \frac{1}{3}$, $x_2 = -\frac{19}{15}$, $x_3 = \frac{46}{15}$

(iii) Given system of linear equations is

$2x_1 - x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 6$

$x_1 - 2x_2 - x_3 = 1$

In matrix form

Let

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 2(-2+4) + (-1-2) + (-2-2) = 4 - 3 - 4 = -3 \neq 0$$

As $|A| \neq 0 \Rightarrow$ solution exists.

Now by Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3}$$

$$= \frac{8(-2+4) + 1(-6-2) + 1(-12-2)}{-3}$$

$$= \frac{16 - 8 - 14}{-3} = \frac{-6}{-3} = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3}$$

$$= \frac{2(-6-2) - 8(-1-2) + 1(1-6)}{-3}$$

$$= \frac{-16 + 24 - 5}{-3} = \frac{3}{-3} = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3}$$

$$= \frac{2(2+12) + 1(1-6) + 8(-2-2)}{-3}$$

$$= \frac{28 - 5 - 32}{-3} = \frac{-9}{-3} = 3$$

So $x_1 = 2$, $x_2 = -1$, $x_3 = 3$.

Q.2 Use matrices to solve the following system:

(i) $x - 2y + z = -1$

$3x + y - 2z = 4$

$y - z = 1$

(ii) $2x_1 + x_2 + 3x_3 = 3$

$x_1 + x_2 - 2x_3 = 0$

$-3x_1 - x_2 + 2x_3 = -4$

(iii) $x + y = 2$

$2x - z = 1$

$2y - 3z = -1$

Solution:

(i) Given system is

$x - 2y + z = -1$

$3x + y - 2z = 4$

$y - z = 1$

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ ..... (1)

where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 1(-1 + 2) + 2(-3 - 0) + 1(3 - 0)$

$= 1 - 6 + 3 = -2 \neq 0$

$|A| \neq 0 \Rightarrow$ inverse of A exists.

Now

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = (-1)^2 (-1 + 2) = 1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = (-1)^3 (-3 - 0) = 3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = (-1)^4(3-0) = 3$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^3(2-1) = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-1-0) = -1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (-1)^5(-1-0) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-1)^4(4-1) = 3$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = (-1)^5(-2-3) = 5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = (-1)^6(1+6) = 7$$

As $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$$= \frac{1}{-2}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{-2}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -3 & 7 \end{bmatrix}$$

Put values in (1)

$$X = -\frac{1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -3 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-2}{-2} \\ \frac{-2}{-2} \\ \frac{0}{-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x = 1,\ y = 1,\ z = 0$$

(ii) Given system is

$$2x_1 + x_2 + 3x_3 = 3$$
$$x_1 + x_2 - 2x_3 = 0$$
$$-3x_1 - x_2 + 2x_3 = -4$$

In matrix form

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$A\ X = B \qquad \text{(say)}$$
$$X = A^{-1} B$$

where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix}$$

$$= 2(2-2) - 1(2-6) + 3(-1+3) = 0 + 4 + 6 = 10 \neq 0$$

$|A| \neq 0 \Rightarrow A^{-1}$ exists.

Now

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = (-1)^2 (2-2) = 0$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = (-1)^3 (2-6) = 4$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^4 (-1+3) = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = (-1)^3 (2+3) = -5$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = (-1)^4 (4+9) = 13$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^5 (-2+3) = -1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^4 (-2-3) = -5$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^5 (-4-3) = 7$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^6 (2-1) = 1$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0-0+20 \\ 12+0-28 \\ 6+0-4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -16 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{20}{10} \\ \frac{-16}{10} \\ \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{8}{5} \\ \frac{1}{5} \end{bmatrix}$$

$\Rightarrow \quad x_1 = 2 \ , \ x_2 = -\frac{8}{5} \ , \ x_3 = \frac{1}{5}$

(iii) Given system is

$$x + y = 2$$
$$2x - z = 1$$
$$2y - 3z = -1$$

In matrix form

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= 1(0+2) - 1(-6-0) \neq 0(4-0) = 2+6+0 = 8 \neq 0$$

$|A| \neq 0 \Rightarrow A^{-1}$ exists.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (-1)^2(0+2) = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = (-1)^3(-6) = 6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (-1)^4(4-0) = 4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = (-1)^3(-3-0) = 3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-1)^4(-3-0) = -3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = (-1)^5(2-0) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-1-0) = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (-1)^5(-1-0) = 1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)^6(0-2) = -2$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{8}{8} \\ \frac{8}{8} \\ \frac{8}{8} \end{bmatrix}$$

$\Rightarrow$ $x = 1$, $y = 1$, $z = 1$.

Q.3 Solve the following systems by reducing their augmented matrices into echelon or reduced echelon from

(i) $x_1 - 2x_2 - 2x_3 = -1$
$2x_1 + 3x_2 + x_3 = 1$
$5x_1 - 4x_2 - 3x_3 = 1$

(ii) $x + 2y + z = 2$
$2x + y + 2z = -1$
$2x + 3y - z = 9$

(iii) $x_1 + 4x_2 + 2x_3 = 2$
$2x_1 + x_2 - 2x_3 = 9$
$3x_1 + 2x_2 - 2x_3 = 12$

Solution:

(i) Given system is

$x_1 - 2x_2 - 2x_3 = -1$
$2x_1 + 3x_2 + x_3 = 1$
$5x_1 - 4x_2 - 3x_3 = 1$

Augmented matrix is

$$A_b = \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$$

Method 1: By Echelon Form:

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2-2(1) & 3-2(-2) & 1-2(-2) & : & 1-2(-1) \\ 5-5(1) & -4-5(-2) & -3-5(-2) & : & 1-5(-1) \end{bmatrix} \text{By} \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 7 & 5 & : & 3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 6 & 7 & : & 6 \end{bmatrix} \text{By } \left(\frac{1}{7}\right) R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 6-6(1) & 7-6\left(\frac{5}{7}\right) & : & 6-6\left(\frac{3}{7}\right) \end{bmatrix} \text{By } R_3 - 6R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & \frac{19}{7} & : & \frac{24}{7} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{By } \left(\frac{7}{19}\right) R_3$$

Now the Augumented matrix is in row echelon form

It can be written as

$x_1 - 2x_2 - 2x_3 = -1$ ..... (1)

$x_2 + \frac{5}{7}x_3 = \frac{3}{7}$ ..... (2)

$x_3 = \frac{24}{19}$ ..... (3)

Put $x_3 = \frac{24}{19}$ in (2)

$$x_2 + \frac{5}{7}\left(\frac{24}{19}\right) = \frac{3}{7}$$

$$x_2 + \frac{120}{133} = \frac{3}{7}$$

$$x_2 = \frac{3}{7} - \frac{120}{133} = \frac{57 - 120}{133} = \frac{-63}{133} = \frac{-9}{19}$$

Put $x_2 = \frac{-9}{19}$ and $x_3 = \frac{24}{19}$ in (1)

$$x_1 - 2\left(-\frac{9}{19}\right) - 2\left(\frac{24}{19}\right) = -1$$

$$\Rightarrow \quad x_1 + \frac{18}{19} - \frac{48}{19} = -1$$

$$\Rightarrow \quad x_1 + \frac{18 - 48}{19} = -1$$

$$\Rightarrow \quad x_1 - \frac{30}{19} = -1$$

$$\Rightarrow \quad x_1 = \frac{30}{19} - 1 = \frac{30 - 19}{19}$$

$$\Rightarrow \quad x_1 = \frac{11}{19}$$

Thus required solution is

$$x_1 = \frac{11}{19}, \quad x_2 = -\frac{9}{19}, \quad x_3 = \frac{24}{19}$$

Method 2: By Reduced Echelon Form

Here we reduce the matrix.

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$ to reduced echelon form

$\Rightarrow$

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2+2(1) & -2+2\left(\frac{5}{7}\right) & : & -1+2\left(\frac{3}{7}\right) \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{ By } R_1 + 2R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -\frac{4}{7} & : & -\frac{1}{7} \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -\frac{4}{7}+\frac{4}{7}(1) & : & -\frac{1}{7}+\frac{4}{7}\left(\frac{24}{19}\right) \\ 0 & 1 & \frac{5}{7}-\frac{5}{7}(1) & : & \frac{3}{7}-\frac{5}{7}\left(\frac{24}{19}\right) \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{ By } \begin{matrix} R_1 + \frac{4}{7}R_3 \\ R_2 - \frac{5}{7}R_3 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & : & \frac{11}{19} \\ 0 & 1 & 0 & : & -\frac{9}{19} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

It can be written as

$$x_1 = \frac{11}{19}, \quad x_2 = -\frac{9}{19}, \quad x_3 = \frac{24}{19}$$

Which is required solution by reduced echelon form.

(ii) Given system is

$x + 2y + z = 2$

$2x + y + 2z = -1$

$2x + 3y - z = 9$

Augmented matrix is given by

$$A_b = \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

Method I: By Echelon form

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 2 & 3 & -1 & 9 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 2-2(1) & 1-2(2) & 2-2(1) & -1-2(2) \\ 2-2(1) & 3-2(2) & -1-2(1) & 9-2(2) \end{array}\right] \text{ By } \begin{array}{l} R_2-2R_1 \\ R_3-2R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -5 \\ 0 & -1 & -3 & 5 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & -1 & -3 & 5 \end{array}\right] \text{By } \left(-\frac{1}{3}\right)R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & -1+1 & -3+0 & 5+\frac{5}{3} \end{array}\right] \text{ By } R_3+R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & -3 & -\frac{20}{3} \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right] \text{ By } \left(-\frac{1}{3}\right)R_3$$

Now it is in echelon form so it can be written as

$x + 2y + z = 2$ ..... (1)

$y = \frac{5}{3}$ ..... (2)

$z = -\frac{20}{9}$ ..... (3)

Put (2) and (3) in equation (1)

$$x + 2\left(\frac{5}{3}\right) + \left(-\frac{20}{9}\right) = 2$$

$$x + \frac{10}{3} - \frac{20}{9} = 2$$

$$x = 2 - \frac{10}{3} + \frac{20}{9} = \frac{18 - 30 + 20}{9}$$

$$x = \frac{8}{9}$$

Thus solution is $x = \frac{8}{9}$, $y = \frac{5}{3}$, $z = -\frac{20}{9}$

Method 2: By Reduced Echelon Form:

Here

We reduce the matrix $\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$ in reduced echelon form.

so $\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2-2(1) & 1-2(0) & : & 2-2\left(\frac{5}{3}\right) \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \text{ By } R_1 - 2R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & : & -\frac{4}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1-1 & : & -\frac{4}{3}+\frac{20}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \text{ By } R_1 - R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{8}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$$

Now

It can be written as

$$x = \frac{8}{9}, \quad y = \frac{5}{3}, \quad z = -\frac{20}{9}$$

which is the required solution.

(iii) Given system is

$$x_1 + 4x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 9$$
$$3x_1 + 2x_2 - 2x_3 = 12$$

Augmented matrix is given by

$$A_b = \begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$$

Method 1 : By Echelon form

$$\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2-2(1) & 1-2(4) & -2-(2) & : & 9-2(2) \\ 3-3(1) & 2-3(4) & -2-3(2) & : & 12-3(2) \end{bmatrix} \text{ By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & -7 & -6 & : & 5 \\ 0 & -10 & -8 & : & 6 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & \frac{5}{7} \\ 0 & -10 & -8 & : & 6 \end{bmatrix} \text{ By } -\left(-\frac{1}{7}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & -10+10(1) & -8+10\left(\frac{6}{7}\right) & : & 6+10\left(-\frac{5}{7}\right) \end{bmatrix} \text{ By } R_3 + 10R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & \frac{4}{7} & : & -\frac{8}{7} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \text{ By } \left(\frac{7}{4}\right) R_3$$

The Augumented matrix is n echelon form it can be written as

$$x_1 + 4x_2 + 2x_3 = 2 \qquad \ldots\ldots\ldots (1)$$

$$x_2 + \frac{6}{7} x_3 = -\frac{5}{7} \qquad \ldots\ldots\ldots (2)$$

$$x_3 = -2 \qquad \ldots\ldots\ldots (3)$$

Put $x_3 = -2$ in equation (2)

$$x_2 + \frac{6}{7}(-2) = -\frac{5}{7}$$

$$x_2 - \frac{12}{7} = -\frac{5}{7}$$

$$x_2 = \frac{12}{7} - \frac{5}{7} = \frac{12-5}{7} = \frac{7}{7} = 1$$

Put $x_2 = 1$ and $x_3 = -2$ in equation (1)

$$x_1 + 4(1) + 2(-2) = 2$$

$$x_1 = 2 + 4 - 4 = 2$$

$$x_1 = 2$$

so required solution is

$$x_1 = 2 \;,\quad x_2 = 1 \;,\quad x_3 = -2$$

Method 2 By Reduced Echelon Form:

Now we reduce $\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$ in reduced form

So $\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4-4(1) & 2-4\left(\frac{6}{7}\right) & : & 2-4\left(-\frac{5}{7}\right) \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{10}{7} & : & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{10}{7}+\frac{10}{7}(1) & : & \frac{34}{7}+\frac{10}{7}(-2) \\ 0 & 1 & \frac{6}{7}-\frac{6}{7}(1) & : & -\frac{5}{7}-\frac{6}{7}(-2) \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \text{ By } \begin{matrix} R_1+\frac{10}{7}R_3 \\ R_2-\frac{6}{7}R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

It can be written as

$x_1 = 2\ ,\quad x_2 = 1\ ,\quad x_3 = -2$

which is the required solution by Reduced Echelon form.

Q.4 Solve the following systems of homogeneous linear equations.

(i)
$$\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - 4x_3 &= 0 \end{aligned}$$

(iii)
$$\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ x_1 + x_2 + 5x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0 \end{aligned}$$

Solution:

(i) Given system is

$x + 2y - 2z = 0$ ..... (1)

$2x + y + 5z = 0$ ..... (2)

$5x + 4y + 8z = 0$ ..... (3)

In matrix form, it can be written as

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$

where $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$= 1(8 - 20) - 2(16 - 25) - 2(8 - 5)$

$= -12 + 18 - 6 = 0$

$\Rightarrow \quad |A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ given system has a non–trivial solution.

Now we sole the system such that from equation (1) and (2)

$x_1 + 4x_2 + 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-12-2} = \frac{-x_2}{-3-4} = \frac{x_3}{1-8}$$

$$\Rightarrow \quad \frac{x_1}{-14} = \frac{-x_2}{-7} = \frac{x_3}{-7}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{x_1}{2} = t \Rightarrow x_1 = 2t$$

$\Rightarrow \quad x_2 = -t \quad , \quad x_3 = t$

The system has infinite many solutions depending upon the value of t.

(ii) Given system is

$x_1 + 2y - 2z = 0$ ..... (1)

$2x + y + 5z = 0$ ..... (2)

$5x + 4y + 8z = 0$ ..... (3)

In matrix form, we have

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

$X = A^{-1} B$

Where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$$= 1(8-20) - 2(16-25) - 2(8-5)$$

$$= -12 + 18 - 6 = 0$$

$\Rightarrow$ $|A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ given system has a non–trivial solution.

Now we solve the system such that from equation (1) and (2)

$x + 2y - 2z = 0$

$2x + y + 5z = 0$

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x}{10+2} = \frac{-y}{5+4} = \frac{z}{1-4}$$

$$\frac{x}{12} = \frac{-y}{9} = \frac{z}{-3}$$

multiplying by (-3)

$$\frac{x}{-4} = \frac{-y}{3} = \frac{z}{1} = t \quad \text{(say)}$$

$\Rightarrow$ $\frac{x}{-4} = t \Rightarrow x = -4t$

$\Rightarrow$ $\frac{y}{3} = t \Rightarrow y = 3t$

and $z = t$

$\Rightarrow$ $x = -4t$, $y = 3t$, $z = t$ is the solution of the system.

The system has infinite many solutions depending upon the value of t.

(iii) Given system is

$x_1 - 2x_2 - x_3 = 0$ ..... (1)

$x_1 + x_2 + 5x_3 = 0$ ..... (2)

$2x_1 - x_2 + 4x_3 = 0$ ..... (3)

In matrix form, we have

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

$X = A^{-1} B$

where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$= 1(4 + 5) + 2(4 - 10) - 1(-1 - 2)$

$= 9 - 12 + 3 = 0$

$|A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ system has non–trivial solution.

Now we find the solution such that from equation (1) and (2)

$x_1 - 2x_2 - x_3 = 0$

$x_1 + x_2 + 5x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-10+1} = \frac{-x_2}{5+1} = \frac{x_3}{1+2}$$

$$\frac{x_1}{-9} = \frac{-x_2}{6} = \frac{x_3}{3}$$

Multiplying by $+3$

$$\frac{x_1}{-3} = \frac{x_2}{-2} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{x_1}{-3} = t \Rightarrow x_1 = -3t$$

$$\Rightarrow \quad \frac{x_2}{-2} = t \Rightarrow x_2 = -2t$$

$$\Rightarrow \quad x_3 = t$$

$\Rightarrow$ $x_1 = -3t$, $x2 = -2t$, $x_3 = t$ is the solution of the system.

The system has infinite many solutions depending upon the value of t.

Q.5 Find the value of λ for which the following systems has a non–trivial solution. Also solve the system for the value of a

(i) $\quad x + y + z = 0$

$\quad 2x + y - \lambda z = 0$

$\quad x + 2y - 2z = 0$

(ii) $\quad x_1 + 4x_2 + \lambda x_3 = 0$

$\quad 2x_1 + x_2 - 3x_3 = 0$

$\quad 3x_1 + \lambda x_2 - 4x_3 = 0$

Solution:

(i) The given system is

$x + y + z = 0$ ..... (1)

$2x + y - \lambda z = 0$ ..... (2)

$x + 2y - 2z = 0$ ..... (3)

In matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

It is given that the system has non–trivial solution.

$\Rightarrow$ $|A| = 0$

$\Rightarrow$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$1(-2 + 2\lambda) - 1(-4 + \lambda) + 1(4 - 1) = 0$

$2\lambda - 2 + 4 - \lambda + 3 = 0$

$\lambda + 5 = 0 \Rightarrow \lambda = -5$

Take equation(1) and (2)

$X + y + z = 0$

$2x + y + 5z = 0$ ¤ $\lambda = 5$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{5-1} = \frac{-y}{5-2} = \frac{z}{1-2}$$

$$\frac{x_1}{4} = \frac{-y}{3} = \frac{z}{-1}$$

multiplying by (-1)

$$\frac{x}{-4} = \frac{y}{3} = \frac{z}{1} = t \quad \text{(say)}$$

$\Rightarrow \quad \frac{x}{-4} = t \Rightarrow x = -4t$

$\Rightarrow \quad \frac{y}{3} = t \Rightarrow y = 3t$

$\Rightarrow$ and $z = t$

$\Rightarrow$ $x = -4t$, $y = 3t$, $z = t$ is the solution of the system.

(ii) Given system is

$x_1 + 4x_2 + \lambda x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$3x_1 + \lambda x_2 - 4x_3 = 0$

In matrix form, we have

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

As it is given that the system has non–trivial solution.

so $|A| = 0$

i.e. $$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-4 + 3\lambda) - 4(-8 + 9) + \lambda(2\lambda - 3) = 0$

$\Rightarrow \quad 3\lambda - 4 - 4 + 2\lambda^2 - 3\lambda = 0$

$\Rightarrow \quad 2\lambda^2 - 8 = 0$

$\Rightarrow \quad \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

When $\lambda = 2$

equation(1) and (2) becomes

$x_1 + 4x_2 + 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-12-2} = \frac{-x_2}{-3-4} = \frac{x_3}{1-8}$$

$$\frac{x_1}{-14} = \frac{-x_2}{-7} = \frac{x_3}{-7}$$

multiplying by (-7)

$$\Rightarrow \quad \frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{x_1}{2} = t \Rightarrow x_1 = 2t$$

$$\Rightarrow \quad \frac{-x_2}{1} = t \Rightarrow x_2 = -t$$

and $\quad x_3 = t$

When $\lambda = -2$

equation(1) and (2) becomes

$x_1 + 4x_2 - 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$\Rightarrow$

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-12+2} = \frac{-x_2}{-3+4} = \frac{x_3}{1-8}$$

$$\frac{x_1}{-10} = \frac{-x_2}{1} = \frac{x_3}{-7}$$

Multiplying by (-7)

$$\frac{7x_1}{10} = \frac{7x_2}{1} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{7x_1}{10} = t \Rightarrow x_1 = \frac{10}{7}t$$

$$\Rightarrow \quad \frac{7x_2}{1} = t \Rightarrow x_2 = \frac{1}{7}t$$

and $\quad x_3 = t$

is required solution.

Q.6 Find the value of λ for which the following system does not possess a unique solution. Also solve the system for value of λ.

$$\mathbf{x_1 + 4x_2 + \lambda x_3 = 2}$$

$$\mathbf{2x_1 + x_2 - 2x_3 = 11}$$

$$\mathbf{3x_1 + 2x_2 - 2x_3 = 16}$$

Solution:

Given system is

$$x_1 + 4x_2 + \lambda x_3 = 2$$

$$2x_1 + x_2 - 2x_3 = 11$$

$$3x_1 + 2x_2 - 2x_3 = 16$$

In matrix form, we have

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 16 \end{bmatrix}$$

$$A\ X = B \qquad \text{(say)}$$

It is given that the system does not possess a unique solution.

$$\Rightarrow \quad |A| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$\Rightarrow \quad 2 - 8 + \lambda = 0$

$\Rightarrow \quad \lambda - 6 = 0 \Rightarrow \lambda = 6$

Put $\lambda = 6$ in the given system it becomes

$x_1 + 4x_2 + 6x_3 = 2$

$2x_1 + x_2 - 2x_3 = 11$

$3x_1 + 2x_2 - 2x_3 = 16$

The augumented matrix of this system is

$$A_b = \left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{array}\right]$$

we reduce this matrix to reduced echelon form, i.e.

$$\left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 2-2(1) & 1-2(4) & -2-2(6) & 11-2(2) \\ 3-3(1) & 2-3(4) & -2-3(6) & 16-3(2) \end{array}\right] \text{By} \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -10 & -20 & 10 \end{array}\right] \text{By} \left(-\frac{1}{7}\right)R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 4-4(1) & 6-4(2) & 2-4(-1) \\ 0 & 1 & 2 & -1 \\ 0 & -10+10(1) & -20+10(2) & 10+10(-1) \end{array}\right] \text{By} \begin{array}{l} R_1 - 4R_2 \\ R_3 - 10R_2 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 0 & -2 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It can be written as

$x_1 + 0x_2 - 2x_3 = 6 \qquad \ldots\ldots\ldots (1)$

$x_2 + 2x_3 = -1 \qquad \ldots\ldots\ldots (2)$

$0x_3 = 0$

Let $x_3 = t$ (arbitrary value)

Put in (2)

$x_2 + 2t = -1$

$x_2 = -2t - 1$

Put $x_3 = t$ in (1)

$x_1 - 2t = 6$

$\Rightarrow \quad x_1 = 2t + 6$

$\Rightarrow \quad x_1 = 2t + 6 \;,\; x_2 = -2t - 1 \;,\; x_3 = t$

This system has infinitely many solutions depending upon the value of t.